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Referees: Prof. Dr. techn. Günther Meschke
Lehrstuhl für Statik und Dynamik
Fakultät für Bau- und Umweltingenieurwissenschaften
Ruhr-Universität Bochum

Prof. Dr.-Ing. Holger Steeb
Lehrstuhl für Kontinuumsmechanik
Fakultät für Bau- und Umweltingenieurwissenschaften
Ruhr-Universität Bochum

Prof. Antonio Gens
Department of Geotechnical Engineering and GeoSciences
Technical University of Catalunya, Barcelona, Spain

Univ.-Prof. Dr.-Ing. Rolf Breitenbücher
Lehrstuhl für Baustofftechnik
Fakultät für Bau- und Umweltingenieurwissenschaften
Ruhr-Universität Bochum
Abstract

In geotechnical applications of artificial ground freezing, safe design and execution require a reliable prediction of the coupled thermo-hydro-mechanical behavior of soils subjected to freezing. In the context of thermo-poro-plasticity, a three-phase Finite Element model of freezing soils is presented in this thesis, considering solid particles, liquid water and crystal ice as separate phases, and mixture temperature, liquid pressure, and solid displacement as primary field variables. Through three fundamental physical laws and corresponding state relations, the model captures the most relevant couplings between the phase transition, the liquid transport within the pores, and the accompanying mechanical deformation. In addition to the volume expansion of water transforming into ice, the contribution of the micro-cryo-suction mechanism to the frost heave phenomenon is described in the model using the theory of premelting dynamics. The different components of the model (e.g. 1D consolidation, phase transition with latent heat effect, and strain development during freezing) are validated separately for each stage of the model development.

For the description of the elastoplastic stress-strain response of freezing soil, a new critical state constitutive model is presented by employing the Clay and Sand Model for the reference unfrozen state, which is then extended to the freezing state by adopting concepts of the enhanced Barcelona Basic Model. In particular, in order to establish a comprehensive model for the prediction of the temperature- and porosity-dependent strength criterion of freezing soils, a novel multi-scale strength homogenization procedure is proposed, which allows to determine the macroscopic cohesion and frictional coefficient based on the current state of the microstructure of freezing soils, i.e. the volume fraction of the soil particles and the temperature-dependent partial saturation of the liquid water or the crystal ice in the pore space. The homogenization method is characterized by a two-step strength upscaling strategy, where in each step a strength homogenization scheme for a two-phase composite of matrix-inclusion morphology is employed based upon an extension of the linear comparison composite method. The validation of the proposed strength homogenization model is accomplished by means of analyses on selected scenarios of composite materials with different strength properties, while the verification of the implemented constitutive model is conducted through four benchmark tests on soils in unfrozen and freezing states.

The performance and applicability of the developed model are demonstrated by means of a one-dimensional soil freezing test exploring the development of frost fringe with frost accretion, together with a case study on artificial ground freezing processes during tunnel excavation with extensive investigation on ground heave prediction, influence of seepage flow on the formation of the desired frost arch, and optimization of freeze pipe arrangement in presence of seepage flow.
The present research work has been carried out between the winter of 2008 and the winter of 2013 at the Institute for Structural Mechanics of Ruhr University Bochum. This period has been a precious and profitable experience in my life.

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Meng-Meng Zhou
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List of Symbols

Abbreviations

CASM  The Clay and Sand Model for saturated soils
CCM  The Cam-clay model for saturated soils
CSH  Calcium Silicate Hydrates
CSL  Critical State Line
DOF  Degree Of Freedom
DP  Drucker-Prager
eBBM  The enhanced BBM for freezing soils
EL  Elliptical
FE  Finite Element
IBVP  Initial Boundary Value Problem
iso-NCL  isotropic Normal Consolidation Line
LCC  Linear Comparison Composite
NE  Number of Elements
NG  Number of GAUSSIAN quadrature points
NN  Number of Nodes
NT  Number of Time steps
OCR  Over Consolidation Ratio
REV  Representative Elementary Volume
SWCC  Soil-Water Characteristic Curve
THM  Thermo-Hydro-Mechanical
TPM  Theory of Porous Media
URL  The Unloading-Reloading Line
Poromechanics

\( \alpha_S \) volumetric thermal dilation coefficient of solid phase \([K^{-1}]\)
\( \alpha_J \) volumetric thermal dilation coefficient of phase \( J \) \([K^{-1}]\)
\( \bar{\rho} \) equivalent pore pressure \([\text{Pa}]\)
\( \beta \) parameter controlling stiffness growth rate in BBM \([\text{Pa}]\)
\( \varepsilon^{\text{rel}} \) effective elastic tangent modulus \([\text{Pa}]\)
\( \mathcal{I} \) symmetric projection tensor \([\text{Pa}]\)
\( \sigma \) tensor of total stresses \([\text{Pa}]\)
\( \varepsilon, \varepsilon_{ij} \) current linearized strain tensor and its components
\( g \) gravity force per unit volume \([\text{N/m}^3]\)
\( q \) heat flux vector \([\text{W/m}^2]\)
\( u \) displacement vector of the solid and crystal \([\text{m}]\)
\( v_L \) Darcy’s velocity \([\text{m/s}]\)
\( w_J \) Eulerian relative mass flow vector \([\text{kg/(m}^2 \text{s})]\)
\( \chi_J \) Eulerian saturation of phase \( J \)
\( \chi \) Bishop parameter
\( \Delta T_{\text{ch}} \) characteristic cooling temperature related to \( R_{\text{ch}} \) \([\text{K}]\)
\( \delta_{ij} \) Kronecker delta
\( d\lambda \) plastic multiplier
\( df_{\Omega_0}, df \) initial and current volume of a representative element \([\text{m}^3]\)
\( \varepsilon^{\text{pl}} \) volumetric plastic strain
\( \varepsilon^{\text{pl}}_S \) volumetric plastic strain of the solid grains
\( \varepsilon \) volumetric strain
\( \eta_0 \) reference viscosity at full saturation \([\text{Pa s}]\)
\( \eta_{\text{rel}} \) temperature dependence function of viscosity
\( \gamma_{\text{CL}}, \gamma_{\text{GL}} \) liquid-crystal and liquid-air interface energy \([\text{kg/s}^2]\)
\( \kappa_{\text{rel}} \) relative permeability \([\text{m}^2]\)
\( \kappa \) swelling index (slope of URL)
\( \lambda_i \) thermal conductivity of material phase \( i \) \([\text{W/(m K)}]\)
\( \lambda_{\text{tot}} \) overall thermal conductivity \([\text{W/(m K)}]\)
\( \lambda_T \) temperature-dependent slope of iso-NCL in the freezing state
\( \lambda \) slope of iso-NCL
\( N \) capillary modulus \([\text{Pa}]\)
\( \mu_{J0} \) specific chemical potential associated in reference condition \([\text{J/kg}]\)
\( \mu_J \) current specific Gibbs chemical potential of phase \( J \) \([\text{J/kg}]\)
\( \nu \) Poisson ratio
\( \dot{m}_{\text{L-C}} \) rate of liquid water mass changing into crystal ice \([\text{kg/(m}^3 \text{s})]\)
\( \phi^{\text{el}} \) elastic LAGRANGIAN porosity
\( \phi^{\text{pl}} \) plastic LAGRANGIAN porosity
\( \phi_{\text{crit}} \) critical porosity

\( \Phi_{\text{CSL}} \) friction angle at the critical state [°]

\( \Phi_{\text{M}} \) mechanical dissipation associated with the viscous liquid flow \([\text{W/m}^3]\)

\( \phi_s \) LAGRANGIAN volume fraction of the solid phase

\( \phi_0, \phi \) initial and current LAGRANGIAN porosity

\( \phi_J \) current LAGRANGIAN partial porosity of phase \( J \)

\( \Psi_S \) free energy of the skeleton \([\text{J/m}^3]\)

\( \rho_{10} \) specific mass density in reference condition \([\text{kg/m}^3]\)

\( \rho_0, \rho_J \) initial mass density and mass density related to phase \( J \) \([\text{kg/m}^3]\)

\( \sigma_{ij}, \sigma^{\prime}_{ij} \) total and effective stress \([\text{Pa}]\)

\( \varepsilon_{ij}^{\text{el}}, \varepsilon_{ij}^{\text{pl}} \) elastic and plastic strains

\( \varepsilon_{ij}^{\text{pl}}, \varepsilon_{ij}^{\text{pl}} \) volumetric and deviatoric parts of plastic strain tensor

\( \vartheta \) LODE’s angle [°]

\( \zeta \) hardening force [Pa]

\( a, a_{\phi} \) volumetric thermal dilation related to the empty porous solid and to the porous volume \([\text{K}^{-1}]\)

\( b \) BIOT coefficient

\( c_{\text{hom}} \) homogenized cohesion for drained partially frozen soils \([\text{Pa}]\)

\( c_s \) specific heat capacity of solid phase \([\text{J/kg K}]\)

\( C_s \) volumetric heat capacity of the skeleton \([\text{J/(K m}^3])\)

\( c_J \) specific heat capacity of phase \( J \) \([\text{J/kg K}]\)

\( c_T \) intersection of the CSL with \( q \)-axis \([\text{Pa}]\)

\( e_{ij} \) deviatoric components of the strain tensor

\( e \) void ratio

\( f \) yield function \([\text{Pa}]\)

\( g \) plastic potential \([\text{Pa}]\)

\( H \) hardening modulus \([\text{Pa}]\)

\( h \) hardening state variable

\( J_3 \) third principal invariant of the effective stress \([\text{Pa}^3]\)

\( k_s, g_s \) bulk and shear modulus of solid particle \([\text{Pa}]\)

\( k_J, g_J \) bulk and shear modulus of phase \( J \) \([\text{Pa}]\)

\( K, G \) effective bulk and shear modulus of the solid skeleton \([\text{Pa}]\)

\( M_{\text{max}} \) slope of the CSL in unfrozen state under triaxial compression

\( M_{\vartheta} \) slope of the CSL in unfrozen state dependent on \( \vartheta \)

\( M_{\vartheta T} \) slope of the CSL in freezing state dependent on \( \vartheta \) and \( T \)

\( m_J \) current mass per unit initial volume of phase \( J \) \([\text{kg/m}^3]\)

\( m \) parameter indicating the pore radius distribution around \( R_{\text{ch}} \)

\( N \) BIOT tangent modulus \([\text{Pa}]\)

\( n \) EULERIAN porosity of current volume

\( nn, rr \) shape parameter and spacing ratio

\( p_{\text{atm}} \) reference atmospheric pressure \([\text{Pa}]\)
\( p_G \) gas pressure \([\text{Pa}]\)  
\( p_{L, C} \) liquid water and crystal ice pressure \([\text{Pa}]\)  
\( p_s \) external pressure applied to solid particle \([\text{Pa}]\)  
\( p_T \) thermo-molecular pressure \([\text{Pa}]\)  
\( p'_0 \) preconsolidation pressure \([\text{Pa}]\)  
\( p', q \) volumetric and deviatoric parts of the effective stress \([\text{Pa}]\)  
\( p_{ch} \) most frequently encountered pore radius \([\text{Pa}]\)  
\( r \) parameter defining max. compressive stiffness  
\( R \) mean curvature radius of the interface \([\text{m}]\)  
\( s_{j0} \) specific entropy density associated in reference condition \([\text{J/(K m}^3])\)  
\( S_f \) freezing entropy per unit of volume \([\text{J/(K m}^3])\)  
\( S_S \) entropy of the solid phase \([\text{J/(K m}^3])\)  
\( s_J \) specific entropy of phase \( J \) \([\text{J/(K m}^3])\)  
\( S_T \) homogenized hydrostatic tensile strength \([\text{Pa}]\)  
\( S \) density of entropy per unit volume \([\text{J/(K m}^3])\)  
\( T_f \) bulk freezing temperature \([\text{K}]\)  
\( T_0, T \) initial and current temperature \([\text{K}]\)  
\( U \) current trapped free energy of the soil skeleton \([\text{J/m}^3])\)  
\( u \) specific volume \([\text{m}^3/kg]\)  
\( W_S \) reversible part of the free energy the soil skeleton \([\text{J/m}^3]\)  

**Finite Element Formulation**

\((\bullet)^*\) variable boundary value  
\((\bullet)_{col}\) \( col \)-th column vector of the matrix  
\((\bullet)_0\) variable initial value  
\( \alpha, \beta, \gamma \) time integration parameters  
\( \Delta \delta d_{n+1} \) incremental update of primary variables at time step end point  
\( \delta d \) test funtions or primary variable variation  
\( \xi^*, \omega^* \) local coordinates and integration weight of the GAUSSIAN point \( * \)  
\( \xi \) local coordinates  
\( d_{n+1} \) primary variables at time step end point  
\( D \) GATEAUX derivatives of \( R_{\text{int}} \) with respect to the their temporal derivatives  
\( d \) primary variables set  
\( J \) JACOBIAN matrix  
\( K \) GATEAUX derivatives of \( R_{\text{int}} \) with respect to the DOFs  
\( R_{\text{ext}} \) external force vector  
\( R_{\text{int}} \) internal force vector  
\( \delta W^o \) variation of balance of equation
\( \dot{d}_{n+1} \) \( \) temporal derivative of primary variables at time step end point

\( \Gamma_{D/N} \) \( \) DIRICHLET/NEUMMAN boundary

\( d_i^e \) \( \) primary variables values at element node \( i \)

\( N_{\eta/l} \) \( \) quadratic/linear shape function of node \( i \)

\( t_{n+1-\alpha} \) \( \) time step generalized midpoint

\( t_{n+1} \) \( \) time step end point

\( t_n \) \( \) time step start point

**Strength upscaling**

\( \sigma \) \( \) microscopic stress tensor

\( \sigma_m, \sigma_d \) \( \) microscopic stress invariants

\( \Sigma \) \( \) macroscopic stress tensor

\( \Sigma_m, \Sigma_d \) \( \) macroscopic stress invariants

\( \dot{\varepsilon} \) \( \) microscopic strain rate

\( \dot{E} \) \( \) macroscopic strain rate

\( \tau_i \) \( \) prestress of material phase \( i \)

\( 1 \) \( \) second-order identity tensors

\( T_{\text{hom}} \) \( \) macroscopic homogenized prestress

\( v \) \( \) microscopic velocity field

\( x \) \( \) microscopic position vector

\( \chi_{L,crit} \) \( \) critical liquid saturation

\( \chi_L \) \( \) partial saturation of liquid water

\( \dot{\varepsilon}_v, \dot{\varepsilon}_d \) \( \) microscopic volumetric and deviatoric strain rate

\( \dot{E}_v, \dot{E}_d \) \( \) macroscopic volumetric and deviatoric strain rate

\( \lambda \) \( \) plastic multiplier

\( \eta^I, \eta^{II} \) \( \) inclusion volume fraction of matrix-inclusion composite at Level 1 and 2

\( C_{\text{hom}} \) \( \) macroscopic homogenized elastic moduli

\( C_i \) \( \) positive-definite elasticity tensor of material phase \( i \)

\( I \) \( \) fourth-order identity tensors

\( K \) \( \) tensor projection

\( A_i \) \( \) fourth-order concentration tensor for the respective material phase \( i \)

\( F_{\text{hom}} \) \( \) macroscopic strength criterion of the composite

\( F_i \) \( \) convex failure criterion of material phase \( i \)

\( H \) \( \) hyperplane in stress space

\( K, G \) \( \) inclusion morphology factors related to elastic modulus

\( V \) \( \) set of kinematically admissible microscopic velocity fields

\( \Omega_i \) \( \) domain occupied by material phase \( i \)

\( \Omega \) \( \) overall domain occupied by the composite
\( \partial G_{\text{hom}} \) the boundary of the macroscopic strength domain
\( \partial G_i \) the boundary of the admissible microscopic stress domain
\( \partial \Omega \) the boundary of the domain occupied by the composite
\( \phi_{\text{crit}} \) critical value of porosity
\( \phi_i \) volume fraction of phase \( i \)
\( \Pi_{\text{hom}} \) macroscopic support function
\( \pi_i \) support function of material phase \( i \)
\( \pi \) microscopic dissipation function
\( \psi_i \) microscopic strain energy density function of material phase \( i \)
\( \Psi \) macroscopic strain rate energy of the linear comparison composite
\( T_1, T_2 \) inclusion morphology factors related to prestress
\( \tilde{\Pi}_{\text{hom}} \) homogenized macroscopic dissipation function
\( A_{\text{hom}}, B_{\text{hom}}, S_{\text{hom}} \) homogenized strength criterion parameters
\( A_1, B_1, S_1 \) strength criterion parameters of material phase \( i \)
\( a \) regularization parameter \( a \to 0 \)
\( C_{\text{hom}} \) macroscopic homogenized cohesion
\( c_i \) cohesion for material phase \( i \)
\( G_{\text{hom}} \) macroscopic strength domain
\( G_i \) admissible microscopic stress domain of material phase \( i \)
\( K_{\text{hom}} \) homogenized bulk modulus
\( k_i, g_i \) bulk and shear modulus of the material phase \( i \)
\( M_{\text{hom}} \) macroscopic homogenized friction coefficient
\( M_i \) frictional coefficient of material phase \( i \)
\( r_g \) shear modulus ratio \( g_{\text{inc}}/g_{\text{mat}} \)
\( Y_i \) phase-wise constant function representing nonlinearity function

Subscript and superscript

0 reference (atmospheric) condition associated with \( p_J = 0 \) and \( T = T_f \)
\( I, II \) strength homogenization step I and II
\( d, ud \) drained and undrained conditions
\( \text{hom} \) homogenized macroscopic properties
\( i \) material phase \( i = S, L, C \) (Solid particle, Liquid water or Crystal ice)
\( J \) pore fluid phase \( J = L, C \)
\( \text{mat, inc} \) matrix and inclusion phases
Chapter 1

Introduction

1.1 Motivation

Over the past few decades, the growing demand for transport networks has given a boost for underground construction projects in urban areas, particularly in the field of mechanized tunneling. Under difficult geological and hydrological ground conditions, for instance, in water-bearing soft ground, auxiliary ground improvement measures such as grouting or artificial ground freezing (AGF) are often applied during tunnel construction for excavation support and groundwater control.

1.1.1 Artificial ground freezing

In contrast to grouting, AGF is an environment-friendly technique since it neither produces dirt or noise, nor brings impurities into the ground. Moreover, the achieved frozen ground is essentially impermeable and almost twice as strong as concrete. The idea of AGF was first born in 1883 by the German scientist F.H. Poetsch for sinking shafts through quicksand during coal mining, and it has been commonly used in the last 20 to 30 years, e.g. during tunnel excavation with low overburden in sensible urban areas, or for groundwater control during geotechnical construction (see e.g. HASS ET AL. (1994), HASS AND SEEGERS (2000), JESSBERGER (1980), JOHANSSON (2009), SCHULTZ ET AL. (2008)).

The ground freezing process converts pore water into ice by withdrawing heat from the soil. With the ice binding the soil particles, the strength and impermeability of the frozen soil body are significantly improved. In order to establish a frozen soil body, a row of vertical, horizontal or inclined freeze pipes are drilled into the required place. An open-ended inner pipe is inserted into the center of the closed-end freeze pipe (see Figure 1.1). The coolant flows into the inner pipe through the supply line, extracts in the freeze pipe the heat from the soil, and then leaves through the return or exhaust line. In the meanwhile, the frost penetrates the soils and a frozen soil body grows around the pipes. Once the frozen soil body has achieved its required thickness, the freezing is operated at a reduced rate in order to maintain the condition. With proper placement of the freeze pipes, the designed shape of the frozen soil body can be obtained.
Depending on the coolant, there are mainly two types of AGF: brine freezing and liquid nitrogen (LN₂) freezing (see Figure 1.2). Brine freezing occurs in a closed circulation system by using refrigeration plants. The brine (usually calcium chloride CaCl₂), cooled typically down to temperatures ranging from −20 °C to −37 °C, flows through a manifold system before returning to the refrigeration plant, where it is chilled and recirculated. Liquid nitrogen freezing extracts the heat from the soil through direct vaporization of the cryogenic fluid (LN₂) in the freeze pipes. The LN₂, usually stored in an insulated pressure vessel, is fed into the inner pipes through a surface manifold system. In the annulus between freeze and inner pipe, it starts to vaporize at −196 °C after withdrawing the heat from the soil on its way up, and is vented directly into the atmosphere.

Owing to the extremely low temperature, LN₂ freezing acts much faster than brine freezing. A frozen soil body can be formed within a few days with LN₂, whereas it takes weeks for the brine freezing system. However, due to its high costs, LN₂ freezing is usually limited to short-term applications or limited volume of frozen soil. For more details please refer to HASS AND SCHAEFERS (2006), SCHULTZ ET AL. (2008).

1.1.2 Potential problems

The AGF can provide a temporarily strengthened ground and groundwater cutoff in geotechnical engineering. However, this technique is connected with a certain risk of damage of the surface infrastructure, since inevitably, AGF may produce frost heave and thaw settlements at the ground surface, which may cause different settlements and, consequently, cracks in existing buildings. Furthermore, the strength and stiffness properties of frozen soil and the time of establishment of a complete frozen soil body with full temporary load carrying capacity are considerably influenced by groundwater or seepage flow, since it provides a continuous source of heat. In case of large seepage flows, a state of thermal equilibrium can be reached, in which freezing stops and the closure of desired frost wall cannot be developed. As a general rule of thumb (SCHULTZ ET AL. 2008),
the critical seepage velocity, above which the formation of desired frost wall cannot be achieved, is 
approximately 2 m/day for brine freezing and 4 ∼ 6 m/day for LN\textsubscript{2} freezing. Evidently, for the sake 
of a safe geotechnical design and construction, application of the soil freezing technique requires a 
reliable prediction of the coupled thermo-hydro-mechanical (THM) processes in porous materials 
upon freezing.

1.2 State of the art

1.2.1 Multiphase modeling of freezing porous materials

Freezing of porous materials is of great interest for a wide range of fields such as geotechnical engi-
neering, food industry, environmental engineering and biomechanics. However, porous solids with 
pores of various sizes behave intriguing and partially counterintuitive when subjected to freezing. A well-known example is the unexpected expansion observed during the freezing of a sealed ce-
ment paste sample fully saturated with benzene, which contracts when solidifying (FABBRI ET AL. 
2006). During freezing, relatively complex fully coupled THM processes are generally involved, in-
cluding phase transition between constituents, and the micro-cryo-suction mechanism which drives 
the liquid towards the frozen sites.

The continuum mechanics description of porous materials as a multi-phase material whose be-
havior is influenced by the interaction of the solid skeleton and the pore fluids can be accomplished 
either using the micro-scale or macro-scale as a point of departure. The Theory of Mixtures (TRUES-
DELL 1965) has been established as a suitable homogenization procedure, which allows to treat 
multi-phase materials as a continuum while each constituent may be described by its own kinemat-
ics and balance equations. The interactions between the constituents are considered by interaction 
terms within the balance equations. Since the Theory of Mixtures contains no microscopic informa-
tion of the mixture, the enhancement by the concept of volume fractions is necessary, which leads 
to the well established concept of the Theory of Porous Media (TPM) (see, e.g. DE BOER (2000), 
EHLERS (1989)). Describing individual phases and their interactions within porous media on an
a priori macroscopic scale of observation, the TPM constitutes a suitable basis for modeling of freezing processes in soils. This theory has been adopted as the basis for several models for porous materials exposed to freezing.

A mathematical freezing model, based upon the mixture theory and basic principles of thermomechanics, has been established for saturated soils by Mikkola and Hartikainen (1999, 2001, 2002). As a mixture of solid grains, liquid water and ice, this freezing soil model is capable of describing cryogenic suction, migration of water to the freezing zone and consequently porosity growth and frost heave by introducing a particular indicator function into the free energy expression together with other characteristic constitutive relations and conservation laws. However, a regularization procedure is needed to resolve the frozen fringe occurred during phase transition, which on the other hand, causes problems regarding convergence and stability.

Similarly, within the framework of the TPM, a three-phase freezing model for saturated porous media, assumed to consist of a solid phase, pore fluid and pore ice, has been developed by Bluhm and Ricken (2007), Kruschwitz and Bluhm (2005). In the context of geometrically nonlinear continuum mechanics, the set of unknown field quantities are determined by solving the set of field equations (balances of mass, momentum and energy) together with an excess saturation condition for the restriction of the motion of the incompressible constituents. The latent-heat incorporated phase transition between water and ice is modeled via a mass exchange term, and the consequent volume expansion during freezing is adequately presented. Nevertheless, this model provides little physical explanation for some involved quantities, as well as for the observed phenomena such as the cryo-suction process that sucks liquid water towards the already frozen sites.

A relatively comprehensive thermoporoelastic theory for the description of mechanical behavior of water-infiltrated materials under freezing conditions has been recently proposed by (Coussy 2005, 2010). The developed theory specifies the multi-scale physics of confined crystallization of ice and provides a more physics-based understanding by means of exploring how the macroscopic properties can be upscaled from the knowledge of properties at lower scales. As far as physics-based modeling is concerned, the premelting dynamics theory developed by Rempe1 et al. (2004), Wettlaufer and Worster (2006) provides an excellent interpretation of the essential contribution of the pre-melted film water to the generation of frost heave. Thus, in order to develop a first numerical model for the description of the coupled behavior of water-infiltrated soft soils upon freezing, the author would adopt the theory of thermoporoelasticity (Coussy 2010) together with the theory of premelting dynamics (Wetttlaufer and Worster 2006).

1.2.2 Elastoplastic constitutive modeling of soil skeleton

According to the effective stress concept introduced by Terzaghi (1943), the deformations of a granular material, caused by rolling and sliding in the contact points of the grains, are almost completely determined by changes of the concentrated forces in these contact points. Therefore, in order to describe the general stress-strain behavior of soils, the constitutive models are necessarily expressed in terms of effective stresses. This approach allows the extension of the well-known Coulomb failure criterion to the Mohr-Coulomb and Drucker-Prager models (Drucker and Prager 1952) that form the basis of classical soil mechanics theory. In the 1950’s, further
improvements of the constitutive soil models have been made by applying the theory of plasticity in
geomechanics, such as introducing the hardening and/or softening plasticity as well as the critical state concept that reveals, that soils will come ultimately into a critical state where shear distortions occur without any further change in the stress or specific volume (Roscoe et al. 1958).

The first elastoplastic constitutive models, framed within the critical state concept and using an associated flow rule, were developed at the University of Cambridge by Roscoe and his co-workers, including the original Cam-Clay model (Roscoe and Schofield 1963) and the modified Cam-Clay model (Roscoe and Burland 1968). On the basis of experimental evidence, the standard Cam-Clay models succeed in predicting a great number of the fundamental aspects of real soils, especially of normally consolidated clays. Nevertheless, the adopted yield surfaces in such critical state models often overestimate significantly the failure stresses on the “dry side”. Besides, by assumption of an associated flow rule, these models are unable to reproduce a commonly-observed important feature for loose sand and normally consolidated clays in undrained tests, which is, that the deviatoric stress often reaches a local peak before approaching the critical state condition.

To overcome these limitations, Yu (1998) have proposed a simple and unified constitutive model, namely Clay and Sand Model (CASM), to describe the overall elastoplastic behavior of both saturated clay and sand observed under both drained and undrained loading conditions. Adopting Rowe’s stress-dilatancy relation as the plastic potential, a non-associated flow rule is assumed. Using a general relation between the stress ratio and state parameter as the yield surface, the CASM builds the basis for the description of the elastoplastic behavior of the soil skeleton, from which other soil models can be extended (see Figure 1.3). In terms of an appropriate choice of state parameters, the standard Cam-Clay models, i.e. the original and modified Cam-Clay models, can be reformulated. The cemented soil model is an extension of the CASM which accounts for the effects of cementation and/or structure following the conceptual approach proposed by Gens and Nova (1993). The double hardening model for saturated soils (Schanz et al. 1999) introduces a second yield surface for plastic deformation on shear dominated load paths on the basis of the CASM. The extension of the saturated to unsaturated conditions was first pioneered by Alonso et al. (1990), Sheng et al. (2004) in the Barcelona Basic Model (BBM) by considering the capillary suction – defined as the difference between the gas and the liquid pressure – as an additional stress variable within the effective stress framework for three-phase continua (Bishop and Blight 1963). Within this conceptual approach, the BBM is characterized by a relatively large flexibility in terms of adaption to different conditions of soft (sandy to clayey) soils. For instance, the reference saturated condition, originally modeled as the modified Cam-Clay model, could be coupled to more complex saturated models such as the CASM.

More recently, Gens and his coworkers (Gens 2010, Nishimura et al. 2009) have proposed an enhanced BBM model for saturated freezing soils after noticing the close analogy between the physics of saturated-freezing (S-L-C) and unsaturated-unfrozen (S-L-G) soils. Using the modified Cam-Clay model for the reference unfrozen condition, their model adopts, in a similar way as in the BBM, two independent governing variables within a unified effective-stress-based framework: the net stress and the suction – defined as the difference between the ice and the liquid pressure, and is able to capture many essential features of the complex mechanical behavior of freezing soils,
including the dependence of shear strength on temperature and porosity.

![Diagram of constitutive soil models](image)

**Figure 1.3:** Hierarchical structure of selected existing constitutive soil models

### 1.2.3 Strength homogenization of heterogeneous materials

The prediction of the strength behaviors of geomaterials such as rocks, soils, and concrete under frost action has yet few attempts. On one hand, in comparison with the multiscale modeling of poroelastic behavior, due to the nonlinear nature of the mechanical principles that underly strength properties, the determination of strength properties of heterogeneous materials remains a challenge. On the other hand, as an important characteristic of freezing geomaterials, their mechanical response changes continuously with varying temperature and applied stress due to the pressure- and temperature-dependent phase transition between ice and water in the pore space. The incorporation of the phase transition, or more specifically the pressure melting of ice, into the strength upscaling procedure is an extremely new idea and no work exists so far.

For the determination of macroscopic properties of heterogeneous materials, appropriate multiscale methods are required as they are governed by the properties, the shape and topology of the individual components (generally denoted as material phases) related often to a large range of spatial scales. Such methods may be based upon computational multiscale methods or on analytical methods such as continuum micromechanics. Computational multiscale methods are attempting to directly numerically resolve the meso- or microstructure of heterogeneous materials by means of numerical discretization methods such as the finite element method and generate macroscopic quan-
1.2. STATE OF THE ART


tities from homogenization over the subscale model (the representative elementary volume). For a
survey of this class of multiscale methods please refer to Fish and Wriggers (2006), Sun et al. (2011). While this class of methods evidently allows a detailed
analysis of the interactions between phases at lower scales, its computational effort is enormous. In
cases, when homogenized properties, such as macroscopic elastic stiffnesses, viscosities, permeabilities
or material strength are required based upon local information from the different phases (this
task will be denoted in the following as "upscaling"), analytical methods may serve as a powerful
conceptual basis. As far as the upscaling of linear properties is concerned, continuum micromechan-
icics provides a well established framework. By now classical homogenization models are available
for the homogenization of elastic properties (e.g. Dormieux et al. 2006, Zaoi 2002), electrical
conductivity (e.g. Herrance 1979, Torquato 1985), and, more recently, for diffusion proper-
ties (Dormieux and Lemarchand 2001, Lemarchand et al. 2003, Pivonka et al. 2004,
Scheiner et al. 2008) and elastic viscosities (e.g. Friebel et al. 2006, Sanahuja 2013).

In contrast, the determination of strength properties of heterogeneous materials remains an ac-
tive research field, resulting from the nonlinear nature of the mechanical principles that underly
strength properties. Among the rare contributions, earlier methods for strength homogenization
were mainly based on limit analysis that provides estimates for the dissipation at plastic collapse
by employing the lower and upper bound theorems of yield design (e.g. Melan 1936, Salencon
1990). By solving a yield design boundary value problem, the strength capacity of various highly
idealized composite materials, such as fiber reinforced composites (Debeuhan and Taliercio
1991), and fluid-saturated porous materials (Debeuhan and Dormieux 1999) can be determined.
An upscaling scheme based on numerical limit analysis was presented in Fuessl et al. (2008)
for the determination of strength envelopes of porous materials, taking localized material failure
into account. An alternative approach was proposed and improved by Ponte Castaneda (1991,
1996, 2002), which is characterized by the use of optimally chosen, so-called "linear comparison
composites" (LCC) to deliver estimates for the effective mechanical properties of porous and rigidly
reinforced composites, that are exact to second-order in the heterogeneity contrast.

Dormieux, Barthelemy and Maghous (Barthelemy and Dormieux 2003, 2004, Maghous
et al. 2009) have proposed an analytical approach for the strength homogenization of cohesive-
frictional matrix materials with pores or rigid inclusions. The main underlying idea of this approach
is to replace the corresponding limit analysis by a sequence of viscoplastic problems. For the re-
sulting homogenized properties at the limit stress or strain state the modified secant method is used.
The model has been applied for the prediction of the macroscopic strength of highly filled composite
materials, such as cement-based mortars, for which the friction coefficient of the composite is higher
than that of the matrix (Heukamp et al. 2005, Lemarchand et al. 2002). Alternatively, Pich-
ler and Hellmich estimate the stiffness and strength of cement paste through an elastic limit analysis,
since in particular for the cement paste, the elastic limit of hydrate govern the overall elastic limits

More recently, Ortega et al. (2011) have developed a strength homogenization method for
cohesive-frictional materials affected by the presence of porosity and rigid-like inclusions. Within
the framework of the yield design theory (Salencon 1990) the linear comparison composite ap-
proach (LOPEZ-PAMIES AND PONTE CASTANEDA 2004, PONTE CASTANEDA 2002) has been extended from the application of nonlinear hyper-elastic composites to elastoplastic matrix-inclusion composites, allowing consideration of the frictional behavior of the matrix material in case that it may be represented by means of a Drucker-Prager-type strength criterion.

1.3 Content of the thesis

1.3.1 Outline and objectives

As a basic framework of the thesis, adopting the theory of thermoporoelasticity (COUSSY 2005, 2010) together with the theory of premelting dynamics (REMPEL ET AL. 2004, WETTLAUFER AND WORSTER 2006) as introduced in Subsection 1.2.1, a three-phase Finite Element (FE) model for porous materials, assumed to consist of the solid skeleton, liquid water and crystal ice, is presented for the description of the coupled THM behavior of water-infiltrated soft soils upon freezing. In addition to the volume expansion of water transforming into ice, the contribution of the micro-cryosuction mechanism to the frost heave phenomenon observed for frost-susceptible soils (TABER 1929, ZHU ET AL. 2000), is described in the model based upon the theory of premelting dynamics. The phase transition between water and ice is incorporated via a temperature-dependent liquid saturation function characterized by two parameters controlling the pore size and pore size distribution. The thermoporoelastic constitutive relations are obtained from an energy approach of poromechanics. By choosing the solid displacement, the liquid water pressure and the mixture temperature as the primary field variables, (1) mass balance of liquid water and crystal ice, (2) overall momentum balance and (3) overall entropy balance are set up as the governing balance laws, within which the interaction terms represent the corresponding THM couplings. Through fundamental balance laws and corresponding state relations, the model is able to capture various couplings among the phase transition, the liquid transport within the pore space, and the accompanying mechanical deformation.

In the context of the FE formulation in a geometrically-linear setting, the basic freezing soil model are formulated as an Initial Boundary Value Problem (IBVP) by transforming the governing balance equations to their variational forms. For the spatial discretization of the IBVP, quadratic LAGRANGIAN shape functions are used for the approximation of the displacement field and linear shape functions are used for the approximations of the liquid pressure and the temperature. With such approximations the BABUSKA-BREZZI stability constraint is fulfilled (BREZZI AND FORTIN 1991). For the temporal discretization, a modified midpoint rule, denoted as the generalized-\(\alpha\) method is used, which ensures unconditional stability and second order accuracy for an appropriate choice of its parameters (KUHL AND CRISFIELD 1999). The discretized weak form, evaluated at the generalized midpoint, yields a highly nonlinear system of equations that is solved iteratively using NEWTON’s method. During the intermediate stage of model development, for the sake of simplicity, the tangent stiffness matrices required to solve the linearized system of algebraic equations are generated numerically according to the methodology presented by LEE AND PARK (2002), whereas at the final stage, they are derived on the basis of consistent linearization. After that, the three-phase freezing model is implemented into the object-oriented FE-code KRATOS (DADVAND ET AL. 2002).
A further improvement of the proposed freezing soil model goes to the extension of the stress-strain behavior of the soil skeleton from elasticity to elastoplasticity. Since the established model for freezing soils should encompass the conditions of unfrozen soil as a special case, it is planned to take an existing elastoplastic model for soft soils as a reference state for the elastoplastic constitutive modeling of freezing soil. A similar approach has been taken by Nishimura et al. (2009) in the enhanced BBM as mentioned in Subsection 1.2.2, within which the modified Cam-Clay model is used as the reference unfrozen state, and two independent stress variables – the net stress and the suction – are adopted for freezing states after exploiting analogies between the physics of frozen-saturated soils and unfrozen-unsaturated soils. Accordingly, in this thesis, the CASM (Yu 1998) is adopted instead as the reference unfrozen condition, in order to capture the overall elastoplastic behavior of both saturated clay and sand. Besides, noticing the linear relationship between the suction and the temperature assumed in the liquid-crystal equilibrium condition, the (freezing) temperature is considered as the second independent variable, instead of the suction, within a similar unified effective-stress-based framework as in the enhanced BBM. Introducing a temperature parameterization on the yield function and the softening/hardening characteristics of the reference CASM, a family of yield functions and critical states will be generated for the freezing states at different temperatures below the bulk freezing point.

Moreover, in order to establish a comprehensive model for the prediction of the temperature- and porosity-dependent strength properties of freezing soils, this thesis proposes a novel multi-scale strength homogenization procedure for a three-phase partially frozen soil composite, where the solid particle phase and the crystal ice phase are assumed to be characterized by two different Drucker-Prager strength criteria and the liquid water phase has, either zero strength capacity under drained condition, or zero shear strength capacity under undrained condition. To this end, the author develops first a basic strength homogenization scheme for a two-phase matrix-inclusion composite by extending the linear comparison composite (LCC) approach proposed by Ortega et al. (2011) for more general classes of heterogeneous materials such as cementitious or geological materials consisting of different material phases, like aggregates or pores. For generating the homogenized strength criterion efficiently in such general cases of matrix-inclusion composites, a novel algorithm is proposed in the thesis, so that the matrix phase can be a quasi frictional material characterized either by a Drucker-Prager-type (hyperbolic) or an elliptical strength criterion, which predicts a strength limit also in hydrostatic compression, while the inclusion phase either may represent empty pores, pore voids filled with a pore fluid, rigid inclusions, or solid inclusions, whose strength characteristics also may be described by a Drucker-Prager-type or an elliptical strength criterion. The validation of the proposed two-phase strength homogenization procedure for selected combinations of strength characteristics of the matrix material and the inclusions is conducted by comparisons with experimental results and alternative existing strength homogenization models. In the subsequent work, the macroscopic strength criterion for drained and undrained freezing soils, as a three-phase composite, is upscaled through a corresponding two-step homogenization process. For each step, the LCC methodology is implemented by estimating the strength criterion of a two-phase nonlinear matrix-inclusion composite in terms of an optimally chosen linear elastic comparison composite with a similar underlying microstructure. In addition to the undrained case, the pressure melting of
ice is further considered by incorporating a temperature- and pressure- dependent phase transition between ice and water. The predicted strength properties under both drained and undrained conditions show qualitatively a good agreement with observed phenomena, such as strengthening of soil during freezing and weakening of soil during pressure melting.

Thereupon, the homogenized temperature- and porosity-dependent strength properties of freezing soils under drained condition is incorporated into the aforementioned concept of the enhanced BBM together with the CASM adopted for the reference unfrozen state, in order to provide an improvement on the model’s strength prediction in the freezing state, whereas in the original enhanced BBM (NISHIMURA ET AL. 2009), for simplification, the slope of the failure criterion, i.e. the critical state line (CSL) remains unchanged, while the cohesion is assumed to increase linearly with decreasing temperature. To be distinguished, this newly-proposed elastoplastic mechanical constitutive model with failure criterion upscaled through strength homogenization will be named as “extended BBM” in this thesis. However, as a limitation, the characteristic strain-rate-dependent strength behavior of frozen soils, mainly due to the presence of ice, is not covered in this thesis, for which a viscous formulation will be required.

The verification and validation of the established model are accomplished by means of selected tests focusing on different aspects of the model behavior. Its performance and applicability are demonstrated by a one-dimensional soil freezing test with focus on exploring the development of frozen fringe with frost heave, together with a re-analysis of a case study on AGF processes during tunnel excavation (ZIEGLER AND BAIER 2007) with extended investigation on ground heave prediction, influence of seepage flow on the formation of desired frost arch, and optimization of freeze pipe arrangement.

1.3.2 Structure

This thesis consists of eight chapters, the remainder of which are organised as follows:

Chapter 2 presents a basic three-phase FE model for freezing soils within thermoporoelasticity by means of setting up three governing balance equations with associated constituent relations. The formulated model is then implemented into the FE-code and validated by means of selected analyses.

Chapter 3 establishes an extended formulation of the LCC methodology for a two-phase composite of matrix-inclusion morphology with more general combinations of the underlying strength criteria for the matrix and inclusion phases, which is then validated by means of comparisons with experimental results and alternative existing strength homogenization models.

Chapter 4 proposes a novel multi-scale strength homogenization procedure for partially frozen soil under both drained and undrained conditions by means of a corresponding two-step strength upscaling strategy, whose prediction is then validated by comparisons with experimental results.

Chapter 5 extends first the framework from poroelasticity to poroplasticity, and then proposes accordingly a new critical state elastoplastic constitutive model for freezing soils.

Chapter 6 demonstrates the performance and applicability of the proposed model by means of two simulation examples: a one-dimensional soil freezing test and a case study concerned with AGF in tunneling.

Chapter 7 concludes this thesis with discussions on results and outlooks to future developments.
This chapter presents a basic three-phase freezing soil model by adopting the theory of thermoporoelasticity (Coussy 2005) together with the theory of pre-melting dynamics (Wettlaufer and Worster 2006). Within these frameworks, the coupled thermo-hydro-mechanical (THM) behavior of soil exposed to frost action is essentially described at the macroscopic scale by the poroelastic constitutive relations, the liquid-crystal equilibrium relation and the liquid saturation curve. By choosing the solid displacement, liquid pressure and mixture temperature as the primary variables, the associated balance equations are formulated and solved using the Finite Element Method (FEM). The established freezing soil model is implemented into the object-oriented FE-code KRATOS, and validated by means of selected analyses focusing on different aspects of the model behavior.

2.1 Theoretical background

The mechanics of porous materials subjected to freezing is considerably complex than simply that of a sealed water-filled bottle as explained in Subsection 1.2.1. For the macroscopic description of the coupled behavior of water-infiltrated materials upon freezing, the theories of poromechanics (Coussy 2005, 2010) and pre-melting dynamics (Rempe et al. 2004, Wettlaufer and Worster 2006), which provide a sound up-scaling framework and allow for a physics-oriented understanding, are adopted. In the proposed model soft soil is modeled as a three-phase porous material, consisting of solid particles (index S), liquid water (index L) and crystal ice (index C). The liquid phase is assumed to contain both pore water and premelted film water (see Figure 2.1).
2.1.1 Premelting dynamics

Premelting mechanism

According to the theory of premelting dynamics (REMPEL ET AL. 2004, WETTLAUFER AND WORSTER 2006), there exist two types of premelting mechanisms that allow water to remain in an unfrozen state at temperatures below the bulk freezing point $T_f$: curvature-induced premelting and interfacial premelting (see Figure 2.2). The former generates supercooled pore water in a porous medium according to the GIBBS–THOMSON law, whereas the latter gives rise to unfrozen film water separating ice from solid particles. The reason for the existence of this liquid film layer in soils lies in the fact that soils are hydrophilic and prefer contacting unfrozen water rather than ice. In other words, the total surface energy of the S-L and L-C interfaces is lower than the surface energy of a S-C interface. The existence of this film layer is a prerequisite for the interpretation of the micro-cryo-suction mechanism, which is described in detail in the upcoming paragraph.

Micro-cryo-suction

Figure 2.3 contains a close look of the frozen fringe, where the phase transition is taking place. Due to interfacial premelting, a thin unfrozen liquid film separates the soil particle from the crystal ice. Assuming the upper side being colder than the lower side, a temperature gradient directed
2.1. THEORETICAL BACKGROUND

downwards exists. According to premelting dynamics (WETTLAUFER AND WORSTER 2006), the thickness of the liquid film increases as the temperature rises, resulting in a drop of the thermo-molecular pressure $p_T$, often denoted as disjoining pressure (REMPEL AND WETTLAUFER 2001). The external pressure applied to the crystal ice $p_C$, identical to that applied to the solid particle $p_S$, equals to the sum of the thermo-molecular pressure $p_T$ and the hydrodynamic pressure $p_L$. If the external pressure is held constant, the force balance requires $p_L$ to decrease at the lower temperature side. As a result, there is a tendency for the liquid water to flow from the warmer side to the colder side. This process is denoted as micro-cryo-suction, which is identified as the driving force of frost heave phenomenon observed for frost-susceptible soils (TABER 1929, ZHU ET AL. 2000).

![Figure 2.3: Schematic illustration of micro-cryo-suction mechanism](image)

**Thermal regelation**

As more liquid water is sucked upward at the frozen fringe yet the thickness of film water remains unchanged under fixed temperature, the crystal ice extends into the pore space and forms progressively a connected network. Meanwhile, the ice pressure increases and eventually can bear the entire load, hence allowing the soil particles to separate and move downwards relative to the ice by means of a process of melting and refreezing referred to as “thermal regelation”– a kind of thermodynamic buoyancy (WETTLAUFER AND WORSTER 2006) (see Figure 2.4a). This initiates the formation of an ice lens, which accretes as more water is drawn upward and is then suppressed by decrease in hydraulic conductivity due to continued freezing within the pore space. Another ice lens can be initialized in time and will cut off completely the water flow to the upper ice lens, resulting in alternating layers of ice lenses and an upward displacement of the ground surface termed as *(secondary)* frost heave (FOWLER AND KRANTZ 1994) (see Figure 2.4b). However, as clays and sands do not exhibit significant frost heave, for simplicity the formation of ice lenses is not considered in this work by assuming a zero ice flow (see Subsection 2.2.1).
2.1.2 Thermoporoelasticity

Lagrangian porosity

Within the framework of unsaturated thermoporoelasticity (Coussy 2004, 2005), an infinitesimal representative element is extracted from the aforementioned three-phase porous continuum. In the initial configuration, the porous volume is assumed to be fully saturated with liquid water. If $dΩ_0$ stands for the volume of the element in this configuration and $ϕ_0$ is the initial porosity, its porous volume can be represented by $ϕ_0 dΩ_0$. In the framework of a geometrically linear theory, the current volume $dΩ$ is related to the initial volume $dΩ_0$ via

$$dΩ = (1 + ϵ) dΩ_0,$$

(2.1)

where $ϵ$ is the volumetric strain. Let $n$ be the Eulerian porosity of the current volume, such that the pores occupy the volume $n dΩ$ in the current configuration. In order to account for the volume change of the porous space, the Lagrangian porosity $ϕ$ and the volume fraction of the solid phase $ϕ_S$ are used, relating the current pore volume and the volume occupied by the solid phase, respectively, to the initial volume $dΩ_0$ according to

$$ϕ dΩ_0 = n dΩ \quad \text{and} \quad ϕ_S dΩ_0 = (1 - n) dΩ,$$

(2.2)

respectively. Combination of Eq. (2.1) and (2.2) leads to an expression for the Lagrangian volume fraction of the solid phase

$$ϕ_S = 1 + ϵ - ϕ.$$

(2.3)

Eulerian liquid and ice saturations

At all time the porous volume is assumed to be filled by water, in both liquid form (L) and crystal form (C). Hence, the current Lagrangian porosity $ϕ$ can be written as

$$ϕ = ϕ_L + ϕ_C,$$

(2.4)
where \( \phi_J \) is the current LAGRANGIAN partial porosity related to phase \( J = L, C \). Once the overall porosity is known, the current partial porosities can be expressed in terms of the degree of saturations as

\[
\phi_J = \phi \chi_J, \quad \text{with} \quad \chi_L + \chi_C = 1, \tag{2.5}
\]

where \( \chi_J \) denotes the EULERIAN saturation, standing for the current partial saturation of phase \( J \) relative to the current deformed porous volume \( \phi \, d\Omega_0 \) (see Figure 2.5b).

\[\text{Figure 2.5: Schematic illustration of LAGRANGIAN porosity (averaging principles are applied for transforming the microscopic model to macroscopic scale (EHLERS AND BLUHM 2000))}\]

### 2.2 Governing balance equations

Adopting linearized geometrical relations and choosing the vector of displacements \( \mathbf{u} \), the liquid water pressure \( p_L \) and the temperature \( T \) of the mixture as primary field variables, the coupled thermo-hydro-mechanical behavior of porous materials during freezing can be described by corresponding balance equations, which form a set of differential equations to be solved by the Finite Element Method (see Section 2.4). For the formulation of the balance equations it is assumed that the pore volume is at all time fully-saturated – partly occupied by ice crystals and partly occupied by the water remaining in liquid form. In the present chapter, the model development is restricted to linear isotropic thermoporoelasticity under the hypothesis of small perturbations

\[
|\varepsilon_{ij}| \ll 1, \quad \left| \frac{\mathbf{u}}{l} \right| \ll 1, \quad \left| \frac{\phi - \phi_0}{\phi_0} \right| \ll 1, \quad \left| \frac{\rho_J - \rho_{J0}}{\rho_{J0}} \right| \ll 1. \tag{2.6}
\]

where \( \varepsilon_{ij} \) are the components of the current linearized strain tensor, \( l \) is the length scaling the dimensions of the porous structure, \( \rho_J \) is the mass density of phase \( J \), and the index \( 0 \) refers to the initial value.

#### 2.2.1 Mass balance of liquid water and crystal ice

Taking into account possible phase transition between liquid water and crystal ice, the mass balance equation relative to each phase can be written as

\[
\frac{d m_L}{d t} + \nabla \cdot \mathbf{w}_L = -\circ m_{L \rightarrow C}, \quad \frac{d m_C}{d t} + \nabla \cdot \mathbf{w}_C = \circ m_{L \rightarrow C}, \tag{2.7}
\]
where \( m_J = \rho_J \phi_J \) stands for the current mass content related to phase \( J \) per unit of initial volume with \( \rho_J \) being the corresponding specific mass density, \( \nabla \cdot (\bullet) \) denotes the divergence operator, \( \mathbf{w}_J \) is the Eulerian relative mass flow vector, and \( \dot{m}_{L\rightarrow C} \) is the rate of liquid water mass changing into crystal ice. In general, the flow of ice with respect to the skeleton occurs much slower than the flow of water such that \( \mathbf{w}_C = 0 \) can be assumed. Summation of these two equations yields the combined mass balance for both the liquid water and the crystal ice phase

\[
\frac{dm_L}{dt} + \frac{dm_C}{dt} + \nabla \cdot \mathbf{w}_L = 0. 
\tag{2.8}
\]

### 2.2.2 Overall momentum balance

Disregarding dynamic effects, the momentum balance equation for the mixture is given as

\[
\nabla \cdot \mathbf{\sigma} + \rho \mathbf{g} = 0, \tag{2.9}
\]

where \( \mathbf{\sigma} \) denotes the tensor of total stresses, \( \rho = (1 - \phi_0) \rho_{S0} + m_L + m_C \) stands for the overall mass density with \( \rho_{S0} \) being the initial mass density of solid particles, and \( \mathbf{g} \) is the gravity force per unit volume.

### 2.2.3 Overall entropy balance

Identifying the spontaneous production of entropy \( \Phi_M \), the second law of thermodynamics provides the entropy balance for the mixture:

\[
T \left( \frac{dS}{dt} + \nabla \cdot (s_L \mathbf{w}_L) \right) + \nabla \cdot \mathbf{q} - \Phi_M = 0 \tag{2.10}
\]

with \( S = S_S + m_L s_L + m_C s_C \) as the overall density of entropy per unit of volume, while \( S_S \) is the entropy of the solid matrix and \( s_J \) the specific entropy related to phase \( J \). \( \mathbf{q} \) is the overall outgoing heat flow vector. Here \( \Phi_M \) stands only for the mechanical dissipation associated with the viscous liquid flow through the porous volume, while the dissipation related to the solid skeleton is assumed to be zero in thermoporoelasticity (Coussy 2004).

The use of the entropy balance (2.10) in this work requires much less efforts than that of an enthalpy balance which is a usual choice in the Porous Media Mechanics, since all the involved thermal quantities can be easily derived within the linear isotropic thermoporoelasticity as described in Section 2.3.

### 2.3 Constitutive equations

#### 2.3.1 Skeleton state equations

Let \( \sigma_{ij} \) be the components of the current stress tensor of the mixture to which the representative element is subjected, and \( \varepsilon_{ij} \) the components of the current linearized strain tensor, relative to an initial configuration free of stress and pore pressure with an initial temperature \( T_0 \). The tensile surface stresses acting along the interfaces between the constituents and their subsequent effects on
the deformation of the solid matrix is ignored since its value is relatively small when compared to the magnitude of the pressures \( p_J \) (COUSSEY AND MONTEIRO 2007). Hence, the equivalent pore pressure can be defined by
\[
\bar{p} = \chi_L p_L + (1 - \chi_L) p_C .
\] (2.11)

According to thermoporoelasticity (COUSSEY 2004), the mechanical dissipation associated with the solid skeleton must be equal to zero:
\[
\sigma_{ij} \, d \varepsilon_{ij} + \bar{p} \, d \phi - S_S \, d T - d \Psi_S = 0
\] (2.12)
where \( \Psi_S \) is the free energy of the skeleton. Defining \( G_S \) as
\[
G_S = \Psi_S - \bar{p} \, \phi ,
\] (2.13)
leads to eventually the state equations in the form
\[
\sigma_{ij} = \frac{\partial G_S}{\partial \varepsilon_{ij}} ; \quad \phi = - \frac{\partial G_S}{\partial \bar{p}} ; \quad S_S = - \frac{\partial G_S}{\partial T} .
\] (2.14)

For an isotropic linear thermoporoelastic material, the energy function \( G_S \) can be expressed as
\[
G_S = - \phi_0 \bar{p} - S_{S0} T + \frac{1}{2} K \epsilon^2 + G \, e_{ij} \, e_{ji} - b \, \bar{p} \, \epsilon - 3 \, a \, K \, (T - T_0) \, \epsilon + 3 \, a_\phi \, \bar{p} \, (T - T_0) - \frac{1}{2} \frac{\bar{p}^2}{N} - \frac{1}{2} \frac{C_S}{T_0} \, (T - T_0)^2 ,
\] (2.15)
resulting in the following expressions for the corresponding state equations
\[
\sigma_{ij} = (K - \frac{2}{3} \, G) \, \epsilon \, \delta_{ij} + 2 \, G \, e_{ij} - b \, \bar{p} \, \delta_{ij} - 3 \, a \, K \, (T - T_0) \, \delta_{ij} ,
\] (2.16)
\[
\phi = \phi_0 + b \, \bar{p} + \frac{\bar{p}}{N} - 3 \, a_\phi \, (T - T_0) ,
\] (2.17)
\[
S_S = S_{S0} + 3 \, a \, K \, \epsilon - 3 \, a_\phi \, \bar{p} + \frac{C_S}{T_0} \, (T - T_0) ,
\] (2.18)
where \( \epsilon = \epsilon_{ii} \) and \( e_{ij} = \epsilon_{ij} - (\epsilon/3) \, \delta_{ij} \) are, respectively, the volumetric dilation and deviatoric components of the strain tensor, with \( \delta_{ij} \) being the KRONECKER delta; \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \); \( K \) and \( G \) are the effective bulk and shear moduli of the empty porous solid with a zero pore pressure (\( \bar{p} = 0 \)); \( b \) and \( N \) are the BIOT coefficient (BIOT 1941) and the BIOT tangent modulus, respectively; \( a \) and \( a_\phi \) are the volumetric thermal dilation coefficients related to the empty porous solid and to the porous volume, respectively; and \( C_S \equiv \rho_{S0} \, (1 - \phi_0) \, c_S \) is the volumetric heat capacity of the skeleton, with \( c_S \) denoting the specific heat capacity of the solid phase. The aforementioned macroscopic properties are linked to the bulk modulus \( k_S \) and the thermal volumetric dilation coefficient \( \alpha_S \) of the solid phase (COUSSEY 2004) via
\[
b = 1 - \frac{K}{k_S} , \quad 1 = \frac{b - \phi_0}{k_S} , \quad a = \alpha_S , \quad a_\phi = \alpha_S \, (b - \phi_0) .
\] (2.19)
For the determination of the effective bulk modulus $K$ and the shear modulus $G$ of the solid skeleton, the MORI-TANAKA scheme for a matrix-inclusion morphology is employed for two extreme cases: the fully unfrozen and fully frozen case (see Figure 2.6). In the present work, isotropy of the material is assumed. Therefore, spherical inclusions are considered. In this case, the bulk and shear moduli for the two extreme cases can be obtained as (LACKNER ET AL. 2008)

\[
K_{S-J} = k_J \left(1 + (1 - \phi) \frac{1 - k_J/k_S}{k_J/k_S + \alpha \phi (1 - k_J/k_S)}\right), \quad \text{with} \quad \alpha = \frac{3 k_J}{3 k_J + 4 g_J}; \quad (2.20)
\]

\[
G_{S-J} = g_J \left(1 + (1 - \phi) \frac{1 - g_J/g_S}{g_J/g_S + \beta \phi (1 - g_J/g_S)}\right), \quad \text{with} \quad \beta = \frac{6 (k_J + 2 g_J)}{5 (3 g_J + 4 g_J)}, \quad (2.21)
\]

where $k_J$ and $g_J$ are the bulk modulus and the shear modulus of phase $J$. Interpolating the two extreme cases with the EULERIAN saturation, the effective bulk and shear moduli can be expressed as

\[
K = \chi_L K_{S-L} + (1 - \chi_L) K_{S-C}, \quad (2.22)
\]

\[
G = \chi_L G_{S-L} + (1 - \chi_L) G_{S-C}. \quad (2.23)
\]

### 2.3.2 The liquid-crystal equilibrium relation

Linear thermoporoelasticity is characterized by a quadratic expression for the energy functionals with regards to their invariants. Accordingly, the current specific GIBBS chemical potential $\mu_J$ of phase $J$ can be written in the quadratic form

\[
\mu_J = \mu_{J0} - (T - T_l) s_{J0} + \frac{p_J}{\rho_{J0}} + \frac{c_J}{2 T_l} (T - T_l)^2 + 3 \alpha_J (T - T_l) \frac{p_J}{\rho_{J0}} - \frac{1}{2} \frac{p_J^2}{\rho_{J0} K_J} \quad (2.24)
\]

where $\mu_{J0}$, $p_{J0}$ and $s_{J0}$ are the specific chemical potential, the specific mass density and the specific entropy associated with the reference (atmospheric) conditions $p_J = 0$ and $T = T_l$, respectively, while $K_J$, $3 \alpha_J$ and $c_J$ denote the bulk modulus, the volumetric thermal dilation coefficient and the heat capacity per mass unit of phase $J$. With the help of Eq. (2.24), the current mass density $\rho_J$ and
the current specific entropy $s_J$ can be expressed as

$$
\frac{1}{\rho_J} = \frac{\partial \mu_J}{\partial T_J} = \frac{1}{\rho_{J0}} \left( 1 + 3 \alpha_J (T - T_l) - \frac{p_J}{K_J} \right),
$$

(2.25)

$$
s_J = -\frac{\partial \mu_J}{\partial T} = s_{J0} + c_J \frac{T - T_l}{T_l} - 3 \alpha_J \frac{p_J}{\rho_{J0}},
$$

(2.26)

Assuming zero dissipation throughout phase transition, i.e. ignoring capillary hysteresis during freeze-thaw cycles (FABBRI ET AL. 2009) (see respective comment in Subsection 2.3.3), the liquid-crystal thermodynamic equilibrium requires the equality of the chemical potentials of both phases $\mu_L = \mu_C$. In addition, owing to the definition of $T_l$, the reference chemical potentials $\mu_{L0}$ and $\mu_{C0}$ are equal. Neglecting the quadratic terms related to the pressure difference $p_C - p_L$ and the cooling temperature $T_l - T$ results in the liquid-crystal equilibrium condition

$$
p_C - p_L = S_f (T_l - T),
$$

(2.27)

where $S_f \equiv \rho_{L0} (s_{L0} - s_{C0}) > 0$ denotes the freezing entropy per unit of volume. Note, that Eq. (2.27) can be used to explain the micro-cryo-suction mechanism, which is identified as the driving force of frost heave phenomenon observed for frost-susceptible soils. As already explained in Subsection 2.1.1, the colder side (with low temperature $T$) of the frozen fringe requires the liquid (or hydrodynamic) pressure $p_L$ to decrease if the crystal pressure $p_C$ is kept constant under force balance, resulting in a suction of the liquid water towards the colder side of the frozen fringe (see Figure 2.3).

2.3.3 The liquid saturation curve

Denoting by $R$ the mean curvature radius of the interface separating the crystal from the adjacent liquid water, and by $\gamma_{CL}$ the liquid-crystal interface energy, the YOUNG-LAPLACE law requires

$$
p_C - p_L = \frac{2 \gamma_{CL}}{R}.
$$

(2.28)

If, for simplicity, a zero contact angle between the liquid and the solid skeleton is assumed, $R$ represents also the pore radius associated with the current liquid-crystal interface. Combining Eqs. (2.27) and (2.28) yields the GIBBS-THOMSON relation

$$
R = \frac{2 \gamma_{CL}}{S_f (T_l - T)},
$$

(2.29)

which implies that all the pores having an entry radius greater than $R$ will freeze at the given temperature $T$, and the remaining pores being filled with liquid water. This allows us to retrieve the existence of a state relation linking the liquid saturation and the temperature. The determination of this relation can be inferred from knowledge of the desorption isotherm from a soil sample partially saturated with liquid water (L) and partially with air (G), which relates the liquid saturation to the capillary pressure. According to the assumption that dissipation during freeze-thaw cycles is neglected in the presented model (see Subsection 2.3.2), this saturation-pressure curve, referred to
as the soil-water characteristic curve (SWCC), is described by the non-hysteretic phenomenological model proposed by van Genuchten

\[ p_G - p_L = \mathcal{N}(\chi_L^{-\frac{1}{m}} - 1)^{1-m}, \quad 0 < m < 1, \]  

(2.30)

where \( p_G, \mathcal{N} \) and \( m \) are the gas pressure, the capillary modulus and a constant representing the shape of the capillary curve, respectively. Similarly to Eq. (2.28), the Young-Laplace law at the gas-liquid interface reads

\[ p_G - p_L = \frac{2 \gamma_{GL}}{R}, \]  

(2.31)

where \( \gamma_{GL} \) is the liquid-air interface energy. Combination of Eqs. (2.30) and (2.31) provides a relationship between the liquid saturation and the pore size

\[ \chi_L = F[R] = \left( 1 + \left( \frac{2 \gamma_{GL}}{\mathcal{N} R} \right)^{\frac{1}{1-m}} \right)^{-m}. \]  

(2.32)

From Eq. (2.32) follows, that the remaining liquid saturation \( \chi_L \) equals to the cumulative fraction \( F[R] \) of pore volume occupied by pores having a pore entry radius smaller than \( R \). Replacing the pore size \( R \) in (2.32) by the Gibbs-Thomson law Eq.(2.29) provides a relationship between the liquid saturation and the temperature

\[ \chi_L = \left( 1 + \left( \frac{T_f - T}{\Delta T_{ch}} \right)^{\frac{1}{1-m}} \right)^{-m}, \]  

(2.33)

where \( \Delta T_{ch} = \frac{\mathcal{N} \gamma_{GL}}{\gamma_{GL}} \) is the characteristic cooling temperature related to the most frequently encountered pore radius \( R_{ch} \), and \( m \) is an index indicating the pore radius distribution around \( R_{ch} \). The influence of \( \Delta T_{ch} \) and \( m \) on the shape of the liquid saturation curve is illustrated in Figure 2.7. Obviously, as a consequence of assuming non-dissipative phase transition and adopting non-hysteretic SWCC (2.30), the obtained “\( \chi_L - T \)” curve (2.33) reduces to a one-to-one relation. In reality, depending on the specific porous network structure, due to capillary hysteresis a freeze-thaw cycle may cause a hysteresis loop in the liquid saturation curve (FABBRI ET AL. 2009). According to an extensive study on hysteresis models for SWCCs (PHAM ET AL. 2005), however, the measurement of a complete set of hysteretic curves is time-consuming and costly, and the choice and mathematical formulation of an adequate hysteresis model, including calibration of the parameters, would require additional considerations beyond the scope of this work. Since for geotechnical applications of the soil freezing method, e.g. in tunneling, the performance of the soil during the freezing phase is generally the relevant aspect for the design of the support measures (see e.g. JESSBERGER (1980), OU ET AL. (2009), SCHULTZ ET AL. (2008)), hysteresis effects are disregarded in this work.

2.3.4 Darcy’s law

Adopting Darcy’s law for describing the transport of the pore fluid, the Eulerian liquid flow can be expressed as

\[ \mathbf{w}_L = \rho_{L0} \mathbf{u}_L = \rho_{L0} \frac{K_0}{\eta_0} \mathcal{R}_{rel}[\chi_L] \left( \nabla p_L + \rho_{L0} \mathbf{g} \right). \]  

(2.34)
2.3. CONSTITUTIVE EQUATIONS

\[ m = 0.7 \]

\[ \Delta T_{ch} = 2 \, ^\circ C \]

**Figure 2.7:** Liquid saturation curve during freezing: Influence of \( \Delta T_{ch} \) (left) and \( m \) (right).

\( \psi_L \) is the Darcy’s velocity, \( \kappa_0 \) is the intrinsic permeability, and \( \kappa_{rel} \) is the saturation dependent relative permeability that can be expressed in a form proposed by Luckner et al. (1989)

\[
\kappa_{rel}[\chi_L] = \sqrt{\chi_L} \left( 1 - \left( 1 - \chi_L^m \right)^m \right)^2 .\tag{2.35}
\]

\( \eta_0 \) is the reference viscosity at \( T = T_f \), while the function \( \eta_{rel} \) accounts for the temperature dependence of the viscosity of supercooled water, which can be well captured by the empirical relation (Grant 2000)

\[
\eta_{rel}[T] = 509.53 \times 10^{-2} \times e^{123.15 - (T_f - T)} .\tag{2.36}
\]

Furthermore, the mechanical dissipation caused by the viscous liquid flow can be expressed as Coussy (2004)

\[
\Phi_M = \frac{\kappa_0 \kappa_{rel}[\chi_L]}{\eta_0 \eta_{rel}[T]} \left( -\nabla p_L + \rho_{L0} g \right) \cdot \left( -\nabla p_L + \rho_{L0} g \right) .\tag{2.37}
\]

### 2.3.5 Fourier’s law

Adopting Fourier’s law, the heat flux is related to the temperature gradient

\[
\mathbf{q} = -\lambda_{tot} \nabla T ,\tag{2.38}
\]

where \( \lambda_{tot} \) stands for the overall thermal conductivity. Considering the solid particles and the ice phase as spherical inclusions embedded in a matrix of water, \( \lambda_{tot} \) can be estimated from homogenization using the Eshelby equivalent inclusion method (Hatta and Taya 1986, Lackner et al. 2005)

\[
\lambda_{tot} = \lambda_L + \frac{\phi_S (\lambda_S - \lambda_L)}{1 + (\lambda_S/\lambda_L - 1)/3} + \frac{\phi_C (\lambda_C - \lambda_L)}{1 + (\lambda_C/\lambda_L - 1)/3} ,\tag{2.39}
\]

\[
\phi_L + \frac{\phi_S}{1 + (\lambda_S/\lambda_L - 1)/3} + \frac{\phi_C}{1 + (\lambda_C/\lambda_L - 1)/3} ,
\]
where \( \lambda_S \), \( \lambda_L \) and \( \lambda_C \) are thermal conductivities of solid particles, liquid water and crystal ice, respectively.

### 2.4 Finite Element formulation

With the chosen set of primary variables \( \mathbf{d} = \begin{bmatrix} \mathbf{u} & p_L & T \end{bmatrix}^T \), the IBVP is formulated in the spatial domain \( \Omega \) for the time period \([0, t]\). This IBVP is to be solved by means of the FEM in the context of the geometrically linear theory.

#### 2.4.1 Weak formulation

The balance equations (2.8-2.10) are transformed to their variational form \( \delta W \) by multiplication with the test functions \( \delta \mathbf{d} \) and integrated over the domain \( \Omega \), leading to the following weak forms

\[
\delta W^1 = \int_{\Gamma_N^1} \delta \mathbf{u} \cdot \mathbf{t}^* \, dA - \int_{\Omega} \nabla \delta \mathbf{u} : \mathbf{\sigma} \, dV + \int_{\Omega} \delta \mathbf{u} \cdot \mathbf{g} \, dV = 0, \\
\delta W^2 = \int_{\Gamma_N^2} \delta p_L \, \mathbf{w}^* \, dA + \int_{\Gamma_N^2} \delta p_L \left( \frac{d m_L}{d t} + \frac{d m_C}{d t} \right) \, dV - \int_{\Omega} \nabla \delta p_L \cdot \mathbf{w}_L \, dV = 0, \\
\delta W^3 = \int_{\Gamma_N^3} \delta T \, \mathbf{q}^* \, dA + \int_{\Gamma_N^3} \delta T \left( \frac{d S_q}{d t} + \frac{d m_C}{d t} (s_C - s_L) + m_L \frac{d s_q}{d t} + m_C \frac{d s_C}{d t} \right) \, dV \\
+ \int_{\Omega} \delta T \, \nabla \mathbf{s}_L \cdot \mathbf{w}_L \, dV - \int_{\Omega} \nabla \delta T : \mathbf{q} \, dV - \int_{\Omega} \delta T \, \Phi_M \, dV = 0, \quad (2.40)
\]

supplemented by the following boundary and initial conditions

<table>
<thead>
<tr>
<th>DIRICHLET boundary</th>
<th>NEUMANN boundary</th>
<th>Initial conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \forall \mathbf{x} \in \Gamma_D^1 ) : ( \mathbf{u} = \mathbf{u}^* ), ( \forall \mathbf{x} \in \Gamma_N^1 ) : ( \mathbf{\sigma} \cdot \mathbf{n} = \mathbf{t}^* ), ( \forall \mathbf{x} \in \Omega ) : ( \mathbf{u} = \mathbf{u}_0 ),</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \forall \mathbf{x} \in \Gamma_D^2 ) : ( p_L = p_L^* ), ( \forall \mathbf{x} \in \Gamma_N^2 ) : ( \mathbf{w}_L \cdot \mathbf{n} = \mathbf{w}^* ), ( \forall \mathbf{x} \in \Omega ) : ( p_L = p_L^0 ),</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \forall \mathbf{x} \in \Gamma_D^3 ) : ( T = T^* ), ( \forall \mathbf{x} \in \Gamma_N^3 ) : ( \mathbf{q} \cdot \mathbf{n} = \mathbf{q}^* ), ( \forall \mathbf{x} \in \Omega ) : ( T = T_0 ),</td>
<td></td>
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</tr>
</tbody>
</table>

#### 2.4.2 Spatial discretization

For the spatial discretization (\( \Omega = \bigcup_{\varepsilon=1}^{\text{NE}} \Omega^\varepsilon \), with NE as the total number of elements) of the IBVP, quadratic shape functions \( \mathbf{N}_q^\varepsilon[\xi] \), \( \xi \): local coordinates) are employed for the approximation of the displacement field on a 3D \text{LAGRANGE} element with \text{NN}_q \text{nodes}, whereas linear shape functions \( \mathbf{N}_l^\varepsilon[\xi] \) are employed for the approximations of the liquid pressure and the temperature field on the finite element with \text{NN}_l \text{nodes} (see Figure 2.8)

\[
\mathbf{u}[\xi] \approx \sum_{i=1}^{\text{NN}_q^\varepsilon} \mathbf{N}_q^\varepsilon[\xi] \mathbf{u}_i^\varepsilon, \quad p_L[\xi] \approx \sum_{i=1}^{\text{NN}_l^\varepsilon} \mathbf{N}_l^\varepsilon[\xi] p_{L_i}^\varepsilon, \quad T[\xi] \approx \sum_{i=1}^{\text{NN}_l^\varepsilon} \mathbf{N}_l^\varepsilon[\xi] T_i^\varepsilon, \quad (2.41)
\]

where \( \mathbf{d}^\varepsilon \) are the values of the field variables at the element nodes, \textit{i.e.} the degrees of freedom (DOFs).
Using a multifield FEM approach, by choosing (different) orders of approximation for the displacements, and the pressure and temperature, respectively, the Babuska-Brezzi stability constraint is fulfilled (Brezzi and Fortin 1991). Introducing the Jacobian matrix \( J = \frac{\partial \xi}{\partial x} \) and applying Gauss-Legendre numerical integration (Cowper 1973), the variational forms in (2.40) are computed by element wise integration over the isoparametric domain using natural coordinates

\[
\delta W = \int_{\Omega} (\cdots) \, dV = \sum_{e=1}^{NE} \int_{\Omega_e} (\cdots) \det(J) \, d\xi = \sum_{e=1}^{NE} \sum_{\ast=1}^{NG} \left\{ (\cdots) \xi_{\ast} \omega_{\ast} \det(J[\xi_{\ast}]) \right\}, \tag{2.42}
\]

with NG as the total number of Gaussian quadrature points, and \( \xi_{\ast} \) and \( \omega_{\ast} \) represent the local coordinates and the associated integration weight of the quadrature point \( \ast \), respectively.

### 2.4.3 Temporal discretization

For the temporal discretization a time stepping scheme, subdividing the total time span into a number of time intervals \( [0, t] = \bigcup_{n=1}^{NT} [t_n, t_{n+1}] \), with NT as the total number of time steps, a modified midpoint rule, denoted as the generalized-\( \alpha \) method, is used. Within this time stepping scheme, the semi-discretized variational forms are evaluated at the generalized midpoint of each time step \( t_{n+1} - \alpha = \alpha t_n + (1 - \alpha) t_{n+1} \) using the following expressions for the primary variable set \( d_{n+1 - \alpha} \) and its temporal derivative \( \dot{d}_{n+1 - \alpha} \):

\[
d_{n+1 - \alpha} = \alpha f d_n + (1 - \alpha f) d_{n+1} \\
\dot{d}_{n+1 - \alpha} = \alpha f \dot{d}_n + (1 - \alpha f) \dot{d}_{n+1}.
\] (2.43)

The velocity at the time step endpoint \( t_{n+1} \) is approximated as

\[
\dot{d}_{n+1} = \frac{\gamma}{\beta} \Delta t (d_{n+1} - d_n) - \frac{\gamma - \beta}{\beta} \dot{d}_n.
\] (2.44)

By using an appropriate choice of the time integration parameters

\[
\alpha_m = \alpha_f = \frac{1}{2}, \quad \beta = \frac{1}{4}, \quad \gamma = \frac{1}{2},
\] (2.45)

unconditional stability and second order accuracy are guaranteed without any additional numerical dissipation (Kuhl 2004, Kuhl and Crisfield 1999). The fully discretized variational forms
evaluated at the generalized midpoint finally can be written as

\[
\delta W_{n+1-\alpha}^1 = \sum_{e} \sum_{i} \sum_{k} (\delta u_{kj})_{n+1-\alpha} \left\{ \int_{\Gamma_{c1}^e} N^g_i (t^*_k)_{n+1-\alpha} \, dA \right. \\
- \sum_{j} \int_{\Omega^e} N^g_i \sigma_{kj}(n+1-\alpha) \, dV + \int_{\Omega^e} N^g_i \rho_{n+1-\alpha} g_k \, dV \left\} \\
= (\delta \mathbf{u}_{n+1-\alpha}) \cdot \left\{ R_{ext}^1 + R_{int}^1 \right\}_{n+1-\alpha} = 0 ,
\]

(2.46)

\[
\delta W_{n+1-\alpha}^2 = \sum_{e} \sum_{i} (\delta p_{Lij})_{n+1-\alpha} \left\{ \int_{\Gamma_{c2}^e} N^l_i \, dA - \sum_{j} \int_{\Omega^e} N^l_i (u_L)_j \, dV \right. \\
+ \int_{\Omega^e} N^l_i \left( \frac{\partial (m_L + m_C)}{\partial \epsilon} \text{div} \mathbf{u} + \frac{\partial (m_L + m_C)}{\partial p_L} \dot{p}_L \\
\left. + \frac{\partial (m_L + m_C)}{\partial T} \dot{T} \right)_{n+1-\alpha} \, dV \right\} \\
= (\delta \mathbf{p}_L)_{n+1-\alpha} \cdot \left\{ R_{ext}^2 + R_{int}^2 \right\}_{n+1-\alpha} = 0 ,
\]

(2.47)

\[
\delta W_{n+1-\alpha}^3 = \sum_{e} \sum_{i} (\delta T^c_{ij})_{n+1-\alpha} \left\{ \int_{\Gamma_{c3}^e} N^l_i q^* \, dA - \sum_{j} \int_{\Omega^e} N^l_i (q_i)_{n+1-\alpha} \, dV \right. \\
+ \int_{\Omega^e} N^l_i \left( T \left( \frac{\partial S_S}{\partial \epsilon} + \frac{\partial m_C}{\partial \epsilon} (s_C - s_L) \right) \text{div} \mathbf{u} \right)_{n+1-\alpha} \, dV \\
+ \int_{\Omega^e} N^l_i \left( T \left( \frac{\partial S_S}{\partial p_L} + \frac{\partial m_C}{\partial p_L} (s_C - s_L) + m_L \frac{\partial s_L}{\partial p_L} + m_C \frac{\partial s_C}{\partial p_L} \dot{p}_L \right) \dot{p}_L \right)_{n+1-\alpha} \, dV \\
+ \int_{\Omega^e} N^l_i \left( T \left( \frac{\partial S_S}{\partial T} + \frac{\partial m_C}{\partial T} (s_C - s_L) + m_L \frac{\partial s_L}{\partial T} + m_C \frac{\partial s_C}{\partial T} \dot{T} \right) \dot{T} \right)_{n+1-\alpha} \, dV \\
+ \sum_{j} \int_{\Omega^e} N^l_i \left( T \left( \frac{\partial s_L}{\partial p_L} \nabla p_L \right)_j + \frac{\partial s_L}{\partial T} \nabla T_j \right) (u_L)_j \right)_{n+1-\alpha} \, dV \\
- \int_{\Omega^e} N^l_i \left( \Phi M \right)_{n+1-\alpha} \, dV \right\} \\
= (\delta \mathbf{T})_{n+1-\alpha} \cdot \left\{ R_{ext}^3 + R_{int}^3 \right\}_{n+1-\alpha} = 0 .
\]

(2.48)

In concise notation, the discretized weak form can be formulated as

\[
\delta W_{n+1-\alpha} = (\delta \mathbf{d})_{n+1-\alpha} \cdot \left\{ \mathbf{R}_{ext} + \mathbf{R}_{int} \right\}_{n+1-\alpha} = 0 ,
\]

(2.49)

where \( \mathbf{R}_{ext} \) and \( \mathbf{R}_{int} \) are external and internal force vectors, respectively.
2.4.4 Linearization

The discretized weak form (2.49) is highly non-linear in terms of the field variables due to the non-linearities involved in the specifications of the constitutive relations such as the liquid saturation function, the saturation-dependent water flow and the mechanical dissipation. For the solution of this highly non-linear problem, the NEWTON-RAPHSON method is applied, which requires the linearization of the discretized weak forms with respect to the DOFs at the end of the time step. With the help of the GATEAUX derivative, the linearization of the discretized weak forms (2.49) is obtained as

\[
\Delta \delta W_{n+1-\alpha} = \delta \hat{d}_{n+1-\alpha} \cdot \left\{ \Delta R_{\text{int}} \right\}_{n+1-\alpha}
\]

\[
= \delta \hat{d}_{n+1-\alpha} \cdot \left\{ \frac{\partial (R_{\text{int}})}{\partial d_{n+1-\alpha}} \Delta d_{n+1-\alpha} + \frac{\partial (R_{\text{int}})}{\partial \dot{d}_{n+1-\alpha}} \Delta \dot{d}_{n+1-\alpha} \right\}
\]

\[
= \delta \hat{d}_{n+1-\alpha} \cdot \left\{ \frac{\partial (R_{\text{int}})}{\partial d_{n+1-\alpha}} (1 - \alpha f) + \frac{\partial (R_{\text{int}})}{\partial \dot{d}_{n+1-\alpha}} (1 - \alpha f) \frac{\gamma}{\beta \Delta t} \right\} \Delta d_{n+1} + \Delta \delta W_{n+1-\alpha}
\]

where \( K \) and \( D \) are the GATEAUX derivatives of \( R_{\text{int}} \) with respect to the DOFs and to their temporal derivatives, respectively. By means of the NEWTON-RAPHSON method, based upon the TAYLOR series expansion of the variational forms \( \delta W_{n+1-\alpha} \), the increments of the nodal vector of primary variable \( \Delta \hat{d} \) at iteration step \( k \) are obtained as

\[
\Delta d_{n+1} = \left( K_{n+1-\alpha}^k (1 - \alpha f) + D_{n+1-\alpha}^k (1 - \alpha f) \frac{\lambda}{\beta \Delta t} \right)^{-1} \left( R_{\text{ext}} + R_{\text{int}}^k \right)_{n+1-\alpha}
\]  

(2.51)

During the intermediate phases of development, e.g. when implementing thermosteoroelastic model, in order to save efforts, the GATEAUX derivatives \( K \) and \( D \), required to solve the linearized system of algebraic equations, are generated numerically according to the methodology presented by LEE AND PARK (2002). Herein, the \( \text{col}-\text{th} \) column vectors of these two matrices are computed via

\[
K_{\text{col}} = (\tilde{K}_{\text{num}})_{\text{col}} = \frac{\mathbf{R}_{\text{int}}[\mathbf{d} + \varepsilon_{\text{num}} \Delta \hat{d}_{\text{col}}, \mathbf{d}] - \mathbf{R}_{\text{int}}[\mathbf{d}, \mathbf{d}]}{\varepsilon_{\text{num}} \Delta \hat{d}_{\text{col}}},
\]

\[
D_{\text{col}} = (\tilde{D}_{\text{num}})_{\text{col}} = \frac{\mathbf{R}_{\text{int}}[\mathbf{d}, \mathbf{d} + \varepsilon_{\text{num}} \Delta \hat{d}_{\text{col}}] - \mathbf{R}_{\text{int}}[\mathbf{d}, \mathbf{d}]}{\varepsilon_{\text{num}} \Delta \hat{d}_{\text{col}}},
\]

(2.52)

where \( \varepsilon_{\text{num}} \) is a small scalar constant which depends on the precision of the computer, and \( \Delta \hat{d}_{\text{col}} \) is a directional vector with all components zero except its \( \text{col}-\text{th} \) element whose value is \( \Delta d_{\text{col}} \).

For the implementation of the final established numerical model, the tangent stiffness matrix \( K \) and damping matrix \( D \) are derived as GATEAUX derivatives of \( R_{\text{int}} \) with respects to the DOFs and to their temporal derivatives by means of consistent linearization, which allows a quadratic convergence speed when solving the linearized equation system. Figure 2.10 and Figure 2.9 show, respectively, a summary of the algorithmic formulation with numerical generation of the GATEAUX
derivatives, and with the \textsc{Gateaux} derivatives derived by consistent linearization. They both provide the basis for the implementation of the proposed three-phase model for freezing soils into the object oriented FE-code \textsc{Kratos} (see e.g. \textsc{Dadvand et al.} (2002)).

\begin{verbatim}
initialize: \quad \mathbf{d}_0 = \begin{bmatrix} \mathbf{u}_0 & p_{L_0} & T_0 \end{bmatrix}^T

LOOP over time steps \quad n : 0 \rightarrow NT

LOOP over iteration steps \quad k

compute: \quad \mathbf{d}_{n+1} = \alpha_f \mathbf{d}_n + (1 - \alpha_f) \mathbf{d}_{n+1}^k
\quad \mathbf{d}_{n+1} = \alpha_f \dot{\mathbf{d}}_n + (1 - \alpha_f) \dot{\mathbf{d}}_{n+1}^k

LOOP over elements \quad e : 0 \rightarrow NE

LOOP over quadrature points \quad \star : 0 \rightarrow NG

compute: \quad \mathbf{d}^\star[\xi^\star], \nabla \mathbf{d}^\star[\xi^\star], \dot{\mathbf{d}}^\star[\xi^\star]

compute: \quad \mathbf{R}^\star_{\text{ext}}[\xi^\star], \mathbf{R}^\star_{\text{int}}[\mathbf{d}^\star, \nabla \mathbf{d}^\star, \dot{\mathbf{d}}^\star]

\quad \mathbf{K}^\star[\mathbf{d}^\star, \nabla \mathbf{d}^\star, \dot{\mathbf{d}}^\star], \mathbf{D}^\star[\mathbf{d}^\star, \nabla \mathbf{d}^\star, \dot{\mathbf{d}}^\star]

integrate: \quad (\mathbf{R}^\star_{\text{ext}})_{n+1-\alpha} = \sum_{\star=1}^{\text{NG}} \omega^\star \mathbf{R}^\star_{\text{ext}}, \quad (\mathbf{R}^\star_{\text{int}})_{n+1-\alpha} = \sum_{\star=1}^{\text{NG}} \omega^\star \mathbf{R}^\star_{\text{int}}
\quad (\mathbf{K}^\star)_{n+1-\alpha} = \sum_{\star=1}^{\text{NG}} \omega^\star \mathbf{K}^\star, \quad (\mathbf{D}^\star)_{n+1-\alpha} = \sum_{\star=1}^{\text{NG}} \omega^\star \mathbf{D}^\star

assemble: \quad (\mathbf{R}_{\text{ext}})_{n+1-\alpha} = \bigcup_{e=1}^{\text{NE}} (\mathbf{R}^e_{\text{ext}})_{n+1-\alpha}, \quad (\mathbf{R}_{\text{int}})_{n+1-\alpha} = \bigcup_{e=1}^{\text{NE}} (\mathbf{R}^e_{\text{int}})_{n+1-\alpha},
\quad \mathbf{K}_{n+1-\alpha} = \bigcup_{e=1}^{\text{NE}} \mathbf{K}^e_{n+1-\alpha}, \quad \mathbf{D}_{n+1-\alpha} = \bigcup_{e=1}^{\text{NE}} \mathbf{D}^e_{n+1-\alpha}

solve: \quad \Delta \mathbf{d}_{n+1} = \left( \mathbf{K}^k (1 - \alpha_f) + \mathbf{D}^k (1 - \alpha_f) \frac{\lambda}{\beta \Delta t} \right)^{-1} \mathbf{R}_{\text{ext}} + \mathbf{R}^k_{\text{int}}
\quad \mathbf{d}_{n+1} = \mathbf{d}_{n+1}^{k+1} + \Delta \mathbf{d}_{n+1}, \quad \dot{\mathbf{d}}_{n+1} = \dot{\mathbf{d}}_{n+1}^{k+1}, \quad \ddot{\mathbf{d}}_{n+1} = \ddot{\mathbf{d}}_{n+1}^{k+1}

update: \quad \mathbf{d}_{n+1}^{k+1} = \mathbf{d}_{n+1}^{k+1} + \Delta \mathbf{d}_{n+1}, \quad \dot{\mathbf{d}}_{n+1}^{k+1}, \quad \ddot{\mathbf{d}}_{n+1}^{k+1}

convergence check: \eta \leq \eta_{\text{crit}}

\begin{tabular}{c c}
TRUE & \text{next time step} \\
FALSE & \text{FALSE} \\
\hline
k & k + 1 \\
n & n + 1
\end{tabular}
\end{verbatim}

Figure 2.9: Algorithmic formulation of the proposed soil model in \textsc{Nassi-Sheiderman} notation with the \textsc{Gateaux} derivatives derived by consistent linearization
initialize: \( \mathbf{d}_0 = \begin{bmatrix} \mathbf{u}_0 & p_{0,0} & T_{0,0} \end{bmatrix}^T \)

LOOP over time steps \( n : 0 \to NT \)

LOOP over iteration steps \( k \)

compute: \( \mathbf{d}_{n+1-\alpha} = \alpha_f \mathbf{d}_n + (1 - \alpha_f) \mathbf{d}_{n+1}^{k+1} \)
\( \mathbf{d}_{n+1-\alpha} = \alpha_f \mathbf{d}_n + (1 - \alpha_f) \mathbf{d}_{n+1}^{k+1} \)

LOOP over quadrature points \( \star : 0 \to \text{NG} \)

compute: \( \mathbf{d}^* [\xi^*], \nabla \mathbf{d}^* [\xi^*], \mathbf{d}^+ [\xi^*] \)

compute: \( \mathbf{R}_{\text{ext}}^*[\xi^*], \mathbf{R}_{\text{int}}^*[\mathbf{d}^*, \nabla \mathbf{d}^*, \mathbf{d}^+] \)

integrate: \( (\mathbf{R}_{\text{ext}}^e)_{n+1-\alpha} = \sum_{s=1}^{\text{NG}} \omega^* \mathbf{R}_{\text{ext}}^e \), \( (\mathbf{R}_{\text{int}}^e)_{n+1-\alpha} = \sum_{s=1}^{\text{NG}} \omega^* \mathbf{R}_{\text{int}}^e \)

LOOP over columns \( \text{col} : 0 \to \text{DOF} \)

compute: \( \mathbf{d}_+ = \mathbf{d} + \varepsilon_{\text{num}} \Delta \mathbf{d}_{\text{col}} \)

LOOP over quadrature points \( \star : 0 \to \text{NG} \)

compute: \( \mathbf{d}^* [\xi^*], \nabla \mathbf{d}^* [\xi^*], \mathbf{d}^+ [\xi^*], \mathbf{d}^* [\xi^*], \nabla \mathbf{d}^* [\xi^*], \mathbf{d}^+ [\xi^*] \)

compute: \( \mathbf{R}_{\text{K}}^e [\mathbf{d}^+], \nabla \mathbf{d}^+ [\mathbf{d}^*], \mathbf{R}_{\text{D}}^e [\mathbf{d}^*, \nabla \mathbf{d}^*, \mathbf{d}^+] \)

integrate: \( \mathbf{R}_{\text{K}}^e = \sum_{s=1}^{\text{NG}} \omega^* \mathbf{R}_{\text{K}}^e \), \( \mathbf{R}_{\text{D}}^e = \sum_{s=1}^{\text{NG}} \omega^* \mathbf{R}_{\text{D}}^e \)

LOOP over rows \( \text{row} : 0 \to \text{DOF} \)

compute: \( \mathbf{K}_{\text{num}}^e \text{row, col} = \frac{(\mathbf{R}_{\text{K}}^e)_{\text{row}} - (\mathbf{R}_{\text{int}}^e)_{\text{row}}}{\varepsilon_{\text{num}} \Delta \mathbf{d}_{\text{col}}} \), \( \mathbf{D}_{\text{num}}^e \text{row, col} = \frac{(\mathbf{R}_{\text{D}}^e)_{\text{row}} - (\mathbf{R}_{\text{int}}^e)_{\text{row}}}{\varepsilon_{\text{num}} \Delta \mathbf{d}_{\text{col}}} \)

assign: \( \mathbf{K}_{n+1-\alpha}^e \approx \mathbf{K}_{\text{num}}^e \), \( \mathbf{D}_{n+1-\alpha}^e \approx \mathbf{D}_{\text{num}}^e \)

assemble: \( (\mathbf{R}_{\text{ext}}^e)_{n+1-\alpha} = \bigcup_{e=1}^{\text{NE}} (\mathbf{R}_{\text{ext}}^e)_{n+1-\alpha} \), \( (\mathbf{R}_{\text{int}}^e)_{n+1-\alpha} = \bigcup_{e=1}^{\text{NE}} (\mathbf{R}_{\text{int}}^e)_{n+1-\alpha} \)
\( \mathbf{K}_{n+1-\alpha}^e = \bigcup_{e=1}^{\text{NE}} \mathbf{K}_{n+1-\alpha}^e \), \( \mathbf{D}_{n+1-\alpha}^e = \bigcup_{e=1}^{\text{NE}} \mathbf{D}_{n+1-\alpha}^e \)

solve: \( \Delta \mathbf{d}_{n+1} = \left( \mathbf{K}_{n+1-\alpha}^e (1 - \alpha_f) + \mathbf{D}_{n+1-\alpha}^e (1 - \alpha_f) \frac{\lambda}{\beta \Delta t} \right)^{-1} (\mathbf{R}_{\text{ext}} + \mathbf{R}_{\text{int}}^k)_{n+1-\alpha} \)

update: \( \mathbf{d}_{n+1}^{k+1} = \mathbf{d}_{n+1}^k + \Delta \mathbf{d}_{n+1}, \mathbf{d}_{n+1}^{k+1}, \mathbf{d}_{n+1}^{k+1} \)

convergence check: \( \eta \leq \eta_{\text{crit}} \)

\( \mathbf{d} \)

Figure 2.10: Algorithmic formulation of the proposed soil model in NASSI-SHNEIDERMAN notation with numerical generation of the GATEAUX derivatives
2.5 Model validation

Once the basic numerical model has been established, a validation procedure has been initiated in order to demonstrate, that the proposed formulation is capable of reproducing the main characteristics of the behavior of freezing porous materials. The adopted validation strategy is characterized by three steps, in which different aspects of the model behavior are analyzed.

2.5.1 Phase transition with latent heat effect

The thermal performance of the model with regards to the phase change behavior and the latent heat effect of freezing soils is investigated by comparing the numerical results of the proposed model with the phase change model presented by Lackner et al. (2005). This numerical model has shown good agreement with the observed data from a freezing experiment performed by the same authors (Lackner et al. 2005) using radiation-type boundary conditions, where the given surrounding temperature is continuously adjusted according to temperature history measured. However, in the FE program used for the present validation analyses, radiation-type boundary conditions with continuously adjusted surrounding temperature cannot be assigned. Therefore, we have implemented the numerical model presented in Lackner et al. (2005) into our FE program and performed comparative analyses, using both numerical models, of freezing test similar to the one as in Lackner’s model, applying, however, flux instead of radiation boundary conditions. To this end, a fully saturated cuboidal sand specimen with a height of 0.09 m and a cross-section of 0.41 × 0.41 m² at an initial temperature \( T_i = 10 \, ^\circ\text{C} \) is considered. Three temperature sensors (A, B and C) have been installed at different positions in the specimen as shown in Figure 2.11a to monitor the spatial and temporal temperature distribution. At time \( t = 0 \, \text{s} \), the top surface is instantaneously subjected to freezing with a constant heat flux \( q^* = -100 \, \text{W/m}^2 \), whereas all the other surfaces are kept thermally isolated. The material parameters involved in this validation test are listed in Table 2.1. Since Lackner et al. have used a different function for the liquid saturation curve

\[
\chi_L = 1 - \frac{\phi_{L,\infty}}{\phi_L} \left( 1 - e^{-\left(T_f-T \right)^2/\bar{T}^2} \right),
\]  

(2.53)

where \( \phi_{L,\infty} = 0.27 \) is the final ice volume fraction and \( \bar{T} = 0.2 \) is a calibration parameter, the parameters \( \Delta T_{ch} \) and \( m \) in Table 2.1 are determined by adjusting the saturation curve used in the proposed model (2.33) to fit best the saturation function (2.53) used in Lackner et al. (2005).

The simulation results for both numerical models are compared in Figure 2.11. According to Figure 2.11a, both models indicate that as soon as phase transition starts the release of latent heat prevents the temperature from dropping. As long as the total released energy is consumed, a rapid temperature decrease is observed. Moreover, as shown in Figure 2.11b, during the freezing process, the freezing front propagates through the specimen from the top to the bottom until the entire specimen is frozen. The comparison shows a good correlation of the numerical results for both temperature and ice saturation evolutions. Only a slight shape difference appears in the ice saturation evolution curves (Figure 2.11b), owing to curve fitting error between the saturation curves (2.33) and (2.53).
Table 2.1: Model validation with regard to phase transition: Material parameters

<table>
<thead>
<tr>
<th>Properties</th>
<th>Symbol</th>
<th>Numerical values</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Porosity</td>
<td>$\phi_0$</td>
<td>0.42</td>
<td>–</td>
</tr>
<tr>
<td>Bulk freezing temperature</td>
<td>$T_f$</td>
<td>273</td>
<td>K</td>
</tr>
<tr>
<td>Freezing entropy</td>
<td>$S_f$</td>
<td>1.2</td>
<td>MPa</td>
</tr>
<tr>
<td>Characteristic cooling</td>
<td>$\Delta T_{ch}$</td>
<td>0.2</td>
<td>°C</td>
</tr>
<tr>
<td>Pore size distribution parameter</td>
<td>$m$</td>
<td>0.7</td>
<td>–</td>
</tr>
<tr>
<td>Initial mass density</td>
<td>$\rho_{s0}$</td>
<td>2650</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td></td>
<td>$\rho_{L0}$</td>
<td>1000</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td></td>
<td>$\rho_{C0}$</td>
<td>913</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>$\lambda_S$</td>
<td>7.694</td>
<td>W/(m K)</td>
</tr>
<tr>
<td></td>
<td>$\lambda_L$</td>
<td>0.611</td>
<td>W/(m K)</td>
</tr>
<tr>
<td></td>
<td>$\lambda_C$</td>
<td>2.222</td>
<td>W/(m K)</td>
</tr>
<tr>
<td>Heat capacity</td>
<td>$c_S$</td>
<td>740</td>
<td>J/(kg K)</td>
</tr>
<tr>
<td></td>
<td>$c_L$</td>
<td>4200</td>
<td>J/(kg K)</td>
</tr>
<tr>
<td></td>
<td>$c_C$</td>
<td>1900</td>
<td>J/(kg K)</td>
</tr>
</tbody>
</table>

Figure 2.11: Model validation with regard to phase transition: a) temperature and b) crystal ice saturation evolution at three sensor positions

2.5.2 Terzaghi’s consolidation problem

As a second validation example, a one dimensional consolidation test is re-analyzed numerically using the developed model. To this end, a soil layer resting on a rigid impervious base at depth $z = 1$ m, while its upper surface at $z = 0$ m remains drained with a reference pressure $p_L = 0$ Pa, is considered. For the sake of simplification, gravity loading is ignored. A vertical constant load $t^* = -1000 \, \text{N/m}^2$ is instantaneously applied at the upper surface. The material parameters involved in this validation test are listed in Table 2.2.

Initially, the applied load is fully carried by the liquid water leading to an over-pressurization owing to its low compressibility. Subsequently this overpressure progressively vanishes as a result of the diffusion process of the fluid towards the boundary of the soil layer which remains drained. The obtained numerical results conform perfectly with the analytical solution (COUSSY 2004) as shown in Figure 2.12. The temporal evolution of the vertical displacement of the upper surface
follows an exponential characteristics, with a value of \( u = -0.09258 \text{ mm} \) at \( t = 1000 \text{s} \). This (nearly asymptotic) value is close to the analytical solution of the soil layer’s drained settlement at \( t = \infty, s_\infty = 0.09286 \text{ mm} \), where the fluid overpressure has dissipated completely.

### 2.5.3 Strain analysis during freezing

For the final validation of the model performance related to the coupled thermo-hydro-mechanical behavior of a cubic specimen (length: 0.1 m), initially fully saturated by liquid water and exposed to uniform cooling, is analyzed. The specimen is assumed as stress-free and undrained, \( i.e., \) no external load is applied, and the total mass of water existing either in solid or liquid form remains constant. The modeling conditions are chosen according to experiments performed by \textsc{Beaudoin and MacInnis} (1974). The material parameters involved in this validation test are listed in Table 2.3, where the values of \( \Delta T_{ch} \) and \( m \) are obtained for the type of mortar with 0\% silica fume based upon its pore size distribution curve \textsc{Coussy and Monteiro} (2008).

\textsc{Coussy and Monteiro} (2008) have shown that there are three contributions to the volumetric dilation during freezing: \( \epsilon = \epsilon_{\Delta \rho} + \epsilon_{th} + \epsilon_{S} \). The main contribution \( \epsilon_{\Delta \rho} \) accounts for the hydraulic effect related to the excess of liquid water expelled from the freezing sites, resulting from the liquid-crystal density difference, which cannot escape from the sample. The other two contributions \( \epsilon_{th} \) and \( \epsilon_{S} \) are consequences of the thermal contraction of the mixture and the micro-cryo-suction process, respectively. However, the impact of the micro-cryo-suction mechanism on the overall volumetric

### Table 2.2: Model validation with regard to consolidation: Material parameters

<table>
<thead>
<tr>
<th>Properties</th>
<th>Symbol</th>
<th>Numerical values</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Porosity</td>
<td>( \phi_0 )</td>
<td>0.3</td>
<td>–</td>
</tr>
<tr>
<td>Bulk modulus</td>
<td>( k_S )</td>
<td>15.53</td>
<td>MPa</td>
</tr>
<tr>
<td></td>
<td>( k_L )</td>
<td>1.79</td>
<td>GPa</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>( g_S )</td>
<td>5.54</td>
<td>MPa</td>
</tr>
<tr>
<td>Intrinsic permeability</td>
<td>( \kappa_0 )</td>
<td>( 1.0 \times 10^{-13} )</td>
<td>m(^2)</td>
</tr>
<tr>
<td>Reference viscosity (at ( T_i ))</td>
<td>( \eta_0 )</td>
<td>( 1.0 \times 10^{-3} )</td>
<td>Pa s</td>
</tr>
</tbody>
</table>

![Figure 2.12: Model validation with regard to consolidation: a) Water pressure profiles over depth for different time instants and b) evolution of vertical displacement of upper surface](image-url)
strains is relatively small under undrained conditions, since the suction process is restricted by the limited amount of liquid water that can be drawn to the frozen fringe. In the drained case, the impact could be larger yet the resultant heaving is not significant as the formation of ice lenses is not covered in this work. In Figure 2.13 the numerical results obtained for the evolution of the volumetric strain $\epsilon$ and its three contributions during freezing are compared to the analytical solutions (indicated as solid lines), showing almost perfect agreement. It has been concluded in COUSSY AND MONTEIRO (2008), that $\epsilon_{S_f}$ is always positive even for liquids that usually contract when they solidify, and would vanish only for a zero freezing entropy. This explains why dilation was observed in the experiments performed by BEAUDOIN AND MACINNIS (1974), in which benzene, which, unlike water, contracts when it solidifies, was used as the saturating liquid. Therein, when the freezing starts, the volume contraction prevails at first contributed by negative values of $\epsilon_{th}$, followed by continuous expansion due to the positive contributions of $\epsilon_{S_f}$ and $\epsilon_{\Delta \rho}$.
Chapter 3

Strength homogenization of matrix-inclusion composites

In order to establish a comprehensive strength prediction for freezing soils, this chapter provides first a reference strength homogenization scheme for a two-phase matrix-inclusion composite by extending the linear comparison composite (LCC) approach (Ortega et al. 2011) to more general combinations of the underlying strength criteria for the matrix and inclusion phases, such as a Drucker-Prager-type or an elliptical solid matrix, intermixed with pore voids, pore fluid, rigid inclusions, or even the same type solid inclusions. The validation of the proposed two-phase strength homogenization approach is conducted by comparisons with experimental results and alternative existing strength homogenization models.

3.1 Linear comparison composite (LCC) method

As a starting point of this chapter, the strength homogenization method proposed by Ortega et al. (2011) based on the application of the LCC theory (Ponte Castaneda 2002) in the context of yield design theory (Salencon 1990), is adopted. In order to motivate the forthcoming developments, the elementary concepts of the yield design theory and the LCC approach are briefly recalled.

3.1.1 Upper bound theorem and yield design

The problem of strength homogenization of a composite material composed of different material phases is framed within the yield design theory, with the focus of determining the macroscopic dissipation capacity through limit analysis. The lower bound theorem based on statically and plastically compatible stress states underestimates the actual dissipation capacity, whereas the upper bound theorem associated with a kinematically compatible velocity field satisfying the normality rule of plastic flow overestimates it. The upper bound theorem is generally preferred against to the
lower bound theorem, because the kinematically compatible velocity field is easier to find than the statically admissible stress field (ULM AND COUSY 2003, chap. 9).

A composite material composed of different material phases is characterized by a smaller length scale as compared to the scale of a representative elementary volume (REV) of the composite. Considering the properties of the individual phases on the grain size level (i.e. the scale of the individual inclusions, denoted in the sequel as "micro-scale"), the strength characteristics of a material phase \( i \) within the composite is assumed to be characterized by an individual convex failure criterion expressed in terms of the Cauchy stress tensor \( \sigma \) at the micro-scale:

\[
F_i[\sigma] \leq 0 \iff \sigma \in G_i.
\]  

(3.1)

\( G_i \) denotes the convex domain of admissible microscopic stress states. Accordingly, at plastic collapse the maximum dissipation capacity of the material phase is defined by the support function \( \pi_i \) of \( G_i \):

\[
\pi_i[\dot{\varepsilon}] = \sup_{\sigma \in G_i} \{ \sigma : \dot{\varepsilon} \},
\]  

(3.2)

where \( \dot{\varepsilon}[v] \) is the strain rate corresponding to the velocity field \( v \), and ‘sup’ denotes the supremum, or least upper bound, of the set \( G_i \). For a given value of \( \dot{\varepsilon} \), the condition \( \sigma : \dot{\varepsilon} = \pi_i[\dot{\varepsilon}] \) defines a hyperplane \( \mathcal{H}[\dot{\varepsilon}] \) in the stress space, which is tangent to the boundary \( \partial G_i \) of the admissible stress domain \( G_i \) at the stress point \( \sigma \), where \( \dot{\varepsilon} \) is normal to \( \partial G_i \) (see Figure 3.1). This is the so-called dual definition of the strength domain \( G_i \) under the condition of associated plasticity (ULM AND COUSY 2003), i.e. \( G_i \) can be defined either through the failure criterion \( F_i \) or the support function \( \pi_i \).

Figure 3.1: Geometrical interpretation of the support function \( \pi_i \) in the stress space (based on DORMIEUX ET AL. (2006))

The main purpose of the yield design approach is the evaluation of the macroscopic support function \( \Pi_{\text{hom}} \), and the determination of the macroscopic stress \( \Sigma \) at the boundary of the macroscopic strength domain \( \partial G_{\text{hom}} \). For a given macroscopic strain rate \( \dot{\mathbf{E}} \), which is the average of the
microscopic strain rate $\dot{\varepsilon}$ over the domain $\Omega$ occupied by the composite,

$$\dot{E} = \ddot{\varepsilon} = \frac{1}{|\Omega|} \int_{\Omega} \dot{\varepsilon} \, d\Omega,$$

(3.3)

the homogenization problem is expressed by the following set of equations

$$\begin{cases}
\text{div} \sigma = 0 & (\Omega) ; \\
\sigma = \frac{\partial \pi}{\partial \dot{\varepsilon}} & (\Omega_i) ; \\
\dot{\varepsilon} = \frac{1}{2} (\text{grad} v + \text{grad}^T v) & (\Omega) ; \\
v[x] = \dot{E} \cdot x & (\partial \Omega),
\end{cases}
$$

(3.4)

where $x$ is the position vector at the microscopic scale, $\Omega_i$ is the domain occupied by the material phase $i$ and $\partial \Omega$ represents the boundary of $\Omega$.

According to the upper bound theorem, associated with kinematically admissible plastic failure mechanisms in situations of associative plastic flow, and by application of Hill’s Lemma which links the microscopic dissipation function $\pi$ with its counterpart at the scale of the REV, the macroscopic dissipation function $\Pi_{\text{hom}}$ can be obtained as

$$\Pi_{\text{hom}}[\dot{E}] = \sup_{\Sigma \in G_{\text{hom}}} \{ \Sigma : \dot{E} \} = \inf_{v' \in V[\dot{E}]} \left[ \pi[\dot{E}[v']] \right].$$

(3.5)

where $\Sigma$ denotes a macroscopic stress tensor defined as the average over the REV:

$$\Sigma = \sigma = \frac{1}{|\Omega|} \int_{\Omega} \sigma \, d\Omega,$$

(3.6)

$G_{\text{hom}}$ is the macroscopic strength domain, $V[\dot{E}]$ is the set of kinematically admissible microscopic velocity fields $v'[x]$, and ‘inf’ is the infimum or the greatest lower bound of $V[\dot{E}]$. Applying the dual definition of the macroscopic strength domain $G_{\text{hom}}$, the macroscopic stress $\Sigma$ at the boundary of $G_{\text{hom}}$ is hence obtained from the dissipation potential:

$$\Sigma = \frac{\partial \Pi_{\text{hom}}[\dot{E}]}{\partial \dot{E}}.$$

(3.7)

### 3.1.2 Linear comparison composite method

The main idea of the LCC method (LOPEZ-PAMIES AND PONTE CASTANEDA 2004, PONTE CASTANEDA 2002) is to estimate the effective behavior of a nonlinear composite by means of an optimally chosen “linear comparison” composite with a similar underlying microstructure. For a linear elastic comparison composite, the strain rate energy is a piecewise-constant function defined as the sum over all material phases $i$:

$$\psi[x, \dot{\varepsilon}] = \sum_i \chi[x] \psi_i[\dot{\varepsilon}], \quad \text{with } \chi[x] = \begin{cases} 
1, & \text{if } x \in \Omega_i; \\
0, & \text{if } x \notin \Omega_i,
\end{cases}$$

(3.8)
where $\psi_i$ is the strain energy density function of the $i^{th}$ phase given in terms of microscopic quantities, considering an initial stress:

$$
\psi_i[\dot{\varepsilon}] = \frac{1}{2} \dot{\varepsilon} : C_i : \dot{\varepsilon} + \tau_i : \dot{\varepsilon} \Rightarrow \sigma = \frac{\partial \psi_i[\dot{\varepsilon}]}{\partial \dot{\varepsilon}} = C_i : \dot{\varepsilon} + \tau_i. \quad (3.9)
$$

Herein, $C_i$ is a positive-definite elasticity tensor and $\tau_i$ stands for a prestress in phase $i$. The macroscopic strain rate energy of the linear comparison composite can be expressed as

$$
\Psi[\dot{\varepsilon}] = \inf_{\dot{\varepsilon}'} \frac{\psi[\dot{x}, \dot{\varepsilon}][\dot{\varepsilon}']}{{\dot{\varepsilon}'}}. \quad (3.10)
$$

Following PONTE CASTANEDA (2002), the fundamental inequality $\inf_x \{f[x] + g[x]\} \geq \inf_x \{f[x]\} + \inf_x \{g[x]\}$ is applied by setting $f = \pi$ and $g = \psi - \pi$:

$$
\inf_{\dot{\varepsilon}'} \frac{\psi[\dot{x}, \dot{\varepsilon}][\dot{\varepsilon}']}{{\dot{\varepsilon}'}} \geq \inf_{\dot{\varepsilon}'} \frac{\pi[\dot{x}, \dot{\varepsilon}][\dot{\varepsilon}']}{{\dot{\varepsilon}'}} + \inf_{\dot{\varepsilon}'} \frac{\psi[\dot{x}, \dot{\varepsilon}'][\dot{\varepsilon}'] - \pi[\dot{x}, \dot{\varepsilon}'][\dot{\varepsilon}']}{{\dot{\varepsilon}'}}. \quad (3.11)
$$

Considering the definitions in Eqs. (3.5) and (3.10), and relaxing the constraint for $\dot{\varepsilon}'$ at the kinematic boundary condition $V[\dot{\varepsilon}]$ the following relationship can be obtained

$$
\Psi[\dot{\varepsilon}] \geq \Pi_{\text{hom}}[\dot{\varepsilon}] + \inf_{\dot{\varepsilon}'} \left(\psi[\dot{x}, \dot{\varepsilon}'][\dot{\varepsilon}'] - \pi[\dot{x}, \dot{\varepsilon}'][\dot{\varepsilon}']\right). \quad (3.12)
$$

For composite materials Eq. (3.12) can be reformulated as

$$
\Psi[\dot{\varepsilon}] \geq \Pi_{\text{hom}}[\dot{\varepsilon}] - \sum_i \phi_i Y_i, \quad (3.13)
$$

where $\phi_i$ is the volume fraction of phase $i$, and $Y_i$ is a phase-wise constant function (ORTEGA ET AL. 2011) defined as

$$
Y_i = \sup_{\dot{\varepsilon}} \{\pi_i[\dot{\varepsilon}] - \psi_i[\dot{\varepsilon}]\}. \quad (3.14)
$$

$Y_i$ contains information on the (local) strength through the material dissipation function $\pi_i$, and on the degree of nonlinearity via the difference between the dissipation function related to (local) plastic collapse and the elastic strain rate energy. Reformulation of Eq. (3.13) leads to an upper bound of the macroscopic dissipation capacity for the comparison composite material:

$$
\Pi_{\text{hom}}[\dot{\varepsilon}] \leq \inf_{C_i, \tau_i} \left\{\Psi[\dot{\varepsilon}] + \sum_i \phi_i Y_i\right\}. \quad (3.15)
$$

Since a strict upper-bound status is difficult to generate for some applications (PONTE CASTANEDA 2002), Eq. (3.15) is generalized by replacing extremal points by stationary points

$$
\Pi_{\text{hom}}[\dot{\varepsilon}] = \text{stat}_{C_i, \tau_i} \left\{\Psi[\dot{\varepsilon}] + \sum_i \phi_i Y_i\right\}. \quad (3.16)
$$
3.2 METHODOLOGY FOR A TWO-PHASE MATRIX-INCLUSION COMPOSITE

The above ‘stat’ operation involves taking the derivatives of the terms inside the curly brackets with respect to the arguments – in the above case, \( C_i \) and \( \tau_i \), solving for the arguments as functions of \( \dot{\varepsilon} \), and substituting the result back inside the brackets to obtain a function of \( \dot{\varepsilon} \). Accordingly, the nonlinearity function \( Y_i \) is re-defined as

\[
Y_i = \text{stat} \left\{ \pi_i[\dot{\varepsilon}] - \psi_i[\dot{\varepsilon}] \right\}.
\]  

(3.17)

In general the resulting estimates obtained from Eq. (3.16) and (3.17) are stationary variational estimates, and not bounds. Nevertheless, in Eq. (3.17) because of the convexity hypothesis on \( \pi_i \) and smoothness hypothesis on \( \psi_i \), the nonlinearity function \( Y_i \) is also convex, resulting in no difference between extremal point and stationarity point. Similarly, the same holds for Eq. (3.16) provided smoothness hypothesis on \( \Psi \). For more details we refer to PONTE CASTANEDA (2002).

3.2 Methodology for a two-phase matrix-inclusion composite

We consider a two-phase composite characterized by a matrix-inclusion morphology as shown in Figure 3.2. The matrix is considered as a cohesive-frictional solid. For the inclusions, different assumptions may be made, including air pores, pores filled with water, inclusions made of cohesive-frictional materials characterized by different strength characteristics, and rigid inclusions (ZHOU AND MESCHKE 2014b). The volume fraction of the inclusion phase is denoted as \( \phi \) and is known a priori. Furthermore, we assume that the strength criteria for both matrix and inclusion phases, \( F_{\text{mat}} \) and \( F_{\text{inc}} \), are known. In order to generate an estimate for the macroscopic strength criterion of the composite \( F_{\text{hom}} \), a nonlinearity function for each material phase based on stationary estimate of the difference between the dissipation function and the strain rate energy according to Eq. (3.17) must be established on the micro-level. Subsequently, at the macro-scale, we add the contributions of the nonlinearity functions \( Y_i \) to the macroscopic strain rate energy \( \Psi \) of a “linear-comparison” composite with a similar underlying microstructure. To generate expressions of the strain rate energy established homogenization methods of continuum micro-mechanics will be used. Having both ingredients established – \( Y_i \) on the micro-scale and \( \Psi \) on the macro-level – the homogenized dissipation function \( \tilde{\Pi}_{\text{hom}} \) of the nonlinear composite, which is associated with the macroscopic strength criterion, can be computed. In the following subsections, the procedure for implementing this LCC methodology is described in detail.

Figure 3.2: Homogenization of a two-phase matrix-inclusion composite
### 3.2.1 Micro-Scale

**Step 1: Microscopic dissipation function**

At the micro-scale, i.e. the scale of the individual material phase $i$, the nonlinear behavior of each phase is characterized by a given strength criterion

$$\mathcal{F}_i[\sigma] = \mathcal{F}_i[\sigma_m, \sigma_d] \leq 0,$$

where $\sigma_m$ and $\sigma_d$ are the mean stress and the norm of the deviatoric stress associated with the stress tensor $\sigma$:

$$\sigma = \sigma_m 1 + s,$$

with

$$\begin{align*}
\sigma_m &= \frac{\text{tr}\sigma}{3}, \\
\sigma_d &= \sqrt{(s : s)/2}.
\end{align*}$$

Introducing the volumetric and deviatoric strain rate invariants $\dot{\varepsilon}_v$ and $\dot{\varepsilon}_d$ defined via

$$\dot{\varepsilon} = \frac{1}{3} \dot{\varepsilon}_v 1 + \delta,$$

with

$$\begin{align*}
\dot{\varepsilon}_v &= \text{tr}\dot{\varepsilon}, \\
\dot{\varepsilon}_d &= \sqrt{\langle\delta : \delta\rangle/2},
\end{align*}$$

and recalling the definition of the support function (3.2), the maximum dissipation capacity the material phase can afford is expressed as

$$\pi_i[\dot{\varepsilon}] = \sup_{\mathcal{F}_i[\sigma_m, \sigma_d] \leq 0} \left\{ \sigma_m \frac{1}{3} \dot{\varepsilon}_v 1 + s : \delta \right\}.$$

For a given value of $s$, the support function reaches a maximum when $\delta$ and $s$ are parallel (ULM AND COUSSY 2003), namely $\delta = (\dot{\varepsilon}_d/\sigma_d) s$. In this case, relation (3.21) takes the form:

$$\pi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d] = \sup_{\mathcal{F}_i[\sigma_m, \sigma_d] \leq 0} \left\{ \sigma_m \dot{\varepsilon}_v + 2 \sigma_d \dot{\varepsilon}_d \right\},$$

which depends only on the two invariants $\dot{\varepsilon}_v$ and $\dot{\varepsilon}_d$ of the strain rate tensor $\dot{\varepsilon}$. On the other hand, for a given strain rate $\dot{\varepsilon}$, the support function reaches a maximum at the stress point $\sigma^*$ where $\dot{\varepsilon}$ is normal to $\partial G_i$, i.e. $\dot{\varepsilon}$ is parallel to $\partial F_i/\partial \sigma[\sigma^*]$:

$$\dot{\varepsilon} = \lambda \frac{\partial F_i}{\partial \sigma}[\sigma^*], \quad \text{or} \quad \begin{align*}
\dot{\varepsilon}_v &= \lambda \frac{\partial F_i}{\partial \sigma_m}[\sigma^*], \\
\delta &= \lambda \frac{\partial F_i}{\partial \sigma_d}[\sigma^*],
\end{align*}$$

where $\lambda \geq 0$ is the plastic multiplier representing the intensity of plastic flow. The flow rule (3.23) obeys the so-called normality rule, i.e. an associated flow rule (DORMIEUX ET AL. 2006), which provides a one-to-one relation between the stress and the strain rates. Applying the chain rule

$$\frac{\partial F_i}{\partial s} = \frac{\partial F_i}{\partial \sigma_d} \frac{\partial \sigma_d}{\partial s}$$

(3.24)
and considering $\delta$ and $s$ are parallel to maximize $s : \delta$, Eq. (3.23) becomes

\[
\begin{align*}
\dot{\varepsilon}_v &= \dot{\lambda} \frac{\partial F_i}{\partial \sigma_m} [\sigma^*_m, \sigma^*_s], \\
\dot{\varepsilon}_d &= \frac{1}{2} \dot{\lambda} \frac{\partial F_i}{\partial \sigma_d} [\sigma^*_m, \sigma^*_s],
\end{align*}
\] (3.25)

where $\sigma^*_m$ and $\sigma^*_d$ are the stress invariants corresponding to the failure state. Solving simultaneously the failure criterion (3.18) and the normality rule (3.25) delivers the plastic multiplier $\dot{\lambda}$ and the stress invariants $\sigma^*_m$ and $\sigma^*_d$ as functions of the strain rate invariants $\dot{\varepsilon}_v$ and $\dot{\varepsilon}_d$. Thus, the microscopic dissipation function (3.22) can be expressed as

\[
\pi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d] = \sigma^*_m \dot{\varepsilon}_v + 2 \sigma^*_d \dot{\varepsilon}_d. 
\] (3.26)

which becomes a unique function of the plastic strain rate invariants $\dot{\varepsilon}_v$ and $\dot{\varepsilon}_d$ considering the failure criterion and the normality rule.

**Step 2: Microscopic strain rate energy**

At the level of the microstructure, a linear comparison solid is defined in the framework of linear thermo-elasticity, characterized by the bulk modulus $k_i$, shear modulus $g_i$, and the prestress $\tau_i$ of the material phase $i$. The microscopic strain rate energy (3.9) at the level of the individual phase can be re-written in the format

\[
\psi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d, k_i, g_i, \tau_i] = \frac{1}{2} k_i \dot{\varepsilon}_v^2 + 2 g_i \dot{\varepsilon}_d^2 + \tau_i \dot{\varepsilon}_v. 
\] (3.27)

For different types of materials, the phase-wise elastic moduli and the prestress, respectively, are summarized below

\[
\begin{align*}
\text{Solid:} & \quad k_i \neq 0, \quad g_i \neq 0, \quad \tau_i \neq 0; \\
\text{Voids:} & \quad k_i = 0, \quad g_i = 0, \quad \tau_i = 0; \\
\text{Rigid:} & \quad k_i \to \infty, \quad g_i \to \infty, \quad \tau_i = 0; \\
\text{Compressible fluid:} & \quad k_i \neq 0, \quad g_i = 0, \quad \tau_i \neq 0.
\end{align*}
\] (3.28)

**Step 3: Nonlinearity function**

According to the LCC method, the nonlinearity function $Y_i$ (3.17) corresponding to material phase $i$ is defined as

\[
Y_i[k_i, g_i, \tau_i] = \text{stat} \left\{ \pi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d] - \psi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d, k_i, g_i, \tau_i] \right\}. 
\] (3.29)

The above stationary condition is solved by setting the partial derivatives with respect to the strain rate invariants equal to zero, i.e.

\[
\begin{align*}
\frac{\partial (\pi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d] - \psi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d, k_i, g_i, \tau_i])}{\partial \dot{\varepsilon}_v} &= 0; \\
\frac{\partial (\pi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d] - \psi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d, k_i, g_i, \tau_i])}{\partial \dot{\varepsilon}_d} &= 0.
\end{align*}
\] (3.30)
Solving these two equations simultaneously leads to the optimal values of the strain rate invariants $\dot{\varepsilon}_v^*$ and $\dot{\varepsilon}_d^*$ as functions of the material moduli $k_i$, $g_i$ and the prestress $\tau_i$. Consequently, the microscopic material nonlinearity function is formulated as
\begin{equation}
Y_i[k_i, g_i, \tau_i] = \pi_i[\dot{\varepsilon}_v^*, \dot{\varepsilon}_d^*] - \psi_i[\dot{\varepsilon}_v^*, \dot{\varepsilon}_d^*, k_i, g_i, \tau_i].
\end{equation}

**Strength criteria at the scale of the material phases**

We now focus on composite materials, for which the matrix phase as well as inclusions are characterized by a cohesive-frictional material behavior. More specifically, two classes of failure criteria suitable for cohesive-frictional materials are considered: A Drucker-Prager-type criterion, formulated here as a regularized hyperbolic criterion, and an elliptical strength criterion, which predicts a strength limit also in hydrostatic compression. This format of strength envelopes is used, e.g. in the Cam-Clay model (WOOD 1991). It should be mentioned that in order to achieve nontrivial solutions for the equation set (3.18, 3.25), the above derivation procedure (see Section 3.2.1-Section 3.2.1) applies only for the case of convex strength domain $G_i$ defined by the convex strength criterion $F_i$. Therefore, the Drucker-Prager strength criterion has to be regularized first to convex function before applying the derivation. Meanwhile, for inclusion phases whose strength criteria cannot be regularized to convex functions, e.g. pore voids, rigid inclusions and fluid-filled pores, their nonlinearity functions are set to zero following ORTEGA ET AL. (2011). The derived dissipation and nonlinearity functions for the Drucker-Prager-type and elliptical strength criteria are summarized below. It is worth to notice that for the above-mentioned special inclusion phases, their nonlinearity functions can be retrieved as zero by evaluating the corresponding limiting cases of the nonlinearity functions for the following Drucker-Prager-type and elliptical criteria.

**Regularized Drucker-Prager (hyperbolic) criterion**

The classical (linear) Drucker-Prager criterion for the $i$-th material phase is expressed in terms of $\sigma_d$ and $\sigma_m$
\begin{equation}
F_i[\sigma_m, \sigma_d] = \sigma_d + M_i (\sigma_m - S_i) \leq 0,
\end{equation}
with the parameters $S_i$ and $M_i$ denoting the hydrostatic tensile strength and the frictional coefficient related to the friction angle $\theta_i$ by $M_i = \tan \theta_i$, respectively (see Figure 3.3a). Eq. (3.32) is reformulated as a hyperbolic strength criterion using the regularization parameter $a$:
\begin{equation}
F_i[\sigma_m, \sigma_d] \approx 1 - \frac{(\sigma_m - S_i)^2}{a} + \frac{\sigma_d^2}{a M_i^2} \leq 0 \quad (0 < a \ll 1).
\end{equation}
For $a \to 0$ the linear Drucker-Prager criterion is retrieved. The dissipation and the nonlinearity functions for the $i$-th phase are obtained in the form
\begin{align*}
\pi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d] &= S_i \dot{\varepsilon}_v - \sqrt{a (\dot{\varepsilon}_v^2 - 4 M_i^2 \dot{\varepsilon}_d^2)}, \\
Y_i[k_i, g_i, \tau_i] &= \frac{M_i^2 (-a (g_i + k_i M_i^2) + g_i (S_i - \tau_i)^2)}{2 g_i (g_i + k_i M_i^2)}. \quad (3.34)
\end{align*}
3.2. METHODOLOGY FOR A TWO-PHASE MATRIX-INCLUSION COMPOSITE

Following Ortega et al. (2011), for the linear comparison composite, the elastic moduli are linked explicitly since $k_i/g_i$ is constant. In the Drucker-Prager case, this relation can be formulated as

$$\frac{k_i}{g_i} = \frac{1}{M_i^2} = \text{constant},$$

(3.35)

resulting in

$$Y_i[g_i, \tau_i] = \frac{M_i^2 \left(-2 a + (S_i - \tau_i)^2 \right)}{4 g_i}.$$  

(3.36)

Elliptical strength criterion

This criterion has the format

$$F_i[\sigma_m, \sigma_d] = \frac{(\sigma_m + S_i)^2}{A_i} + \frac{\sigma_d^2}{B_i} - 1 \leq 0.$$  

(3.37)

An illustration of the parameters $A$, $B$ and $S$ is shown in Fig. 3.3b). The dissipation and the nonlinearity functions are obtained as

$$\pi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d] = -S_i \dot{\varepsilon}_v + \sqrt{A_i \dot{\varepsilon}_v^2 + 4 B_i \dot{\varepsilon}_d^2},$$

$$Y_i[k_i, g_i, \tau_i] = \frac{B_i \left(A_i g_i - B_i k_i - g_i \left(S_i + \tau_i\right)^2 \right)}{2 g_i \left(A_i g_i - B_i k_i\right)}.$$  

(3.38)

Figure 3.3: Strength criteria of material phase $i$: a) regularized hyperbolic Drucker-Prager criterion with $a = 10^{-8}$; b) elliptical criterion

3.2.2 Macro-Scale

Step 4: Macroscopic strain rate energy

At the macro-scale, the macroscopic strain rate energy $\Psi(\dot{E})$ of the fictitious linear comparison composite needs to be computed. The composite is assumed to be composed of a matrix material having
solid inclusions. To this end, homogenization concepts of continuum micromechanics (Ortega et al. 2011, ZAOUI 2002) are adopted. Considering a continuous description of the microscopic stress field within a matrix-inclusion composite $\Omega$:

$$\sigma[x] = C[x] : \varepsilon[x] + \tau[x] \quad \forall x \in \Omega,$$

(3.39)

the elastic modulus $C[x]$ and the prestress $\tau[x]$ are specified separately for the matrix and the inclusions, respectively:

$$C = \begin{cases} C_{\text{mat}} = 3 k_{\text{mat}} \mathbb{I} + 2 g_{\text{mat}} \mathbb{K} & (\Omega_{\text{mat}}), \\ C_{\text{inc}} = 3 k_{\text{inc}} \mathbb{I} + 2 g_{\text{inc}} \mathbb{K} & (\Omega_{\text{inc}}) \end{cases},$$

(3.40)

$$\tau = \begin{cases} \tau_{\text{mat}} = \tau_{\text{mat}} \mathbb{1} & (\Omega_{\text{mat}}), \\ \tau_{\text{inc}} = \tau_{\text{inc}} \mathbb{1} & (\Omega_{\text{inc}}). \end{cases}$$

(3.41)

$\Omega_{\text{mat}} = (1 - \phi) \Omega$, $\Omega_{\text{inc}} = \phi \Omega$ are the domains occupied by the matrix and inclusion space, respectively; $\mathbb{I}$ and $\mathbb{J}$ are the second- and fourth-order identity tensors; $J_{ijkl} = \frac{1}{3} \delta_{ij} \delta_{kl}$ and $\mathbb{K} = \mathbb{I} - \mathbb{J}$ are tensor projections. According to continuum micromechanics, the corresponding macroscopic stress-strain relation is obtained from homogenization as:

$$\Sigma = C_{\text{hom}} : \dot{E} + T_{\text{hom}},$$

(3.42)

where $C_{\text{hom}}$ and $T_{\text{hom}}$ are the homogenized (macroscopic) elastic moduli and prestress, respectively. Applying the Mori-Tanaka scheme (L1 and Wang 2008) for the case of perfect adherence between the matrix-inclusion interfaces, the macroscopic elastic moduli and the prestress $C_{\text{hom}}$ and $T_{\text{hom}}$ in Eq. (3.42) can be estimated through

$$C_{\text{hom}} = (1 - \phi) C_{\text{mat}} : A_{\text{mat}} + \phi C_{\text{inc}} : A_{\text{inc}} : (1 - \phi) A_{\text{mat}} + \phi A_{\text{inc}})^{-1},$$

(3.43)

$$T_{\text{hom}} = (1 - \phi) \tau_{\text{mat}} : A_{\text{mat}} + \phi \tau_{\text{inc}} : A_{\text{inc}} : (1 - \phi) A_{\text{mat}} + \phi A_{\text{inc}})^{-1},$$

(3.44)

where $A_{i}$ is a fourth-order concentration tensor for the respective material phase $i$. Assuming spherical inclusions and inserting the respective concentration tensor for this case (see ZAOUI (2002)) into Eqs. (3.43) and (3.44), the macroscopic elastic stiffness tensor and the prestress tensor are obtained as

$$C_{\text{hom}} = 3 K_{\text{hom}} \mathbb{J} + 2 G_{\text{hom}} \mathbb{K} = 3 g_{\text{mat}} K \mathbb{J} + 2 g_{\text{mat}} G \mathbb{K},$$

(3.45)

$$T_{\text{hom}} = T_{\text{hom}} \mathbb{1} = \mathbb{T}_1 \left( \phi \tau_{\text{inc}} + \mathbb{T}_2 (1 - \phi) \tau_{\text{mat}} \right) \mathbb{1}.$$

(3.46)
The inclusion morphology factors $K$, $G$, $T_1$ and $T_2$ can be specified as functions of the porosity $\phi$ and the ratio of the shear moduli of the inclusion and the matrix:

$$K = \frac{4 M_{\text{inc}}^2 (1 - \phi) + (3 + 4 M_{\text{mat}}^2 \phi) r_g}{(3 \phi + 4 M_{\text{mat}}^2) M_{\text{inc}}^2 + 3 M_{\text{mat}}^2 (1 - \phi) r_g},$$  \hfill (3.47)

$$G = \frac{(9 + 8 M_{\text{mat}}^2) (1 - \phi + \phi r_g) + (6 + 12 M_{\text{mat}}^2) r_g}{9 + 8 M_{\text{mat}}^2 + (6 + 12 M_{\text{mat}}^2) (n + (1 - \phi) r_g)},$$  \hfill (3.48)

$$T_1 = \frac{4 + 3 / M_{\text{mat}}^2}{4 + 3 / M_{\text{mat}}^2 + 3 (1 - \phi) r_g / M_{\text{inc}}^2},$$  \hfill (3.49)

$$T_2 = \frac{4 + 3 r_g / M_{\text{inc}}^2}{4 + 3 / M_{\text{mat}}^2},$$  \hfill (3.50)

with

$$r_g = \frac{g_{\text{inc}}}{g_{\text{mat}}} = \text{constant}. \hfill (3.51)$$

For a given composite, the porosity $\phi$ and the shear ratio $r_g$ are known.

Continuum micromechanics for poro-elastic materials also provides the macroscopic strain rate energy function of the two-phase composite in the form

$$\Psi_{\text{hom}}[\dot{\mathbf{E}}] = \frac{1}{2} \dot{\mathbf{E}} : \mathbf{C}_{\text{hom}} : \dot{\mathbf{E}} + T_{\text{hom}} : \dot{\mathbf{E}} - \frac{(\tau_{\text{mat}} - \tau_{\text{inc}})^2}{2 N},$$  \hfill (3.52)

with

$$N = \frac{k_{\text{mat}}}{b - \phi}; \quad b = 1 - \frac{K_{\text{hom}}}{k_{\text{mat}}},$$  \hfill (3.53)

where $b$ and $\phi$ are the Biot coefficient and the Biot modulus, respectively. The last term on the right hand side of Eq. (3.52) is derived from the prestress difference of the matrix-inclusion interface; for details we refer to (Dormieux et al. 2006).

Denoting $\dot{E}_v = \text{tr}(\dot{\mathbf{E}})$ and $\dot{E}_d = \sqrt{\Delta : \Delta}/2$ as the macroscopic strain rate invariants with $\Delta = \dot{\mathbf{E}} - (\dot{E}_v/3) \mathbf{1}$, inserting Eqs. (3.45), (3.46) and (3.53) into (3.52) leads to the macroscopic strain rate energy density

$$\Psi_{\text{hom}}[\dot{E}_v, \dot{E}_d, g_{\text{mat}}, \tau_{\text{mat}}, \tau_{\text{inc}}] = g_{\text{mat}} \left( \frac{1}{2} K (\dot{E}_v)^2 + 2 G (\dot{E}_d)^2 \right) + T_1 \left( \phi \tau_{\text{inc}} + T_2 (1 - \phi) \tau_{\text{mat}} \right) \dot{E}_v$$

$$+ \left( M_{\text{mat}}^2 K - (1 - \phi) \right) M_{\text{mat}}^2 \frac{(\tau_{\text{inc}} - \tau_{\text{mat}})^2}{2 g_{\text{mat}}}. \hfill (3.54)$$

**Step 5: Macroscopic dissipation function**

Employing the generated expressions for the nonlinearity function (3.31) and the strain rate energy (3.54), the next step consists in evaluating the stationary condition (3.16) for the homogenized
dissipation function of the matrix-inclusion composite:

\[ \Pi_{\text{hom}}[\dot{E}_v, \dot{E}_d] = \text{stat}_{g_{\text{mat}}, \tau_{\text{mat}}, \tau_{\text{inc}}} \left\{ \dot{\Pi}_{\text{hom}}[\dot{E}_v, \dot{E}_d, g_{\text{mat}}, \tau_{\text{mat}}, \tau_{\text{inc}}] \right\} \]

(3.55)

The stationarity condition (3.55) implies solving simultaneously the following set of equations:

\[ \frac{\partial \dot{\Pi}_{\text{hom}}}{\partial g_{\text{mat}}} = 0, \quad \frac{\partial \dot{\Pi}_{\text{hom}}}{\partial \tau_{\text{mat}}} = 0 \quad \text{and} \quad \frac{\partial \dot{\Pi}_{\text{hom}}}{\partial \tau_{\text{inc}}} = 0 \]

(3.56)

to determine the optimal modulus and prestress parameters \( g_{\text{mat}}^*, \tau_{\text{mat}}^* \) and \( \tau_{\text{inc}}^* \) as functions of the macroscopic strain rate invariants \( \dot{E}_v \) and \( \dot{E}_d \). Finally, the macroscopic dissipation function \( \Pi_{\text{hom}} \) is obtained in the form

\[ \Pi_{\text{hom}}[\dot{E}_v, \dot{E}_d] = \Pi_{\text{hom}}[\dot{E}_v, \dot{E}_d, g_{\text{mat}}^*, \tau_{\text{mat}}^*, \tau_{\text{inc}}^*] \]

(3.57)

**Step 6: Macroscopic strength criterion**

Having the macroscopic dissipation function (3.57) specified for the given strength criteria for the matrix and the inclusions, the macroscopic bounding stresses providing the strength criterion for the two-phase composite can be determined by selecting one of the following approaches according to the specified form of \( \dot{\Pi}_{\text{hom}} \).

**Case-1** If the specified form of the macroscopic dissipation function \( \dot{\Pi}_{\text{hom}} \) can be expressed by the homogeneous function of degree 1 given by:

\[ \dot{\Pi}_{\text{hom}}^*[\dot{E}_v, \dot{E}_d] = S \dot{E}_v + \sqrt{A (\dot{E}_v)^2 + 4 B (\dot{E}_d)^2}, \]

(3.58)

the standard approach can be used. It determines the macroscopic strength criterion by means of taking the derivative of the homogenized dissipation function with respect to the strain rates according to Eq. (3.7):

\[ \left\{ \begin{array}{l}
\Sigma_m[\dot{E}_v, \dot{E}_d] = \frac{\partial \dot{\Pi}_{\text{hom}}[\dot{E}_v, \dot{E}_d]}{\partial \dot{E}_v}, \\
\Sigma_d[\dot{E}_v, \dot{E}_d] = \frac{1}{2} \frac{\partial \dot{\Pi}_{\text{hom}}[\dot{E}_v, \dot{E}_d]}{\partial \dot{E}_d}
\end{array} \right. \]

(3.59)

where \( \Sigma_m = \text{tr}(\Sigma)/3 \) and \( \Sigma_d = \sqrt{(\Sigma : \Sigma)/2} \) are the macroscopic stress invariants with \( \Sigma = \Sigma - \Sigma_m 1 \). Accordingly, the macroscopic stress invariants can be obtained as functions of volumetric and deviatoric strain rate invariants in the form:

\[ \left\{ \begin{array}{l}
\Sigma_m = S \pm \frac{AD_v}{\sqrt{AD_v^2 + 4B D_d^2}}, \\
\Sigma_d = \pm \frac{2B D_d}{\sqrt{AD_v^2 + 4B D_d^2}}
\end{array} \right. \]

(3.60)
3.2. METHODOLOGY FOR A TWO-PHASE MATRIX-INCLUSION COMPOSITE

which allows to set up a relationship between $\Sigma_m$ and $\Sigma_d$ analytically by eliminating the dependence on $E_v$ and $E_d$. This relationship provides eventually the homogenized strength criterion $F_{\text{hom}}$ in the format:

$$F_{\text{hom}}[\Sigma_m, \Sigma_d] = \pm \left( \frac{(\Sigma_m - S)^2}{A} + \frac{\Sigma_d^2}{B} \right) - 1 = 0.$$  \hfill (3.61)

**Case-2**  If the specified form of the macroscopic dissipation function $\tilde{\Pi}_{\text{hom}}$ can be expressed by the homogeneous function of degree 1, yet possessing not necessarily the same format as in (3.58), we propose a new approach in this paper to generate the homogenized strength criterion $F_{\text{hom}}$. Assuming proportional loading allows to adopt a linear relation between the macroscopic strain rate invariants $E_v = r E_d$. Setting the general form of the macroscopic dissipation function defined as

$$\tilde{\Pi}_{\text{hom}} = \Sigma_m E_v + 2 \Sigma_d E_d.$$  \hfill (3.62)

and its specified polynomial form $\tilde{\Pi}_{\text{hom}}^\ast$ equal, leads to

$$\tilde{\Pi}_{\text{hom}}[E_v, r] = \tilde{\Pi}_{\text{hom}}^\ast[\dot{E}_v, r].$$  \hfill (3.63)

Since $\dot{E}_v$ and $\dot{E}_d$ are of the same order in both expressions of $\tilde{\Pi}_{\text{hom}}$ and $\tilde{\Pi}_{\text{hom}}^\ast$, the dependence of $\dot{E}_v$ in (3.63) is hence eliminated. As a result, solving Eq. (3.63) leads to two solutions $r_1$ and $r_2$ for the factor $r$, as functions of $\Sigma_m$ and $\Sigma_d$. For associated plastic flow the dissipation function is a unique function of the strain rate at the failure state (ULLM AND COUSSEY 2003). In other words, when $F = 0$, the strain rate tensor is unique, i.e., only one solution for $r$ is allowed. This is equivalent to the condition:

$$r_1[\Sigma_m, \Sigma_d] = r_2[\Sigma_m, \Sigma_d].$$  \hfill (3.64)

Accordingly, the homogenized strength criterion $F_{\text{hom}}$ can be generated from

$$F_{\text{hom}}[\Sigma_m, \Sigma_d] = r_1[\Sigma_m, \Sigma_d] - r_2[\Sigma_m, \Sigma_d] = 0,$$  \hfill (3.65)

which provides a relationship between $\Sigma_m$ and $\Sigma_d$ at the failure state. This approach works more efficiently and robustly, and would lead to the same result as in the standard approach according to Case-1 for macroscopic dissipation functions possessing the format of (3.58).

**Case-3**  If the specified form of the macroscopic dissipation function $\tilde{\Pi}_{\text{hom}}$ cannot be expressed by the homogeneous function of degree 1, a more general approach is presented. It derives first the macroscopic strain invariants $E_v$ and $E_d$ as functions of the macroscopic stress invariants $\Sigma_m$ and $\Sigma_d$ by solving simultaneously the equation set (3.59), and then generates the homogenized strength criterion $F_{\text{hom}}$ via

$$F_{\text{hom}}[\Sigma_m, \Sigma_d] = \tilde{\Pi}_{\text{hom}} - \tilde{\Pi}_{\text{hom}}^\ast = 0.$$  \hfill (3.66)

However, due to the complex dependencies of $\dot{E}_v$ and $\dot{E}_d$ on $\Sigma_m$ and $\Sigma_d$ according to Eq. (3.60), the generated form of the homogenized strength criterion $F_{\text{hom}}$ would be rather complicated.
3.3 Application to selected matrix-inclusion morphologies

In this section, the LCC method is applied for strength homogenization of matrix-inclusion composites characterized by various combinations of matrix and inclusion materials. The matrix phase can be a solid which may either be described by a Drucker-Prager (DP) or an elliptical (EL) strength criterion, while the inclusion phase can be either pore voids, pores filled with fluid, rigid inclusions, or inclusions made of a cohesive frictional material which also may be described by either a Drucker-Prager (DP) or an elliptical (EL) strength criterion, however with different strength parameters as compared to the matrix material.

For the solid matrix phase, the strength criteria for both cases are summarized below (see also Eqs. (3.33) and (3.37) in Section 3.2):

\[
F_{\text{DP}}^{\text{mat}}[\sigma_m, \sigma_d] = 1 - \frac{(\sigma_m - S_{\text{mat}})^2}{a} + \frac{\sigma_d^2}{a M_{\text{mat}}^2} \leq 0 \quad (a \to 0); \\
F_{\text{EL}}^{\text{mat}}[\sigma_m, \sigma_d] = 1 - \frac{(\sigma_m + S_{\text{mat}})^2}{A_{\text{mat}}} - \frac{\sigma_d^2}{A_{\text{mat}} M_{\text{mat}}^2} \leq 0.
\]

Note, that for simplicity, the relation \( B_{\text{mat}} = A_{\text{mat}} M_{\text{mat}}^2 \) will be used in the sequel for the elliptical strength criterion since both \( A_{\text{mat}} \) and \( B_{\text{mat}} \) are positive (see Figure 3.3b). To establish the macroscopic strain rate energy of the linear comparison solid (see Section 3.2.2) the Mori-Tanaka scheme (LI AND WANG 2008), assuming perfect adherence between the matrix-inclusion interfaces, will be applied.

3.3.1 Solid matrix with pore voids

As a first application the macroscopic strength behavior of a two-phase composite formed by a solid frictional matrix material and pore voids, is established. For the matrix material, two strength criteria are investigated: a hyperbolic (Drucker-Prager-type) and an elliptical strength criterion. This composite material represents the typical situation of an unsaturated quasi-brittle porous material.

As summarized in Eq. (3.28), the spatial distribution of the elastic stiffness and the prestress for this composite can be described as follows:

\[
\begin{align*}
C_{\text{mat}} &= 3 k_{\text{mat}} J + 2 g_{\text{mat}} K, \\
C_{\text{inc}} &= 0, \\
\tau_{\text{mat}} &= \tau_{\text{mat}}^{\text{hom}} \mathbf{1} \quad (\Omega_{\text{mat}}); \\
\tau_{\text{inc}} &= 0 \quad (\Omega_{\text{inc}}),
\end{align*}
\]

Hence, the ratio of the shear moduli of the inclusion and the matrix is obtained as \( r_g = g_{\text{inc}} / g_{\text{mat}} = 0 \). Since the pore voids have no strength capacity, the corresponding nonlinearity function for the pore domain equals to zero (\( Y_{\text{inc}} = 0 \)). Following the procedure outlined in Section 3.2 and skipping intermediate results yields the estimate for the macroscopic strength criterion of the composite for both cases as

\[
F_{\text{hom}}[\Sigma_m, \Sigma_d] = 1 + \frac{(\Sigma_m + S_{\text{hom}})^2}{A_{\text{hom}}} + \frac{\Sigma_d^2}{B_{\text{hom}}} \leq 0,
\]
3.3. APPLICATION TO SELECTED MATRIX-INCLUSION MORPHOLOGIES

with

\[
\begin{align*}
A_{\text{DP}}^\text{hom} &= \mathcal{K} M_{\text{mat}}^2 (1 - \phi) S_{\text{mat}}^2 \frac{1 - \phi - \mathcal{K} M_{\text{mat}}^2}{1 - \phi - 2 \mathcal{K} M_{\text{mat}}^2} \\
B_{\text{DP}}^\text{hom} &= \mathcal{G} M_{\text{mat}}^2 (1 - \phi) \frac{1 - \phi - \mathcal{K} M_{\text{mat}}^2}{1 - \phi - 2 \mathcal{K} M_{\text{mat}}^2} \\
C_{\text{DP}}^\text{hom} &= \mathcal{K} M_{\text{mat}}^2 S_{\text{mat}}^2 \frac{1 - \phi}{1 - \phi - 2 \mathcal{K} M_{\text{mat}}^2}
\end{align*}
\]

for the case of a Drucker-Prager-type strength criterion and

\[
\begin{align*}
A_{\text{EL}}^\text{hom} &= \mathcal{K} M_{\text{mat}}^2 ((1 - \phi) A_{\text{mat}} - (1 - \phi - \mathcal{K} M_{\text{mat}}^2) S_{\text{mat}}^2) \\
B_{\text{EL}}^\text{hom} &= \mathcal{G} M_{\text{mat}}^2 ((1 - \phi) A_{\text{mat}} - (1 - \phi - \mathcal{K} M_{\text{mat}}^2) S_{\text{mat}}^2) \\
C_{\text{EL}}^\text{hom} &= \mathcal{K} M_{\text{mat}}^2 S_{\text{mat}}
\end{align*}
\]

for the case of an elliptical type strength criterion for the matrix material.

Figure 3.4: Estimate of the macroscopic strength criteria of a two-phase composite formed by a solid matrix and pore voids for different porosities (\(\phi = 0.1, 0.5, 0.9\)). The matrix material is characterized by a) a Drucker-Prager-type strength criterion and b) an elliptical strength criterion.

The graphical interpretation of the macroscopic strength criteria in the meridional plane obtained for both cases is shown in Figure 3.4. When the porosity is approaching zero, the composite is mainly composed of the matrix phase. Therefore, as expected, the predicted macroscopic strength criterion of the composite approaches the strength criterion of the matrix phase. With increasing porosity, the strength domain shrinks and eventually converges to a point at the origin, i.e. the strength criterion of pore voids, when the porosity approaches one.

It is interesting to note, that for the case of Drucker-Prager-type solid matrix, with increasing porosity, the macroscopic strength criterion transforms from a hyperbolic to an elliptical criterion when a critical value of porosity \(\phi_{\text{crit}}\) is passed (Figure 3.4). From the mathematical point of view, this is due to the sign change of \(B_{\text{hom}}/A_{\text{hom}}\) from negative to positive with increasing porosity (see Figure 3.5a). By setting \(B_{\text{hom}}/A_{\text{hom}} = 0\), the critical porosity is evaluated as

\[
\phi_{\text{crit}} = \frac{4}{3} M_{\text{mat}}^2 \in [0, 1].
\]
which means for $M_{\text{mat}} > \sqrt{3}/2$ the macroscopic strength criterion will always be a hyperbolic or Drucker-Prager-type criterion.

In the case of an elliptical criterion used for the matrix, the characteristics of the macroscopic strength criterion does not change since the sign of $B_{\text{hom}}/A_{\text{hom}}$ is always positive. Figure 3.5b-d shows the influences of the porosity $\phi$ on the predicted effective macroscopic friction coefficient $M_{\text{hom}} = \sqrt{|B_{\text{hom}}/A_{\text{hom}}|}$, the cohesion $C_{\text{hom}}$, and the compressive hydrostatic strength limit $\tilde{\Sigma}_{-m}$ for a given matrix friction coefficient $M_{\text{mat}} = 0.5$ for both Drucker-Prager-type (DP) and elliptical (EL) strength criteria for the matrix.

![Graphs showing the influences of porosity on macroscopic strength criteria](image)

**Figure 3.5:** Influence of porosity $\phi$ on the predicted a) ratio of $B_{\text{hom}}$ over $A_{\text{hom}}$, b) friction coefficient $M_{\text{hom}}$, c) cohesion $C_{\text{hom}}$, and d) compressive hydrostatic strength limit $\tilde{\Sigma}_{-m}$ of a porous solid material whose matrix material is characterized by a Drucker-Prager-type (DP) and an elliptical (EL) strength criterion with $M_{\text{mat}} = 0.5$.

### 3.3.2 Solid matrix with fluid-filled pores

The second application of the strength up-scaling procedure is concerned with the generation of the macroscopic strength behavior of a two-phase composite formed by a solid matrix, characterized by either a hyperbolic or elliptical strength criterion, and pores completely filled with a pore fluid. This is the typical situation of a fully saturated porous material. The spatial distribution of the stiffness and the prestress for this composite can be described as follows:

$$
\left\{
\begin{array}{l}
C_{\text{mat}} = 3\, k_{\text{mat}} \mathbb{J} + 2\, g_{\text{mat}} \mathbb{K}, \\
\tau_{\text{mat}} = \tau_{\text{mat}}^1 \Omega_{\text{mat}} \\
C_{\text{inc}} = 3\, k_{\text{inc}} \mathbb{J}, \\
\tau_{\text{inc}} = \tau_{\text{inc}}^1 \Omega_{\text{inc}}
\end{array}
\right.
$$

(3.73)
Since the fluid phase has no shear strength, the shear modulus ratio (3.51) becomes zero: \( r_g = 0 \). As explained in Subsection 3.2.1 the corresponding nonlinearity function for the fluid inclusion domain also equals zero: \( Y_{inc} = 0 \). Following again the procedure outlined in Section 3.2, omitting intermediate results, the following estimates for the macroscopic strength criterion of the composite are obtained for the two cases:

\[
F_{\text{hom}}[\Sigma_m, \Sigma_d] = 1 + \frac{(\Sigma_m + S_{\text{hom}})^2}{A_{\text{hom}}} + \frac{\Sigma_d^2}{B_{\text{hom}}} \leq 0 ,
\]

with

\[
\begin{align*}
A_{\text{hom}}^{DP} &= -a \frac{1 - \phi - K M_m^2 (1+n^2)}{1 - \phi - K M_m^2} \\
B_{\text{hom}}^{DP} &= a G M_m^2 (1 - \phi) \\
S_{\text{hom}}^{DP} &= -S_m
\end{align*}
\]

for the case of a Drucker-Prager-type solid matrix and

\[
\begin{align*}
A_{\text{hom}}^{EL} &= \frac{A_m (1 - \phi (K M_m^2 (1+n)-1))}{1 - \phi - K M_m^2} \\
B_{\text{hom}}^{EL} &= -A_m G M_m^2 (1 - \phi) \\
S_{\text{hom}}^{EL} &= S_m
\end{align*}
\]

for the case of an elliptical type strength criterion for the solid matrix.

Figure 3.6 illustrates the generated macroscopic strength criteria for both cases for different porosities. In contrast to the previously investigated unsaturated case (matrix with air voids), the homogenized strength criterion always adopts the characteristics of the matrix material (i.e. it always has Drucker-Prager characteristics for the case shown in Figure 3.6a and it always has elliptical characteristics for the case shown in Figure 3.6b. When the porosity is approaching zero, in both cases the predicted macroscopic strength criterion of the composite approaches to the strength criterion of the matrix phase. With increasing porosity, the strength domain shrinks, predicting a decreasing friction coefficient of the composite and eventually converges to the horizontal axis (\( \Sigma_d = 0 \)), corresponding to the strength criterion of the fluid phase, when the porosity approaches one.

Figure 3.7 shows the influences of porosity \( \phi \) on the predicted macroscopic friction coefficient and cohesion for the two matrix cases.

### 3.3.3 Solid matrix with rigid inclusions

In the third case the strength behavior of a two-phase composite formed by a solid matrix, characterized by either hyperbolic or elliptical strength criterion, with rigid inclusions is established. The phase-wise distribution of the stiffness and prestress is given as:

\[
\begin{align*}
C_{\text{mat}} &= 3 k_m + 2 g_m \mathbb{I}, \\
\tau_{\text{mat}} &= \tau_{\text{mat}} \mathbf{1} (\Omega_{\text{mat}}); \\
C_{\text{inc}} &\to \infty, \\
\tau_{\text{inc}} &= 0 (\Omega_{\text{inc}}),
\end{align*}
\]
In this case, the shear modulus ratio (3.51) \( r_g \to \infty \) and the nonlinearity function of the rigid inclusion phase is considered as zero \( (Y_{\text{inc}} = 0) \). For a matrix with rigid inclusions the expression for the macroscopic prestress (3.46) adopting the Mori-Tanaka scheme yields

\[
T_{\text{hom}} = T_1 T_2 (1 - \phi) \tau_{\text{mat}},
\]

where the inclusion morphology factors \( T_1 \) and \( T_2 \) have been derived earlier in Eq. (3.47). The LCC scheme provides an estimate for the macroscopic strength criterion of the composite in the format

\[
F_{\text{hom}}[\Sigma_m, \Sigma_d] = 1 + \frac{(\Sigma_m + S_{\text{hom}})^2}{A_{\text{hom}}} + \frac{\Sigma_d^2}{B_{\text{hom}}} \leq 0,
\]

with

\[
\begin{cases}
A_{\text{DP}}^{\text{hom}} = K M_{\text{mat}}^2 \frac{(1-\phi)^2}{1-\phi-2K M_{\text{mat}}} \\
B_{\text{DP}}^{\text{hom}} = G M_{\text{mat}}^2 (1-\phi) \frac{(1-\phi-K M_{\text{mat}}^2)}{1-\phi-2K M_{\text{mat}}} \\
S_{\text{DP}}^{\text{hom}} = K M_{\text{mat}}^2 S_{\text{mat}} \frac{(1-\phi)}{1-\phi-2K M_{\text{mat}}} 
\end{cases}
\]
for the case of a Drucker-Prager-type strength criterion and

\[
\begin{align*}
A_{\text{EL}}^{\text{hom}} &= K M^2_{\text{mat}} \left( (1 - K M^2_{\text{mat}}) S^2_{\text{mat}} - A_{\text{mat}} (1 - \phi) \right) \\
B_{\text{EL}}^{\text{hom}} &= G M^2_{\text{mat}} \left( (1 - K M^2_{\text{mat}}) S^2_{\text{mat}} - A_{\text{mat}} (1 - \phi) \right) \\
S_{\text{EL}}^{\text{hom}} &= K M^2_{\text{mat}} S_{\text{mat}}
\end{align*}
\]  

(3.81)

for the case of an elliptical strength characteristics of the solid matrix.

The macroscopic strength criteria obtained from the proposed upscaling approach for different values of the volume fraction \(\phi\) of the rigid inclusions are shown in Figure 3.8 for both the Drucker-Prager (Figure 3.8a) and the elliptical criterion (Figure 3.8b) assumed for the matrix material. When \(\phi\) approaches zero, the composite is mainly composed of the matrix phase and the predicted macroscopic strength criterion of the composite approaches to the strength criterion of the matrix phase. With increasing volume fraction \(\phi\), the macroscopic strength domain expands due to the reinforcement with rigid inclusions. Eventually, when \(\phi \to 1\) the strength estimate predicts an infinite macroscopic friction coefficient, which is in accordance with a purely rigid material. Figure 3.8 illustrates the collapse of the two branches in the meridional plane of the regularized hyperbolic Drucker-Prager strength envelope and the elliptical shape of the macroscopic strength envelope, respectively, to a line collinear with the \(\Sigma_{d}\)-axis the when \(\phi \to 1\). Note, that in both cases

Figure 3.8: Estimate of the macroscopic strength criteria of a two-phase composite formed by a solid matrix and rigid inclusions for different volume fractions of the inclusions (\(\phi = 0.1, 0.5, 0.9\)). The matrix material is characterized by a) a Drucker-Prager-type strength criterion and b) an elliptical strength criterion.

the friction coefficient increases faster with increasing volume fraction of inclusions \(\phi\) compared to the cohesion. Although finally the correct infinite asymptotic strength is predicted as \(\phi \to 1\) for compressive states of stresses, the tensile strength decreases with increasing volume fraction of rigid inclusions. This counter-intuitive artifact of the LCC-based upscaling procedure results from the assumption of zero prestress for the inclusion phase \((\tau_{\text{inc}} = 0)\). As will be shown in the next Subsection, this assumption is identical to the assumption of infinite friction coefficient \(M_{\text{inc}}\) and zero cohesion \(c_{\text{inc}}\). This artifact can be eliminated by assuming a strength criterion for the inclusions with infinite friction coefficient as well as infinite cohesion (see Subsection 3.3.4). However, such
an infinite prestress for the inclusion phase is mathematically restricted during the implementation of the LCC methodology, as it will lead to an infinite value of $\Psi_{\text{hom}}$ in Eq. (3.54) and all subsequent steps will fail.

### 3.3.4 Solid matrix with solid inclusions

As a final application, the strength criterion of a two-phase composite, characterized by different quasi-frictional materials for the matrix and the solid inclusions, respectively, is established. Two cases are investigated: One scenario assumes that the matrix and the solid inclusions can be described by a Drucker-Prager-type strength criterion, and another case assumes that the strength behavior of both matrix and inclusions can be represented by an elliptical strength criterion. For this case, the phase-wise distribution of elastic stiffness and prestress is formulated as

\[
\begin{align*}
C_{\text{mat}} &= 3 k_{\text{mat}} J + 2 g_{\text{mat}} K, \\
C_{\text{inc}} &= 3 k_{\text{inc}} J + 2 g_{\text{inc}} K, \\
\tau_{\text{mat}} &= \tau_{\text{mat}} 1 (\Omega_{\text{mat}}), \\
\tau_{\text{inc}} &= \tau_{\text{inc}} 1 (\Omega_{\text{inc}}).
\end{align*}
\]  

(3.82)

In this case the shear modulus ratio $r_g = g_{\text{inc}} / g_{\text{mat}} \neq 0$. The Drucker-Prager-type and the elliptical strength envelopes for the solid inclusions are given as

\[
\begin{align*}
F_{\text{DP inc}}[\sigma_m, \sigma_d] &= 1 - (\sigma_m - S_{\text{inc}})^2 a + \frac{\sigma_d^2}{a M_{\text{inc}}^2} \leq 0, \\
F_{\text{EL inc}}[\sigma_m, \sigma_d] &= \left(\frac{\sigma_m + S_{\text{inc}}}{A_{\text{inc}}}\right)^2 + \frac{\sigma_d^2}{A_{\text{inc}} M_{\text{inc}}^2} - 1 \leq 0,
\end{align*}
\]  

(3.83)

where, similar to the matrix phase in Eq. (3.67), the relation $B_{\text{inc}} = A_{\text{inc}} M_{\text{inc}}^2$ is adopted for the elliptical criterion used for the inclusion. Applying the LCC-based upscaling method, again admitting details, the macroscopic strength criterion of the composite is obtained for the two cases in the form

\[
F_{\text{hom}}[\Sigma_m, \Sigma_d] = 1 + \frac{(\Sigma_m + S_{\text{hom}})^2}{A_{\text{hom}}} + \frac{\Sigma_d^2}{B_{\text{hom}}} \leq 0,
\]  

(3.84)

with

\[
\begin{align*}
A_{\text{hom}}^{\text{DP}} &= \left( M_{\text{inc}}^2 M_{\text{mat}}^2 \phi \eta (S_{\text{inc}} - S_{\text{mat}})^2 Q_1 - K M_{\text{mat}}^4 r_g \right) Q_2 - 2 Q_1, \\
&\quad + \frac{2 (n + \eta T_2)^2 (Q_1 - K M_{\text{mat}}^4 r_g)}{Q_2} - \phi \eta \left( \phi r_g M_{\text{mat}}^2 + \eta M_{\text{inc}}^2 \tau_{\text{inc}}^2 \right) \frac{2 T_2^2 r_g}{Q_2} - K, \\
B_{\text{hom}}^{\text{DP}} &= G \left( a Q_1 r_g - M_{\text{inc}}^2 M_{\text{mat}}^2 \phi \eta \frac{Q_1 - K M_{\text{inc}}^4 r_g}{Q_2} (S_{\text{inc}} - S_{\text{mat}})^2 \right), \\
S_{\text{hom}}^{\text{DP}} &= T_1 \left( S_{\text{inc}} (n + \eta T_2) \left( 3 M_{\text{inc}}^2 M_{\text{mat}}^2 \phi \eta r_g - Q_2 \right) - \left( \phi (S_{\text{inc}} + 2 S_{\text{mat}}) + \eta (2 S_{\text{inc}} + S_{\text{mat}}) T_2 \right) M_{\text{inc}}^2 M_{\text{mat}}^2 \phi \eta r_g / Q_2 \right)
\end{align*}
\]  

(3.85)
for the case of Drucker-Prager-type strength criteria for both the solid matrix and the inclusions, and

\[
\begin{align*}
A_{\text{hom}}^\text{EL} &= -K \left( (\eta A_{\text{mat}} - (\eta - K M_{\text{mat}}^2) (S_{\text{inc}} - S_{\text{mat}})^2) \\
M_{\text{mat}}^2 r_g + A_{\text{inc}} M_{\text{inc}}^2 \phi \right) / r_g \\
B_{\text{hom}}^\text{EL} &= -G \left( (\eta A_{\text{mat}} - (\eta - K M_{\text{mat}}^2) (S_{\text{inc}} - S_{\text{mat}})^2) \\
M_{\text{mat}}^2 r_g + A_{\text{inc}} M_{\text{inc}}^2 \phi \right) / r_g \\
S_{\text{hom}}^\text{EL} &= T_1 (\phi S_{\text{inc}} + \eta S_{\text{mat}} T_2)
\end{align*}
\] (3.86)

for the case of elliptical strength criteria for both the solid matrix and the inclusions, respectively, where

\[
\begin{align*}
Q_1 &= \phi M_{\text{inc}}^2 + \eta M_{\text{mat}}^2 r_g \\
Q_2 &= 2 Q_1 (Q_1 - K M_{\text{mat}}^4 r_g) - \phi \eta M_{\text{inc}}^2 M_{\text{mat}}^2 r_g \\
\eta &= 1 - \phi .
\end{align*}
\] (3.87)

The resulting macroscopic strength envelopes are shown in Figure 3.9 for different volume fractions of the solid inclusions for both cases. In the present example, it is assumed that the strength properties (cohesion and friction angle) of the inclusions is smaller than the strength of the matrix material. The material parameters used for matrix and the inclusions are summarized in Table 3.1.

When the volume fraction $\phi$ of the inclusions approaches zero, the predicted macroscopic strength criterion of the composite degenerates to the strength criterion of the matrix phase. With increasing $\phi$, the strength domain changes gradually and converges eventually to the strength envelope of the inclusion phase as $\phi \to 1$.

![Figure 3.9](image.png)

**Figure 3.9:** Estimate of the macroscopic strength criteria of a two-phase composite formed by a solid matrix and solid inclusions for different volume fractions of the inclusions ($\phi = 0.1, 0.5, 0.9$). The matrix as well as the inclusions are characterized by a) a Drucker-Prager-type strength criterion, and b) an elliptical strength criterion.

It is worth noting that the case of pore void inclusions (see Subsection 3.3.1) can be retrieved alternatively as the limit case of an elliptical type criterion for solid inclusions setting $A_{\text{inc}}$, $B_{\text{inc}} \to 0$.
Table 3.1: Strength properties of the material phases involved in Figure 3.9

<table>
<thead>
<tr>
<th>Parameters</th>
<th>DP-DP</th>
<th>EL-EL</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{\text{mat}}$</td>
<td>$10^{-8}$</td>
<td>150</td>
<td>MPa</td>
</tr>
<tr>
<td>$A_{\text{inc}}$</td>
<td>$10^{-8}$</td>
<td>50</td>
<td>MPa</td>
</tr>
<tr>
<td>$M_{\text{mat}}$</td>
<td>0.7</td>
<td>0.7</td>
<td>–</td>
</tr>
<tr>
<td>$M_{\text{inc}}$</td>
<td>0.3</td>
<td>0.3</td>
<td>–</td>
</tr>
<tr>
<td>$S_{\text{mat}}$</td>
<td>8</td>
<td>8</td>
<td>MPa</td>
</tr>
<tr>
<td>$S_{\text{inc}}$</td>
<td>0</td>
<td>8</td>
<td>MPa</td>
</tr>
<tr>
<td>$r_g$</td>
<td>$(M_{\text{inc}}/M_{\text{mat}})^2$, $(M_{\text{inc}}/M_{\text{mat}})^2$</td>
<td>–</td>
<td></td>
</tr>
</tbody>
</table>

and $S_{\text{inc}}$, $\tau_{\text{inc}} = 0$; the case of pore fluid inclusion (see Subsection 3.3.2) can be retrieved from the case of Drucker-Prager-type solid inclusion for $M_{\text{inc}} \to 0$ and $c_{\text{inc}} = 0$; while the result obtained for rigid inclusions (see Subsection 3.3.3) can be retrieved as a Drucker-Prager-type solid inclusion for $M_{\text{inc}} \to \infty$ and $c_{\text{inc}} = 0$.

Setting $c_{\text{inc}} = 0$, however, leads to the counterintuitive result discussed in Subsection 3.3.3, that the strength envelope would shrink in the tensile regime for increasing volume fraction of rigid inclusions. Obviously, for rigid inclusions, the cohesion of the inclusions should be set to infinity, $c_{\text{inc}} \to \infty$. In this case, in contrast to the results in Subsection 3.3.3, the strength of the composite provides physically correct asymptotic values (reaching infinity) both in compression and in tension when the volume fraction of rigid inclusions approaches to one ($n \to 1$).

### 3.4 Validation of the LCC approach

In this section, the predicted results of strength homogenization for selected scenarios of composite materials as presented in the previous section are validated.

#### 3.4.1 Cohesive-frictional (Drucker-Prager) matrix with pore voids

For the validation of the macroscopic strength criterion predicted by the LCC method for a matrix material, characterized by a Drucker-Prager strength criterion, intermixed with pore voids, the hydrostatic strength limits $\Sigma_m^\pm$ in compression and traction are compared, with the analytical results obtained from the 1D hollow sphere thought model, which provides reasonable strength estimates of an isotropic material based on equilibrium and resistance considerations (see details in DORMIEUX ET AL. (2006)), and with the homogenization results obtained by MAGHOUS ET AL. (2009) based upon the modified secant method.

According to the predicted macroscopic strength criterion from the LCC approach (3.69) the limit hydrostatic strengths in compression and in tension are obtained as functions of the porosity $\phi$:

\[
\Sigma_m^- = S_{\text{mat}} \frac{M_{\text{mat}} (1 - \phi)}{M_{\text{mat}} - \sqrt{\frac{3}{4}} \phi}; \quad \Sigma_m^+ = S_{\text{mat}} \frac{M_{\text{mat}} (1 - \phi)}{M_{\text{mat}} - \sqrt{\frac{3}{4}} \phi}.
\]  

(3.88)
In comparison, Maghous et al. (2009) have obtained the following results:

\[
\bar{\Sigma}_m^- = \frac{S_{\text{mat}}}{M_{\text{mat}}} \left( 1 - \phi \right) ; \quad \bar{\Sigma}_m^+ = \frac{S_{\text{mat}}}{M_{\text{mat}}} \left( 1 - \phi \right) - \sqrt{\frac{\phi}{2}}.
\]  

(3.89)

in case of associated plastic flow by implementing a nonlinear homogenization technique based upon the modified secant method, while the corresponding analytical expressions for the hollow sphere model with Drucker-Prager solid are derived by Dormieux et al. (2006) as:

\[
\Sigma^*_- = S_{\text{mat}} \left( 1 - \phi \sqrt{\frac{2}{3}} M_{\text{mat}}^{-1} \right) ; \quad \Sigma^*_+ = S_{\text{mat}} \left( 1 - \phi \sqrt{\frac{2}{3}} M_{\text{mat}}^{+1} \right).
\]  

(3.90)

Figure 3.10 shows the hydrostatic compressive and tensile strength vs. porosity according to the hollow sphere model, Maghous’ modified secant model, and our LCC-based prediction. The friction coefficient of the matrix is set to \(M_{\text{mat}} = 0.3\). As far as the compressive strength is concerned, all three models conform well for porosities larger than its critical value \(\phi_{\text{crit}}\) (see Figure 3.10a), defined as the transition from infinite to finite strength in compression (see also Subsection 3.3.1). However, for \(\phi \leq \phi_{\text{crit}}\), both the proposed LCC method and Maghous’ modified secant model fail to predict correctly the compressive strength. As already explained in Dormieux et al. (2006), Maghous et al. (2009), the reason lies in the fact that in case of low porosities, a single effective strain rate for the whole solid matrix phase, inherent to the LCC approach (Ponce Castaneda 1991) which was adopted by both models, is not accurate enough for capturing the strain heterogeneity around the pores. As can be noticed from Figure 3.10a, the critical porosity \(\phi_{\text{crit}} = (2/3) M_{\text{mat}}^2 = 0.06\) obtained from Maghous’s model is half of the value \(\phi_{\text{crit}} = 0.12\) obtained from the proposed LCC model according to Eq. (3.72) if the Mori-Tanaka scheme is applied. This may be explained by the quadratic averaging rule used in Maghous’ model based on the modified secant method (Barthelemy and Dormieux 2004) when defining the effective strain rate, in contrast to the linear averaging rule adopted in the proposed model in Eq. (3.3). On the other hand, the comparison of the estimated tensile strength in Figure 3.10b shows relatively an excellent agreement for any value of porosities.

Figure 3.10: Comparison of the hydrostatic strength limits predicted by the LCC method with Maghous’ model in case of associated plastic flow, and the hollow sphere model, for a Drucker-Prager matrix \((M_{\text{mat}} = 0.3)\) with air pores: a) compression; b) tension.
3.4.2 Cohesive-frictional (Drucker-Prager) matrix with rigid inclusions

The validation of the up-scaled macroscopic strength criterion generated for a composite characterized by a frictional (Drucker-Prager-type) matrix material with rigid inclusions is based on a comparison of the estimated homogenized macroscopic friction coefficient with results from experiments performed by PEDRO (2004) and with predictions obtained by MAGHOUS ET AL. (2009) based on the modified secant model (see Subsection 3.4.1). In the experiments performed by PEDRO (2004), the strength properties of Fontainebleau sand samples, reinforced by randomly distributed gravel inclusions with varying volume fractions, have been investigated. Since the grain size of the Fontainebleau sand is much finer than the gravel sand, the sand can be modeled as a homogenized matrix reinforced by gravel inclusions, which have, relative to the sand, a considerably larger stiffness and therefore can be considered as rigid.

According to the predicted macroscopic strength criterion (3.79), the homogenized friction coefficient of the sand-gravel composite is determined as

$$\tilde{M}_{\text{hom}} = \sqrt{-\frac{B_{\text{DP}}^{\text{hom}}}{A_{\text{DP}}^{\text{hom}}}} = \frac{M_{\text{mat}}}{1 - \phi} \sqrt{\frac{\left(3(1-\phi^2)+(6+8 M_{\text{mat}}^2) \phi\right) \left(6+9 \phi+(12+8 \phi) M_{\text{mat}}^2\right)}{6(1+2 M_{\text{mat}}^2)(3+4 \phi M_{\text{mat}}^2)}}. \quad (3.91)$$

In the Maghous’ modified secant model the macroscopic friction coefficient of composites with a Drucker-Prager matrix and rigid inclusions is established as

$$M_{\text{hom}}^* = M_{\text{mat}} \sqrt{1 + \frac{3}{2} \phi} \frac{1 - \frac{3}{2} \phi M_{\text{mat}}^2}{1 - \frac{3}{2} \phi M_x M_{\text{mat}}}, \quad (3.92)$$

where $M_x \in [0, M_{\text{mat}}]$ is the dilatancy coefficient of the matrix phase. Introducing $M_x$ allows for consideration of a non-associative plastic behavior: $M_x = 0$ represents a plastic incompressible case, and the limit case $M_x = M_{\text{mat}}$ corresponds to an associated plastic flow rule (the normality rule).

Figure 3.11 contains a comparison of the predictions from the presented LCC-based model with the homogenization results obtained by MAGHOUS ET AL. (2009) and the experimental results from PEDRO (2004). The predictions obtained from the Maghous’ modified secant method fit almost perfectly with the Pedro’s experimental data for the case $M_x = 0$ (plastic incompressible case), whereas for the case $M_x = M_{\text{mat}}$ (associated flow rule) Maghous’ model, as well as the proposed LCC-based homogenization model, overestimate the macroscopic friction coefficient to a large extent. This confirms that the adoption of the associated flow rule in the LCC method, on which the upper bound theorem relies on, always results in an overprediction of dilatancy for geomaterials such as soils and rocks. Besides, although both the presented LCC method and Maghous’ modified secant method are framed within the (variational) LCC approach proposed by PONTE CASTANEDA (1991, 2002), they differ in the averaging rule used for defining the effective strain rate (see Subsection 3.4.1) and the regularization technique applied for resolving the high singularity of the support function, e.g. in the case of the Drucker-Prager-type criterion. These differences contribute to the disparity observed in Figure 3.11a between Maghous’ model ($M_x = M_{\text{mat}}$), using an associated flow rule, and the proposed LCC method. On the other hand, when the porosity is increased to one, the macroscopic friction coefficient is expected to asymptotically reach infinity when the whole domain is occupied.
3.4. VALIDATION OF THE LCC APPROACH

by the rigid phase. This limit case is well captured by the prediction from the LCC model as shown in Figure 3.11b, while the modified secant model reaches a finite value.

![Graph showing validation of LCC approach](image)

**Figure 3.11:** Comparison of LCC-based predictions of the friction coefficient of a frictional matrix material with rigid inclusions with model predictions by MAGHOUSS ET AL. (2009) and experimental data by PEDRO (2004)

3.4.3 Cohesive-frictional matrix with solid (cohesive-frictional) inclusions

In this subsection, the predictive capabilities of the LCC-based strength upscaling for composite materials characterized by a cohesive-frictional (Drucker-Prager-type) matrix material with solid inclusions, whose material behavior is also characterized by a cohesive-frictional behavior represented by the Drucker-Prager strength criterion are investigated. As the basis for the validation of the macroscopic strength predicted by the LCC approach, experiments conducted by HEUKAMP ET AL. (2003), LEMARCHAND ET AL. (2002) on unleached and leached cement pastes and mortars are adopted. For the unleached test, the mortar is characterized by a water-cement-sand mass ratio of \( w/c/s = 1/2/4 \) using an ordinary Type I Portland cement and a fine Nevada sand, which corresponds roughly to an inclusion volume fraction of \( \phi = 0.5 \). After the calcium leaching process, the calcium silicate hydrates (CSH) and Portlandite (\( \text{Ca(OH)}_2 \)) are decalcified in the cement paste, resulting in an increase of the volume fraction of the aggregates to \( \phi \approx 0.72 \) and a reduction of the stiffness and strength of the remaining solid matrix. All relevant material and strength properties listed in Table 3.2 can be determined experimentally from the tests on the unleached and leached specimens (HEUKAMP ET AL. 2003) and from tests on quartz sand (GOMES ET AL. 2010, PICHLER AND HELLMICH 2011). The mortar material (subscript M) is considered as a composite material, consisting of a cohesive-frictional (Drucker-Prager-type) matrix phase – the cement paste (subscript CP) – and a solid (Drucker-Prager-type) inclusion phase – the fine sand grains (subscript S). The macroscopic strength characteristics of both the unleashed and the leached mortar material predicted by the LCC method are compared with the experimental data.

Specifying Eq. (3.84) using the material parameters of the matrix and inclusion phases listed in Table 3.2, the cohesion and the frictional coefficient characterizing the macroscopic strength criteria
Table 3.2: Experimentally-determined material and strength properties of unleached and leached mortar specimens (M) composed of a cement paste matrix (CP) and fine sand inclusions (S)

<table>
<thead>
<tr>
<th>Properties</th>
<th>Unleached</th>
<th>Leached</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inclusion vol. fraction $\phi$</td>
<td>0.5</td>
<td>0.72</td>
<td>–</td>
</tr>
<tr>
<td>Cohesion $c_M$</td>
<td>9.82</td>
<td>0.96</td>
<td>MPa</td>
</tr>
<tr>
<td>$c_{CP}$</td>
<td>17.11</td>
<td>1.15</td>
<td>MPa</td>
</tr>
<tr>
<td>$c_S$</td>
<td>0.1</td>
<td>0.1</td>
<td>MPa</td>
</tr>
<tr>
<td>Friction coefficient $M_M$</td>
<td>1.02</td>
<td>0.81</td>
<td>–</td>
</tr>
<tr>
<td>$M_{CP}$</td>
<td>0.82</td>
<td>0.56</td>
<td>–</td>
</tr>
<tr>
<td>$M_S$</td>
<td>1.08</td>
<td>1.08</td>
<td>–</td>
</tr>
<tr>
<td>Shear modulus $g_{CP}$</td>
<td>13.5</td>
<td>3.2</td>
<td>GPa</td>
</tr>
<tr>
<td>$g_S$</td>
<td>44.3</td>
<td>44.3</td>
<td>GPa</td>
</tr>
</tbody>
</table>

of the unleached (superscript UL) and leached (superscript L) mortar specimens are estimated as

$$\begin{align*}
\hat{c}_M^{UL} &= 12.87 \text{ MPa} \\
\hat{M}_M^{UL} &= 0.96
\end{align*}$$

and

$$\begin{align*}
\hat{c}_M^{L} &= 1.06 \text{ MPa} \\
\hat{M}_M^{L} &= 0.85
\end{align*}$$

(3.93)

When comparing with the experimental results for $c_M$ and $M_M$ in Table 3.2, the values obtained for the cohesion and the frictional coefficient correlate well, both in regards to the overall tendency from the unleached to the leached state, as well as quantitatively.

Figure 3.12: Comparison of model prediction and experimental results obtained for the macroscopic strength envelope for: a) unleached; b) leached mortar specimens composed of a cement paste matrix (CP) and fine sand inclusions (S)

Figure 3.12 contains a comparison of the strength envelopes for both (unleashed and leached) mortars. The predictions by the proposed LCC approach are included as dashed lines, and the best fit of experimental data as solid lines. Apparently, the decrease in cohesion and the increase in the friction coefficient of both unleached and leached mortar, relative to the cement paste matrix, which
results from the changing volume fraction of the sand inclusions acting as a reinforcement of the cement paste, are captured well by the model predictions. In particular for the leached case (see Figure 3.12b), the predicted macroscopic strength criterion $\tilde{F}_{\text{hom}}$ shows an excellent agreement with the experimental result $F_{\text{hom}}^*$. 
Chapter 4

Strength upscaling for freezing soils

On the basis of the extended formulation of the LCC methodology for two-phase composite of matrix-inclusion morphology presented in the previous chapter, this chapter proposes a novel multi-scale strength homogenization procedure for freezing soils regarded as partially frozen soils under both drained and undrained conditions. For the validation of the proposed upscaling strategies, the predicted strength properties for fully and partially frozen fine sand are compared with experimental results, with focus on the investigation of the influences of porosity and liquid saturation degree on the predicted strength criteria.

4.1 Upscaling strategies

In this chapter, a novel multi-scale strength homogenization procedure for freezing soils composite under both drained and undrained conditions is presented with the help of two different multi-scale thought models. For each homogenization step, the proposed extended formulation of the LCC methodology for selected matrix-inclusion morphologies in the previous chapter is applied. Besides, the incorporation of pressure melting is considered as a complementary step for the undrained case. In the proposed framework, freezing soil is modeled as a three-phase cohesive-frictional composite material, consisting of liquid water (index L; with different strength criteria $F^d_L$ and $F^{ud}_L$ under, respectively, drained and undrained conditions) and crystal ice (index C; with Drucker-Prager (Drucker-Prager) strength criterion $F_C$) as spherical inclusions embedded in a matrix of solid particles (index S; with Drucker-Prager strength criterion $F_S$). Defining $\phi$ as the total porosity and $\chi_L$ as the partial saturation of liquid water among the inclusions, the volume fractions of all three phases can be expressed respectively as

$$\phi_S = 1 - \phi, \quad \phi_L = \phi \chi_L, \quad \phi_C = \phi (1 - \chi_L)$$

(4.1)

satisfying $\phi_S + \phi_L + \phi_C = 1$. The macroscopic strength properties of partially frozen soils under drained and undrained conditions are upscaled through a two-step homogenization process based
upon the multi-scale thought model shown in, respectively, Figure 4.1 and Figure 4.2, in which the strain rate dependency of strength for freezing soil is not considered.

**Figure 4.1:** Multi-scale thought model of strength homogenization for drained freezing soil

**Figure 4.2:** Multi-scale thought model of strength homogenization for undrained freezing soil considering pressure melting of ice

In the first homogenization step for both drained and undrained freezing soils (see Figure 4.1 and Figure 4.2), assuming at sufficiently low temperature, the macroscopic strength of fully frozen soil composed of a Drucker-Prager solid matrix intermixed with Drucker-Prager crystal inclusions, is to be determined. Based upon the work done by ZHOU AND MESCHE (2014b), after regularization of the Drucker-Prager criteria of material phases to hyperbolic, such matrix-inclusion morphology
will lead to a homogenized hyperbolic strength criterion $F_{\text{hom}}^I$ for the solid-crystal composite if the Linear Comparison Composite (LCC) method (Ortega et al. 2011) is applied. The detailed implementation of the LCC methodology to deliver $F_{\text{hom}}^I$ is presented in Section 4.2.

In the second homogenization step of drained freezing soil (see Figure 4.1), when temperature increases resulting in partial melting of the ice, the obtained hyperbolic matrix of fully frozen soil is then modeled to be intermixed with drained liquid water, which can be equivalently considered as pore voids with zero strength capacity. According to Ortega et al. (2011), Zhou and Meschke (2014b), a hyperbolic solid matrix and pore voids will result in a hyperbolic or elliptical strength criterion. The detailed implementation of the LCC methodology to deliver the homogenized strength criterion $F_{\text{hom}}^{II,d}$ of drained partially frozen soil is presented in Section 4.3.

In the second homogenization step of undrained freezing soil (see Figure 4.2), with temperature increases resulting in partial melting of the ice, the obtained hyperbolic matrix of fully frozen soil is then modeled to be intermixed with undrained liquid water with zero shear strength but infinite hydrostatic compressive strength, or equivalently a Drucker-Prager-type solid inclusion with both cohesion and friction coefficient approaching zero. According to Zhou and Meschke (2014b), a hyperbolic solid matrix and pore fluid (or equivalently a hyperbolic solid inclusion with zero cohesion and friction coefficient) will result in a new hyperbolic strength criterion. The detailed implementation of the LCC methodology to deliver the homogenized strength criterion $F_{\text{hom}}^{II,ud}$ of undrained partially frozen soil is presented in Section 4.4.

Especially for the undrained case, the applied hydrostatic compressive stress can be transmitted to the liquid phase. Adopting the temperature-dependent liquid saturation function for freezing soil derived in Zhou and Meschke (2013) and introducing further the pressure-dependency of the bulk freezing point, the pressure melting of ice can be incorporated and will be presented in Section 4.5.

4.2 Homogenization step I: fully frozen soil

4.2.1 Input and output

The strength upscaling from Level 0 to Level 1 consists of determining the macroscopic strength criterion $F_{\text{hom}}^I$ of fully frozen soil composed of a Drucker-Prager solid matrix intermixed with Drucker-Prager crystal inclusions. As input, the inclusion volume fraction $\eta^I$ of the solid-crystal composite at Level 1 equals to the given porosity $\phi$ at sufficiently low temperature since

$$\eta^I = \frac{\phi_C}{\phi_S + \phi_C} = \phi, \quad \text{for} \quad \chi_L = 0,$$

and the microscopic strength criteria for both material phases are given as

$$\begin{align*}
F_{\text{mat}}^I &= F_S[p, q] = q - M_S (p + S_S) \leq 0, \\
F_{\text{inc}}^I &= F_C[p, q] = q - M_C (p + S_C) \leq 0,
\end{align*}$$

4.2. Homogenization step I: fully frozen soil
where \( p \) and \( q \) are, respectively, the volumetric stress and the norm of the deviatoric stress associated with the stress tensor \( \sigma \):

\[
\sigma = -p \mathbf{1} + \mathbf{s}, \quad \text{with} \quad \begin{cases} p = -\text{tr}\sigma/3, \\ q = \sqrt{(\mathbf{s} : \mathbf{s})/2}. \end{cases}
\]  

(4.4)

Because the normal stresses in soils are usually compressive, it is more practical to consider compressive stresses positive (WOOD 1990), just opposite to the sign convention of classical continuum mechanics. The involved material parameter \( M_i \) denotes the frictional coefficient related to the friction angle \( \theta_i \) by \( M_i = \tan \theta_i \), while \( S_i \) represents the hydrostatic tensile strength related to the cohesion \( c_i \) via \( S_i = c_i/M_i \) (see Figure 4.3).

### 4.2.2 Homogenization procedure

**Regularized microscopic strength criteria** At the micro-scale, i.e. the scale of the individual material phase \( i \), in order to apply the LCC approach, the strength criteria that characterize the non-linear behavior of both phases have to be regularized to convex functions. Using the regularization parameter \( a \) \((0 < a \ll 1)\), the equation set (4.3) is re-formulated as hyperbolic strength criteria

\[
\begin{cases}
F_{\text{mat}}^I[p, q] \approx 1 - \frac{(p + S_S)^2}{a} + \frac{q^2}{a M_S^2} \leq 0, \\
F_{\text{inc}}^I[p, q] \approx 1 - \frac{(p + S_C)^2}{a} + \frac{q^2}{a M_C^2} \leq 0.
\end{cases}
\]

(4.5)

which circumvent the singular points at \((p = -S_i, \ q = 0)\) as illustrated in Figure 4.3. For \( a \to 0 \) the linear Drucker-Prager criterion can be retrieved.

![Figure 4.3: Regularized hyperbolic strength criterion (blue dashed curve) of material phase \( i \) based upon Drucker-Prager criterion (black line)](image)
4.2. HOMOGENIZATION STEP I: FULLY FROZEN SOIL

Microscopic dissipation functions Introducing the microscopic volumetric and deviatoric strain-rate invariants \( \dot{\varepsilon}_v \) and \( \dot{\varepsilon}_d \) defined via

\[
\dot{\varepsilon} = \frac{1}{3} \dot{\varepsilon}_v \mathbf{1} + \dot{\delta}, \quad \text{with} \quad \begin{cases} 
\dot{\varepsilon}_v = \text{tr} \dot{\varepsilon}, \\
\dot{\varepsilon}_d = \sqrt{\langle \delta : \delta \rangle}/2
\end{cases}
\] (4.6)

and recalling the dual definition of the strength domain (Ulm and Coussy 2003, Zhou and Meschke 2014b), the corresponding support function of material phase \( i \), i.e. the maximum dissipation capacity the material phase can afford at plastic collapse takes the form:

\[
\pi_i(\dot{\varepsilon}_v, \dot{\varepsilon}_d) = \sup_{F_i[p, q] \leq 0} \{ -p \dot{\varepsilon}_v + 2q \dot{\varepsilon}_d \},
\] (4.7)

according to the upper bound theorem within yield design framework (Ortega et al. 2011). Besides, the adoption of the associated flow rule (or the normality rule), in which the upper bound theorem relies on (Dormieux et al. 2006), reads

\[
\dot{\varepsilon} = \lambda \frac{\partial F_i}{\partial \sigma} [\sigma^*], \quad \text{or} \quad \begin{cases} 
\dot{\varepsilon}_v = -\lambda \frac{\partial F_i}{\partial p}[p^*, q^*], \\
\dot{\varepsilon}_d = \frac{1}{2} \lambda \frac{\partial F_i}{\partial q}[p^*, q^*]
\end{cases}
\] (4.8)

where \( \lambda \geq 0 \) is the plastic multiplier representing the intensity of plastic flow, and \( \sigma^* \) is the stress point at which the support function reaches maximal, with \( p^* \) and \( q^* \) denoting the corresponding stress invariants.

Solving simultaneously the failure criterion (4.5) and the normality rule (4.8) delivers the plastic multiplier \( \dot{\lambda} \) and the stress invariants \( p^* \) and \( q^* \) as functions of the strain rate invariants \( \dot{\varepsilon}_v \) and \( \dot{\varepsilon}_d \). Thus, the microscopic dissipation function (4.7) becomes a unique function of the plastic strain rate invariants \( \dot{\varepsilon}_v \) and \( \dot{\varepsilon}_d \):

\[
\pi_i(\dot{\varepsilon}_v, \dot{\varepsilon}_d) = \sigma_m^* \dot{\varepsilon}_v + 2 \sigma_d^* \dot{\varepsilon}_d.
\] (4.9)

or in the specified forms:

\[
\begin{align*}
\pi_i^{\text{mat}}[\dot{\varepsilon}_v, \dot{\varepsilon}_d] &= -S_i \dot{\varepsilon}_v - \sqrt{a (\dot{\varepsilon}_v^2 - 4 M_i^2 \dot{\varepsilon}_d^2)}, \\
\pi_i^{\text{inc}}[\dot{\varepsilon}_v, \dot{\varepsilon}_d] &= -S_i \dot{\varepsilon}_v - \sqrt{a (\dot{\varepsilon}_v^2 - 4 M_i^2 \dot{\varepsilon}_d^2)}.
\end{align*}
\] (4.10)

Microscopic strain rate energy At the level of the microstructure, a linear comparison solid is defined in the framework of linear thermo-elasticity, characterized by the bulk modulus \( k_i \), shear modulus \( g_i \), and the prestress \( \tau_i \) of material phase \( i \). The microscopic strain rate energy for both phases can be re-written in the format

\[
\psi_i^{\text{mat}}[\dot{\varepsilon}_v, \dot{\varepsilon}_d, k_i, g_i, \tau_i] = \frac{1}{2} k_i \dot{\varepsilon}_v^2 + 2 g_i \dot{\varepsilon}_d^2 + \tau_i \dot{\varepsilon}_v,
\] (4.11)
 \[
\psi_i^{\text{inc}}[\dot{\varepsilon}_v, \dot{\varepsilon}_d, k_i, g_i, \tau_i] = \frac{1}{2} k_i \dot{\varepsilon}_v^2 + 2 g_i \dot{\varepsilon}_d^2 + \tau_i \dot{\varepsilon}_v.
\] (4.11)
Nonlinearity functions  According to the LCC method, the nonlinearity function \( Y_i \) (3.17) corresponding to material phase \( i \) is defined as

\[
Y_i[k_i, g_i, \tau_i] = \text{stat} \left\{ \pi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d] - \psi_i[\dot{\varepsilon}_v, \dot{\varepsilon}_d, k_i, g_i, \tau_i] \right\}.
\]  

(4.12)

The above stationary condition is solved by setting the partial derivatives of its argument with respect to the strain rate invariants equal to zero, i.e.

\[
\begin{align*}
\frac{\partial Y_i}{\partial \dot{\varepsilon}_v} &= -k_i \dot{\varepsilon}_v - \frac{a \dot{\varepsilon}_v}{\sqrt{a (\dot{\varepsilon}_v^2 - 4 M_i^2 \dot{\varepsilon}_d^2)}} - S_i - \tau_i = 0, \\
\frac{\partial Y_i}{\partial \dot{\varepsilon}_d} &= -4 g_i \dot{\varepsilon}_d + \frac{4 a M_i^2 \dot{\varepsilon}_d}{\sqrt{a (\dot{\varepsilon}_v^2 - 4 M_i^2 \dot{\varepsilon}_d^2)}}.
\end{align*}
\]  

(4.13)

Solving simultaneously these two equations leads to the optimal values of the strain rate invariants \( \dot{\varepsilon}_v^* \) and \( \dot{\varepsilon}_d^* \) as functions of the material moduli \( k_i, g_i \) and the prestress \( \tau_i \). Consequently, the microscopic material nonlinearity function yields

\[
Y_i[k_i, g_i, \tau_i] = \pi_i[\dot{\varepsilon}_v^*, \dot{\varepsilon}_d^*] - \psi_i[\dot{\varepsilon}_v^*, \dot{\varepsilon}_d^*, k_i, g_i, \tau_i].
\]

\[
= M_i^2 \left( -a (g_i + k_i M_i^2) + g_i (S_i - \tau_i)^2 \right)
\]

\[\frac{2 g_i (g_i + k_i M_i^2)}{4 g_i (g_i + k_i M_i^2)} \]  

(4.14)

Following ORTEGA ET AL. (2011), for the linear comparison composite the elastic moduli are linked explicitly since \( k_i / g_i = \text{constant} \). In the Drucker-Prager case, this relation writes

\[
\frac{k_i}{g_i} = \frac{1}{M_i^2} = \text{constant} ,
\]

(4.15)

resulting in

\[
\begin{align*}
Y_{\text{mat}}^I[g_S, \tau_S] &= \frac{M_S^2}{4 g_S} \left( -2 a + (S_S + \tau_S)^2 \right), \\
Y_{\text{inc}}^I[g_C, \tau_C] &= \frac{M_C^2}{4 g_C} \left( -2 a + (S_C + \tau_C)^2 \right).
\end{align*}
\]  

(4.16)

Macroscopic strain rate energy  At the macro-scale, the macroscopic strain rate energy \( \Psi(\dot{\mathbf{E}}) \) of the fictitious linear comparison composite needs to be computed. Adopting homogenization concepts of continuum micromechanics (ORTEGA ET AL. 2011, ZAOUI 2002), the corresponding macroscopic stress-strain relation is obtained from homogenization as

\[
\Sigma = C^I_{\text{hom}} : \dot{\mathbf{E}} + T^I_{\text{hom}},
\]

(4.17)

where \( C^I_{\text{hom}} \) and \( T^I_{\text{hom}} \) are, respectively, the homogenized (macroscopic) elastic moduli and prestress. Applying the Mori-Tanaka scheme (LI AND WANG 2008) for the case of perfect adherence
between matrix-inclusion interfaces, the macroscopic modulus tensor and prestress in Eq. (4.17) can be derived as

\[ C_{\text{hom}}^I = 3 K_{\text{hom}}^I \mathbb{I} + 2 G_{\text{hom}}^I \mathbb{K} = 3 g_S K_1 \mathbb{I} + 2 g_S G_1 \mathbb{K}, \]

\[ T_{\text{hom}}^I = T_1^I = T_2^I = \left( \eta^I \tau_C + T_2^I (1 - \eta^I) \tau_S \right) \mathbb{1}, \]  

(4.18)

(4.19)

where \( \mathbb{1} \) and \( \mathbb{I} \) are the second- and fourth-order identity tensors, \( J_{ijkl} = \frac{1}{3} \delta_{ij} \delta_{kl} \) and \( \mathbb{K} = I - \mathbb{I} \) are tensor projections, and \( K, G, T_1^I \) and \( T_2^I \) are the inclusion morphology factors expressed as:

\[ K_1^I = \frac{4 M_C^2 (1 - \eta^I) + (3 + 4 M_S^2 \eta^I) r_g}{(3 \eta^I + 4 M_S^2) M_C^2 + 3 M_S^2 (1 - \eta^I) r_g}, \]

(4.20)

\[ G_1^I = \frac{(9 + 8 M_S^2) (1 - \eta^I + \eta^I r_g) + (6 + 12 M_S^2) r_g}{9 + 8 M_S^2 + (6 + 12 M_S^2) (\eta^I + (1 - \eta^I) r_g)}, \]

(4.21)

\[ T_1^I = \frac{4 + 3 M_S^2}{4 + 3 \eta^I / M_S^2 + 3 (1 - \eta^I) r_g / M_C^2}, \quad T_2^I = \frac{4 + 3 r_g / M_S^2}{4 + 3 / M_S^2}, \]  

(4.22)

with \( r_g \) being the ratio of the shear moduli between the crystal and the solid phases \( r_g = \frac{g_C}{g_S} \) constant. The macroscopic strain rate energy function of the two-phase composite defined in Eq. (3.52) is specified in the form

\[ \Psi_{\text{hom}}^I[\dot{E}] = \frac{1}{2} \dot{E} : C_{\text{hom}}^I : \dot{E} + T_{\text{hom}}^I : \dot{E} - \frac{(\tau_S - \tau_C)^2}{2 N} \]  

(4.23)

with

\[ N = \frac{k_S}{b - \eta^I}; \quad b = 1 - \frac{K_{\text{hom}}^I}{k_S} \]  

(4.24)

where \( b \) and \( \phi \) are, respectively, the Biot coupling coefficient and the Biot tangent modulus. The last term is derived from the prestress difference of the matrix-inclusion interface. For details we refer to (Dormieux et al., 2006). Denoting \( \dot{E}_v = \text{tr}(\dot{E}) \) and \( \dot{E}_d = \sqrt{(\Delta : \Delta)/2} \) as the macroscopic strain rate invariants where \( \Delta = \dot{E} - (\dot{E}_v / 3) \mathbb{1} \), inserting Eqs. (4.18), (4.19) and (4.24) into (4.23) leads to the macroscopic strain rate energy density:

\[ \Psi_{\text{hom}}^I[\dot{E}_v, \dot{E}_d, g_S, \tau_S, \tau_C] = g_S \left( \frac{1}{2} K_1^I (\dot{E}_v)^2 + 2 G_1^I (\dot{E}_d)^2 \right) + T_1^I \left( \eta^I \tau_C + T_2^I (1 - \eta^I) \tau_S \right) \dot{E}_v \]

\[ + \left( M_S^2 K_1^I - (1 - \eta^I) \right) M_S^2 \frac{(\tau_C - \tau_S)^2}{2 g_S}. \]  

(4.25)

Macroscopic dissipation function

Employing the generated expressions for the nonlinearity function (4.16) and the strain rate energy (4.25), the next step consists in evaluating the stationary condition (3.16) for the homogenized dissipation function of the matrix-inclusion composite:

\[ \tilde{\Pi}_{\text{hom}}^I[\dot{E}_v, \dot{E}_d] = \text{stat} \left\{ \Psi_{\text{hom}}^I[\dot{E}_v, \dot{E}_d, g_S, \tau_S, \tau_C] \right. \]

\[ \left. + (1 - \eta^I) Y_S[g_S, \tau_S] + \eta^I Y_C[g_S, \tau_C] \right\}. \]  

(4.26)
The stationarity condition (4.26) implies solving simultaneously the following set of equations

\[
\frac{\partial \hat{\Pi}_{\text{hom}}^I}{\partial g_S} = 0, \quad \frac{\partial \hat{\Pi}_{\text{hom}}^I}{\partial \tau_S} = 0 \quad \text{and} \quad \frac{\partial \hat{\Pi}_{\text{hom}}^I}{\partial \tau_C} = 0, \tag{4.27}
\]

to determine the optimal modulus and prestress parameters \( g_S^\star, \tau_S^\star \) and \( \tau_C^\star \) as functions of the macroscopic strain rate invariants \( \dot{E}_v \) and \( \dot{E}_d \). Finally, the macroscopic dissipation function \( \hat{\Pi}_{\text{hom}}^I \) is expressed by an equivalent polynomial function of degree-1 in terms of the arguments \( \dot{E}_v \) and \( \dot{E}_d \):

\[
\hat{\Pi}_{\text{hom}}^I[\dot{E}_v, \dot{E}_d] = \hat{\Pi}_{\text{hom}}^I[\dot{E}_v, \dot{E}_d, g_S^\star, \tau_S^\star, \tau_C^\star] = x_1 \dot{E}_v + x_2 \sqrt{x_3 (\dot{E}_v)^2 + x_4 (\dot{E}_d)^2}, \tag{4.28}
\]

where \( x_1, x_2, x_3 \) and \( x_4 \) are terms independent of the strain rate invariants.

**Macroscopic strength criterion** Once the homogenized dissipation function (4.28) is obtained, the macroscopic strength criterion of the solid-crystal composite can be determined according to the yield design definition (ORTega et al. 2011):

\[
\tilde{\Pi}_{\text{hom}}^I = -P \dot{E}_v + 2Q \dot{E}_d, \tag{4.29}
\]

where \( P = -\text{tr}(\Sigma)/3 \) and \( Q = \sqrt{(S : S)/2} \) are the macroscopic stress invariants with \( S = \Sigma - P \mathbf{1} \). Assuming proportional loading allows to adopt a linear relation between the macroscopic strain rate invariants \( \dot{E}_d = r \dot{E}_v \). Setting the general form (4.29) and the specified form (4.28) of the macroscopic dissipation function equal writes

\[
-P \dot{E}_v + 2Q (r \dot{E}_v) = x_1 \dot{E}_v + x_2 \sqrt{x_3 (\dot{E}_v)^2 + x_4 (r \dot{E}_v)^2}, \tag{4.30}
\]

in which the dependence of \( \dot{E}_v \) can be eliminated:

\[
\left( \frac{-P - x_1 + 2Q r}{x_2} \right)^2 = \frac{x_3 + x_4 r^2}{x_2^2}. \tag{4.31}
\]

As a result, solving Eq. (4.31) leads to two solutions \( r_1 \) and \( r_2 \) for the factor \( r \), as functions of \( P \) and \( Q \). On the other hand, within the framework of associated plasticity (UlM and CouSSy 2003) the dissipation function is a unique function of the strain rate at the failure state, that is to say, only one solution for \( r \) is allowed. This is equivalent to the condition:

\[
r_1[P, Q] = r_2[P, Q]. \tag{4.32}
\]

The relationship between \( P \) and \( Q \) at the failure state is hence defined as

\[
\mathcal{F}_{\text{hom}}^I[P, Q] = r_1[P, Q] - r_2[P, Q] = 0, \tag{4.33}
\]

resulting in the following expression for the homogenized strength criterion \( \mathcal{F}_{\text{hom}}^I \) of the solid-crystal composite at Level 1 (fully frozen soil):

\[
\mathcal{F}_{\text{hom}}^I[P, Q] = 1 + \frac{(P + S_{\text{hom}})}{A_{\text{hom}}}^2 - \frac{Q^2}{B_{\text{hom}}} \leq 0, \tag{4.34}
\]
4.2. HOMOGENIZATION STEP I: FULLY FROZEN SOIL

where

\[ A_{I}^{l} = \left( M_{C}^{2} M_{S}^{2} \eta I X_{3} (S_{C} - S_{S})^{2} X_{1} - K M_{S}^{4} r_{g} \right) \frac{X_{1}}{r_{g}} - \frac{a X_{1}}{r_{g}} \]

\[ - \left( 2 (\eta I + X_{3} T_{2})^{2} (X_{1} - K M_{S}^{4} r_{g}) \right) \frac{T_{2} r_{g}}{X_{3} X_{2} - K} \],

\[ B_{I}^{l} = G \left( a X_{1} r_{g} - M_{C}^{2} M_{S}^{2} \eta I \right) X_{3} \frac{X_{1} - K M_{S}^{4} r_{g}}{X_{3} X_{2} - K} \],

\[ S_{I}^{l} = - T_{1} \left( S_{C} (\eta I + X_{3} T_{2}) \left( 3 M_{C}^{2} M_{S}^{2} \eta I X_{3} r_{g} - X_{2} \right) \right) \frac{(\eta I) (S_{C} + 2 S_{S}) + X_{3} (2 S_{C} + S_{S}) T_{2}}{M_{C}^{2} M_{S}^{2} \eta I X_{3} r_{g} / X_{2} \right),

with

\[ X_{1} = \phi M_{C}^{2} + (1 - \phi) M_{S}^{2} r_{g} \],

\[ X_{2} = 2 X_{1} (X_{1} - K M_{S}^{4} r_{g}) - \phi (1 - \phi) M_{C}^{2} M_{S}^{2} r_{g} \],

\[ X_{3} = 1 - \eta I \].

4.2.3 Evaluation of strength prediction on fully frozen soil

The resulting macroscopic strength envelopes of the homogenized solid-crystal composite is shown in Figure 4.4 for different values of the inclusion volume fraction \( \eta I \), i.e. the porosity \( \phi \) in the fully frozen case of soils. The relevant strength and material properties are summarized in Table 4.1 based upon experimental tests. For the solid phase, reasonable strength parameters of dry sand-clay are assumed, while for the crystal phase the strength parameters of pure ice at temperature \( T = -40^{\circ}C \) and strain rate \( \dot{\varepsilon} = 5 \times 10^{-3} \) are used (Fish and Zaretsky 1997). The shear modulus ratio is computed according to its definition: \( r_{g} = g_{C} / g_{S} = 0.1 \). When the porosity \( \phi \) approaches zero, the predicted macroscopic strength criterion \( F_{I}^{l} \) of the solid-crystal composite degenerates to the hyperbolic strength criterion of the solid phase \( F_{S} \). With increasing porosity \( \phi \) the strength domain changes gradually and converges eventually to the strength envelop of the crystal phase \( F_{C} \) as porosity goes to one.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Symbols</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cohesion</td>
<td>( c_{S} )</td>
<td>0.1</td>
<td>MPa</td>
</tr>
<tr>
<td></td>
<td>( c_{C} )</td>
<td>12.93</td>
<td>MPa</td>
</tr>
<tr>
<td>Friction coefficient</td>
<td>( M_{S} )</td>
<td>0.9</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>( M_{C} )</td>
<td>0.35</td>
<td>–</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>( g_{S} )</td>
<td>37.5</td>
<td>GPa</td>
</tr>
<tr>
<td></td>
<td>( g_{C} )</td>
<td>3.8</td>
<td>GPa</td>
</tr>
</tbody>
</table>
The influence of porosity $\phi$ on the predicted macroscopic friction coefficient $M_{\text{hom}}^I$ and cohesion $c_{\text{hom}}^I$ computed from:

$$M_{\text{hom}}^I = \sqrt{\frac{B_{\text{hom}}^I}{A_{\text{hom}}^I}} \quad \text{and} \quad c_{\text{hom}}^I = \sqrt{B_{\text{hom}}^I \left(1 + \left(\frac{S_{\text{hom}}^I}{A_{\text{hom}}^I}\right)^2\right)},$$

(4.37)

is illustrated in Figure 4.5, which provides logical description of the porosity-dependent strength of frozen soils. In the limit case of pure soil grains (dry sand or clay) when $\phi = 0$, the strength mainly consists of inter-particle friction and particle interference – relatively large friction coefficient, and becomes almost zero in unconfined condition – very small cohesion. In the other limit case when $\phi = 1$, the strength is governed by that of pure ice with a much larger cohesion due to adhesive ice bonds and smaller friction coefficient in comparison with soil grains. For the solid-ice mixture, the homogenized friction coefficient $M_{\text{hom}}^I$ decreases monotonically between these two limit cases when the porosity $\phi$ varies from 0 to 1 as shown in Figure 4.5a. On the other hand, the homogenized cohesion $c_{\text{hom}}^I$ (see Figure 4.5b) has a higher value than that of pure ice at high porosity, i.e low concentration of soil grains. This can be explained by the synergistic strengthening effects between the solid grains and ice matrix that prevent the collapse of the structure (namely structural hindrance). With decreasing porosity, i.e. the solid grain concentration is further increased, the homogenized cohesion reaches its maximum value due to the establishment of particle contact, and then drops rapidly toward that for a dry sand or clay. This prediction conforms qualitatively well with the study conducted by Goughnour and Andersland (1968) on the influence of sand concentration on the strength of sand-ice mixture, as well as the test results reported by Kaplan (1971).

Figure 4.4: Predicted macroscopic strength criteria $F_{\text{hom}}^I$ (green line) of fully frozen soil at different porosities ($\phi = 0.01, 0.5, 0.99$)

### 4.3 Homogenization step II: drained partially frozen soil

#### 4.3.1 Input and output

The strength upscaling from Level 1 to Level 2 for the drained case illustrated in Figure 4.1 consists of determining the macroscopic strength behavior of partially frozen soil $F_{\text{hom}}^{II,d}$ composed of
4.3. HOMOGENIZATION STEP II: DRAINED PARTIALLY FROZEN SOIL

Figure 4.5: Influence of porosity $\phi$ on the predicted macroscopic a) friction coefficient $M_{\text{hom}}^I$ [MPa], and b) cohesion $c_{\text{hom}}^I$ of fully frozen soil

4.3.1 Homogenization procedure

Regularized microscopic strength criterion

At the micro-scale, the strength criterion of the solid-crystal matrix phase $F_{\text{mat}}^{II}$ obtained from last homogenization step is a concave hyperbolic function as illustrated in Figure 4.6 (black curve). In order to apply the LCC approach, the matrix strength criterion is first regularized to a convex hyperbolic function (blue dashed line in Figure 4.6) in the light of maintaining the same friction coefficient $M_{\text{hom}}^I$ and cohesion $c_{\text{hom}}^I$ as derived in (4.37). Using the same regularization parameter $a$, the matrix strength criterion in Eq. (4.39) is re-formulated as

$$F_{\text{mat}}^{II} [p, q] \approx 1 - \frac{(p + \tilde{S}_{\text{hom}}^I)^2}{a} + \frac{q^2}{a (M_{\text{hom}}^I)^2} \leq 0 ,$$  

(4.40)
where the regularized hydrostatic tensile strength $\tilde{S}_{\text{hom}}^I$ is redefined as
\begin{equation}
\tilde{S}_{\text{hom}}^I = c_{\text{hom}}^I / M_{\text{hom}}^I.
\end{equation}

**Figure 4.6:** Regularized convex hyperbolic strength criterion (blue dashed curve) based upon the homogenized concave strength criterion (black line) of the solid-crystal matrix phase.

**Microscopic dissipation function** Similar as in Subsection 4.2.2 after evoking the failure criterion and the normality rule, the regularized dual definition of the microscopic strength domain of the matrix phase in terms of the dissipation function leads accordingly to
\begin{equation}
\pi_{\text{mat}}^H[\dot{\varepsilon}_v, \dot{\varepsilon}_d] = -\tilde{S}_{\text{hom}}^I \dot{\varepsilon}_v - \sqrt{4 (M_{\text{hom}}^I)^2 \dot{\varepsilon}_d^2 - (\dot{\varepsilon}_v^2)}.
\end{equation}

**Microscopic strain rate energy** At the level of the microstructure, a linear comparison solid of the matrix phase characterized by the bulk modulus $K_{\text{hom}}^I$, shear modulus $G_{\text{hom}}^I$, and the prestress $T_{\text{hom}}^I$ is defined within linear thermo-elasticity in Eqs. (4.18-4.19), resulting in the following expression of the microscopic strain rate energy for the matrix phase:
\begin{equation}
\psi_{\text{mat}}^H[\dot{\varepsilon}_v, \dot{\varepsilon}_d, K_{\text{hom}}^I, G_{\text{hom}}^I, T_{\text{hom}}^I] = \frac{1}{2} K_{\text{hom}}^I \dot{\varepsilon}_v^2 + 2 G_{\text{hom}}^I \dot{\varepsilon}_d^2 + T_{\text{hom}}^I \dot{\varepsilon}_v.
\end{equation}

For the inclusion phase of liquid-filled pores under drained condition, the elastic moduli and pre-stress are assumed to be zero: $k_L = g_L = \tau_L = 0$.

**Nonlinearity function** Following the same procedure as in Subsection 4.2.2 the microscopic non-linearity function of the matrix phase yields
\begin{equation}
Y_{\text{mat}}^H[G_{\text{hom}}^I, T_{\text{hom}}^I] = \frac{(M_{\text{hom}}^I)^2}{4 G_{\text{hom}}^I} \left(-2a + (\tilde{S}_{\text{hom}}^I + T_{\text{hom}}^I)^2\right).
\end{equation}
in which the relation between the bulk and shear moduli:

\[ \frac{K_{\text{hom}}^{I}}{G_{\text{hom}}^{I}} \approx \frac{1}{(M_{\text{hom}}^{I})^2} = \text{constant}, \]  

(4.45)

are assumed for the hyperbolic case according to LCC approach (Ortega et al., 2011).

**Macroscopic strain rate energy**  At the macro-scale, the macroscopic strain rate energy \(\Psi(\dot{E})\) of the fictitious linear comparison composite for partially frozen soil needs to be computed. Adopting homogenization concepts of continuum micromechanics (Ortega et al., 2011, Zouoi 2002), the corresponding macroscopic stress-strain relation is obtained:

\[ \Sigma = C_{\text{hom}}^{II, d} \cdot \dot{E} + T_{\text{hom}}^{II, d}, \]  

(4.46)

where \(C_{\text{hom}}^{II, d}\) and \(T_{\text{hom}}^{II, d}\) are, respectively, the homogenized (macroscopic) elastic moduli and prestress. Applying the Mori-Tanaka scheme (Li and Wang 2008) for the case of perfect adherence between solid matrix-pore inclusion interfaces, the macroscopic modulus tensor and prestress in Eq. (4.46) can be derived as

\[ C_{\text{hom}}^{II, d} = 3 K_{\text{hom}}^{II, d} I + 2 G_{\text{hom}}^{II, d} \mathbb{K} = 3 G_{\text{hom}}^{I} K_{\text{hom}}^{II, d} I + 2 G_{\text{hom}}^{I} G_{\text{hom}}^{II, d} \mathbb{K}, \]  

(4.47)

\[ T_{\text{hom}}^{II, d} = T_{\text{hom}}^{I, d} I = K_{\text{hom}}^{II, d} (M_{\text{hom}}^{I})^2 T_{\text{hom}}^{I, d} I, \]  

(4.48)

with

\[ K_{\text{hom}}^{II, d} = \frac{4 (1 - \eta_{II})}{3 \eta_{II} + 4 (M_{\text{hom}}^{I})^2}, \]  

(4.49)

\[ G_{\text{hom}}^{II, d} = \frac{9 + 8 (M_{\text{hom}}^{I})^2}{9 + 8 (M_{\text{hom}}^{I})^2 + (6 + 12 (M_{\text{hom}}^{I})^2) \eta_{II}}. \]  

(4.50)

Thus, the macroscopic strain rate energy function of the two-phase composite takes the form

\[ \Psi_{\text{hom}}^{I, d} \left[ \dot{E}_v, \dot{E}_d, G_{\text{hom}}^{I}, T_{\text{hom}}^{I} \right] = G_{\text{hom}}^{I} \left[ \frac{1}{2} K_{\text{hom}}^{II, d} (\dot{E}_v)^2 + 2 G_{\text{hom}}^{II, d} (\dot{E}_d)^2 \right] + K_{\text{hom}}^{II, d} (M_{\text{hom}}^{I})^2 T_{\text{hom}}^{I} \dot{E}_v \]  

\[ + \left( (M_{\text{hom}}^{I})^2 K_{\text{hom}}^{II, d} (1 - \eta_{II}) \right) \left( \frac{(M_{\text{hom}}^{I})^2}{2 G_{\text{hom}}^{I}} \right) (T_{\text{hom}}^{I})^2. \]  

(4.51)

**Macroscopic dissipation function**  Evaluating the stationary condition (3.16) for the homogenized dissipation function of the solid/crystal-pore composite

\[ \Pi_{\text{hom}}^{I, d} \left[ \dot{E}_v, \dot{E}_d \right] = \text{stat} \left\{ \Psi_{\text{hom}} \left[ \dot{E}_v, \dot{E}_d, G_{\text{hom}}^{I}, T_{\text{hom}}^{I} \right] + (1 - \eta_{II}) Y_{\text{mat}} \left[ G_{\text{hom}}^{I}, T_{\text{hom}}^{I} \right] \right\}. \]  

(4.52)

leads to the following expression for the macroscopic dissipation function:

\[ \Pi_{\text{hom}}^{I, d} \left[ \dot{E}_v, \dot{E}_d \right] = y_1 \dot{E}_v + y_2 \sqrt{y_3 (\dot{E}_v)^2 + y_4 (\dot{E}_d)^2}, \]  

(4.53)

which is an equivalent polynomial function of degree-1 in terms of the arguments \(\dot{E}_v\) and \(\dot{E}_d\). Herein, \(y_1, y_2, y_3\) and \(y_4\) are terms independent of the strain rate invariants.
Macroscopic strength criterion  Following the same approach as in homogenization step 1, the macroscopic strength criterion of the solid/crystal-pore composite is determined accordingly:

\[ F_{\text{hom}}^{II, d}(P, Q) = 1 - \frac{(P + S_{\text{hom}}^{II, d})^2}{A_{\text{hom}}^{II, d}} + \frac{Q^2}{B_{\text{hom}}^{II, d}} = 0, \]  

(4.54)

where

\[
\begin{align*}
A_{\text{hom}}^{II, d} &= K_{\text{hom}}^{II, d} (M_{\text{hom}}^{I})^2 (1 - \eta^{II})^2 / Y_2 \left( a - (\tilde{S}_{\text{hom}}^{II})^2 Y_1 / Y_2 \right), \\
B_{\text{hom}}^{II, d} &= G_{\text{hom}}^{II, d} (M_{\text{hom}}^{I})^2 (1 - \eta^{II}) \left( a - (\tilde{S}_{\text{hom}}^{II})^2 Y_1 / Y_2 \right), \\
S_{\text{hom}}^{II, d} &= -K_{\text{hom}}^{II, d} (M_{\text{hom}}^{I})^2 \tilde{S}_{\text{hom}}^{II} (1 - \eta^{II}) / Y_2,
\end{align*}
\]  

(4.55)

with

\[
\begin{align*}
Y_1 &= K_{\text{hom}}^{II, d} (M_{\text{hom}}^{I})^2 (1 - \eta^{II}) , \\
Y_2 &= 2 K_{\text{hom}}^{II, d} (M_{\text{hom}}^{I})^2 (1 - \eta^{II}).
\end{align*}
\]  

(4.56)

4.3.3 Evaluation of strength prediction on drained partially frozen soil

The predicted macroscopic strength envelopes of the drained partially frozen soil, i.e. the solid/crystal-pore composite at porosity \( \phi = 0.5 \) and \( \phi = 0.99 \) are shown, respectively, in Figure 4.7 and Figure 4.8. For each case three different values of liquid saturation (\( \chi_{L} = 0.01, 0.5, 0.99 \)) are compared and the same material properties as listed in Table 4.1 are used.

For the case of porosity \( \phi = 0.5 \) in Figure 4.7, the variation of liquid saturation from 0 to 1 results in a transition from fully frozen to unfrozen state. When the liquid saturation \( \chi_{L} \to 0 \), the predicted macroscopic strength criterion \( F_{\text{hom}}^{II, d} \) (black line) degenerates to the regularized hyperbolic strength criterion of the matrix phase \( F_{\text{hom}}^{I} \) as expressed in Eq. (4.40). With increasing liquid saturation, the predicted cohesion of the freezing soil defined as

\[ c_{\text{hom}}^{II, d} = \sqrt{B_{\text{hom}}^{II, d} \left( \frac{S_{\text{hom}}^{II, d}}{A_{\text{hom}}^{II, d}} - 1 \right)}, \]  

(4.57)

is decreasing and tries to approach zero in the unfrozen state (see also Figure 4.9b, blue curve), whereas the predicted friction coefficient of the freezing soil

\[ M_{\text{hom}}^{II, d} = \sqrt{B_{\text{hom}}^{II, d} / A_{\text{hom}}^{II, d}}, \]  

(4.58)

undergoes a slight drop from the fully frozen state to unfrozen state (see also Figure 4.9a, blue curve). The predicted homogenized strength criterion in the unfrozen state is close to the Mohr-Coulomb failure criterion, which is a good criterion for sand and probably for clay as well (VERRUIJT 2012). It is important to note here, that since in the unfrozen state the strength properties of different soil types (e.g. sand, clay and silt) can be measured, we could use this as input and run a back-analysis to calibrate the afore-assumed strength properties of the solid material phase \( c_S \) and
4.3. HOMOGENIZATION STEP II: DRAINED PARTIALLY FROZEN SOIL

Figure 4.7: Predicted macroscopic strength criteria $F_{II, d}^{\text{hom}}$ (black line) of drained partially frozen soil at porosity $\phi = 0.5$ for different liquid saturations ($\chi_L = 0.01, 0.5, 0.99$)

$M_S$ listed in Table 4.1 which are unusually impossible to be determined experimentally and would differ for different soil types.

For the case of porosity $\phi = 0.99$ in Figure 4.8, the variation of liquid saturation from 0 to 1 results in a transition from pure ice to pure water under drained condition. When the liquid saturation $\chi_L \to 0$, the predicted macroscopic strength criterion $F_{II, d}^{\text{hom}}$ (black line) degenerates to that of ice $F_C$. With increasing liquid saturation, i.e. partial melting of ice into water, the predicted cohesion of the ice-water mixture decreases and eventually goes to zero as expected (see also Figure 4.9b, red curve). In fact, the strength domain shrinks and eventually converges to a point at the origin, i.e. the strength criterion of pure water under drained condition when the liquid saturation approaches one. During this evolution, a transition of the homogenized strength envelop from hyperbolic to elliptic is obtained, resulting in a hydrostatic strength limit in compression for the ice-water mixture. From the mathematical point of view, this is due to the sign change of $B_{II, d}^{\text{hom}} / A_{II, d}^{\text{hom}}$ from negative to positive with increasing liquid saturation. The critical liquid saturation which characterizes this transition is computed as $\chi_{L, \text{crit}} = 4/3 (M_C)^2 = 0.077$, and can be recognized in Figure 4.9a (red curve).

Figure 4.8: Predicted macroscopic strength criteria $F_{II, d}^{\text{hom}}$ (black line) of drained partially frozen soil at porosity $\phi = 0.99$, i.e. ice-water mixture, for different liquid saturations ($\chi_L = 0.01, 0.5, 0.99$)
CHAPTER 4. STRENGTH UPSCALING FOR FREEZING SOILS

Besides, the influence of porosity \( \phi \) on the predicted macroscopic friction coefficient \( M_{\text{hom}}^{II,d} \) and cohesion \( c_{\text{hom}}^{II,d} \) is illustrated in Figure 4.10 for three different cases of liquid saturation (\( \chi_L = 0.01, 0.5, 0.99 \)). In the fully frozen state (\( \chi_L \to 0 \)), the homogenized results are identical to that as plotted in Figure 4.5 (green curves) from the first homogenization step. With increasing liquid saturation, i.e. partial melting of ice into water, a drop off in the predicted cohesion \( c_{\text{hom}}^{II,d} \) is observed and eventually it converges to zero in the unfrozen state when \( \chi_L \to 1 \) (see Figure 4.10b). Increase of liquid saturation for cases of large porosity, i.e. larger liquid volume fraction \( \phi_L \), can lead to a transition of the homogenized strength envelop from hyperbolic to elliptic (see Figure 4.10a, discontinuity points), resulting in a hydrostatic compressive strength limit.

4.4 Homogenization step II: undrained partially frozen soil

4.4.1 Input and output

The strength upscaling from Level 1 to Level 2 for the undrained case illustrated in Figure 4.2 consists of determining the macroscopic strength behavior of partially frozen soil \( F_{\text{hom}}^{II,ud} \) composed of a hyperbolic solid-crystal matrix, obtained from the first homogenization step (see Section 4.2).
4.4. HOMOGENIZATION STEP II: UNDRAINED PARTIALLY FROZEN SOIL

As input, the inclusion volume fractions \( \eta^I \) at Level 1 and \( \eta^{II} \) at Level 2 are defined via Eq. (4.38) the same as for the drained case. Having the same regularized microscopic strength criterion of the matrix phase \( F^I_{\text{hom}} \) as for the undrained case, the only difference lies in the microscopic strength criterion for the liquid phase \( F^L_{\text{ud}} \):

\[
\begin{align*}
F^I_{\text{mat}} &= F^I_{\text{hom}}[p, q] \approx 1 - \frac{(p + \tilde{S}^I_{\text{hom}})^2}{a} + \frac{q^2}{a(M^I_{\text{hom}})^2} \leq 0, \\
F^I_{\text{inc}} &= F^I_{\text{ud}}[p, q] = q^2 \leq 0 ,
\end{align*}
\]

which is characterized with zero shear strength but infinite hydrostatic compressive strength. Similarly the nonlinearity function for the inclusion phase is again set to zero: \( Y^I_{\text{inc}} = 0 \) following the LCC approach (ZHOU AND MESCHKE 2014b).

4.4.2 Homogenization procedure

The strength upsampling of the undrained freezing soil at Level II follows almost the same procedure as at Level I (see Section 4.2), where both matrix and inclusion phases can be characterized by Drucker-Prager-type (or hyperbolic) strength criteria. By assumption of the following elastic moduli for the inclusion phase: \( k_L \rightarrow \infty \) and \( g_L = 0 \) (or \( M_L = 0 \) and \( r_g = 0 \) according to Eq. (4.15) and (3.51)), the corresponding inclusion morphology factors of the composite can be specified as:

\[
\begin{align*}
K^{II, \text{ud}} &= \frac{4 (1 - \eta^{II})}{4 (M^I_{\text{hom}})^2 + 3 \eta^{II}}, \\
G^{II, \text{ud}} &= \frac{(9 + 8 (M^I_{\text{hom}})^2) (1 - \eta^{II})}{9 + 8 (M^I_{\text{hom}})^2 + (6 + 12 (M^I_{\text{hom}})^2) \eta^{II}}, \\
\mathcal{T}^{II, \text{ud}}_1 &= \frac{4 (M^I_{\text{hom}})^2 + 3}{4 (M^I_{\text{hom}})^2 + 3 \eta^{II}}, \quad \mathcal{T}^{II, \text{ud}}_2 = \frac{4 (M^I_{\text{hom}})^2}{4 (M^I_{\text{hom}})^2 + 3}.
\end{align*}
\]

Skipping the intermediate steps, eventually the homogenized strength criterion \( F^{II, \text{ud}}_{\text{hom}} \) of the solid/crystal-liquid composite at Level 2 (undrained partially frozen soil) can be obtained as:

\[
F^{II, \text{ud}}_{\text{hom}}[P, Q] = 1 - \frac{(P + S^{II, \text{ud}}_{\text{hom}})^2}{A^{II, \text{ud}}_{\text{hom}}} + \frac{Q^2}{B^{II, \text{ud}}_{\text{hom}}} \leq 0,
\]

where

\[
\begin{align*}
A^{II, \text{ud}}_{\text{hom}} &= a \frac{1 - \eta^{II} - K^{II, \text{ud}} (M^I_{\text{hom}})^2 (1 + (\eta^{II})^2)}{1 - \eta^{II} - K^{II, \text{ud}} (M^I_{\text{hom}})^2}, \\
B^{II, \text{ud}}_{\text{hom}} &= a \frac{G^{II, \text{ud}} (M^I_{\text{hom}})^2 (1 - \eta^{II})}{1 - \eta^{II} - K^{II, \text{ud}} (M^I_{\text{hom}})^2}, \\
S^{II, \text{ud}}_{\text{hom}} &= \tilde{S}^I_{\text{hom}}.
\end{align*}
\]

The above expressions (4.64) are independent of \( \mathcal{T}^{II, \text{ud}}_1 \) and \( \mathcal{T}^{II, \text{ud}}_2 \) since they were canceled out due to the following relations:

\[
\left( \eta^{II} + \mathcal{T}^{II, \text{ud}}_2 (1 - \eta^{II}) \right) \mathcal{T}^{II, \text{ud}}_1 = 1 \quad \text{and} \quad (1 - \eta^{II}) \mathcal{T}^{II, \text{ud}}_2 - \mathcal{T}^{II, \text{ud}}_1 (M^I_{\text{hom}})^2 = 0.
\]
4.4.3 Evaluation of strength prediction on undrained partially frozen soil

The predicted macroscopic strength envelopes of the undrained partially frozen soil, i.e. the solid/crystal-liquid composite at porosity $\phi = 0.5$ and $\phi = 0.99$ are shown, respectively, in Figure 4.11 and Figure 4.12. For each case three different values of liquid saturation ($\chi_L = 0.01, 0.5, 0.99$) are compared and the same material properties as listed in Table 4.1 are used.

For the case of porosity $\phi = 0.5$ in Figure 4.11, the variation of liquid saturation from 0 to 1 results in a smooth transition from fully frozen to unfrozen state which is quite similar as that for the drained case (see Figure 4.7).

For the case of porosity $\phi = 0.99$ in Figure 4.12, the variation of liquid saturation from 0 to 1 results in a smooth transition from pure ice to pure water under drained condition. With increasing liquid saturation, i.e. partial melting of ice into water, the predicted cohesion and friction coefficient of the ice-water mixture computed as:

$$
\begin{align*}
c^{H,\text{ud}}_{\text{hom}} &= \sqrt{B^{H,\text{ud}}_{\text{hom}} \left( \frac{c^{H,\text{ud}}_{\text{hom}}}{A^{H,\text{ud}}_{\text{hom}}} - 1 \right)} \\
M^{H,\text{ud}}_{\text{hom}} &= \sqrt{B^{H,\text{ud}}_{\text{hom}}} / A^{H,\text{ud}}_{\text{hom}},
\end{align*}
$$

both decrease and eventually approach zero, i.e. the strength properties of undrained water (see also Figure 4.13, red curve). However, unlike the drained case (see Figure 4.10a), the predicted friction coefficient $M^{H,\text{ud}}_{\text{hom}}$ plotted in Figure 4.14a has no discontinuity point encountered, which means the predicted strength criterion for undrained partially frozen soil is always hyperbolic (or Drucker-Prager-type) and has hence infinitive hydrostatic strength in compression.

Besides, the influence of porosity $\phi$ on the predicted macroscopic cohesion $c^{H,\text{ud}}_{\text{hom}}$ illustrated in Figure 4.14b for three different cases of liquid saturation is similar to that of the drained case (see Figure 4.14b). However, unlike the drained case (see Figure 4.14a), the evolutions of predicted macroscopic friction coefficient $M^{H,\text{ud}}_{\text{hom}}$ over porosity illustrated in Figure 4.14a undergo always smooth transitions without any discontinuity point, which means the predicted strength criterion for undrained partially frozen soil is always hyperbolic (or Drucker-Prager-type).

![Figure 4.11: Predicted macroscopic strength criteria $F^{II,\text{d}}_{\text{hom}}$ (black line) of partially frozen soil at porosity $\phi = 0.5$ for different liquid saturations ($\chi_L = 0.01, 0.5, 0.99$)](image)
Figure 4.12: Predicted macroscopic strength criteria $F_{\text{hom}}^{\text{II}, \text{ud}}$ (black line) of undrained partially frozen soil at porosity $\phi = 0.99$, i.e. ice-water mixture, for different liquid saturations ($\chi_L = 0.001, 0.5, 0.999$)

Figure 4.13: Influence of liquid saturation $\chi_L$ on the predicted macroscopic a) cohesion $c_{\text{hom}}^{\text{II}, \text{ud}}$, and b) friction coefficient $M_{\text{hom}}^{\text{II}, \text{ud}}$ [MPa] of undrained partially frozen soil at different porosities ($\phi = 0.5, 0.99$)

Figure 4.14: Influence of porosity $\phi$ on the predicted macroscopic a) cohesion $c_{\text{hom}}^{\text{II}, \text{ud}}$, and b) friction coefficient $M_{\text{hom}}^{\text{II}, \text{ud}}$ [MPa] of undrained partially frozen soil at different liquid saturations ($\chi_L = 0.01, 0.5, 0.99$)
4.5 Consideration of pressure melting

4.5.1 Pressure- and temperature-dependent liquid saturation

This section is considered as a complementary step for the strength homogenization of undrained freezing soil, in which the applied hydrostatic compressive stress can be transmitted to the liquid phase. In order to incorporate ice pressure melting into our model, the pressure-dependency of the bulk freezing/melting point $T_f$ of phase transition between crystal ice and liquid water is considered:

$$T_f[P] = T_{f0} \left(1 - \frac{P}{395.2}\right)^{1/9},$$  \hspace{1cm} (4.67)

where $T_{f0} = 273.15$K is the reference temperature associated with the reference (atmospheric) conditions. Recalling further the temperature-dependent and hysteresis-free liquid saturation function (2.33) derived in Subsection 2.3.3 leads to:

$$\chi_L[P, T] = \left(1 + \left(\frac{T_f[P] - T}{\Delta T_{ch}}\right)^{1/m}\right)^{-m},$$  \hspace{1cm} (4.68)

where $T_f$ is the bulk freezing or melting point under given hydrostatic pressure, $\Delta T_{ch}$ and $m$ are two parameters related to the pore size and pore size distribution of the porous material, the obtained macroscopic strength of undrained partially frozen soil from previous step (4.63) is eventually derived as:

$$F_{II, ud}^{H, hom}[P, Q] \xrightarrow{\chi_L[P, T]} F_{II, ud}^{H, hom}[P, Q, \chi_L[P, T]] \xleftarrow{\chi_L[P, T]} F_{II, ud}^{H, hom}[P, Q, T]$$  \hspace{1cm} (4.69)

in which the pressure melting of ice is as well incorporated. The liquid saturation curves over freezing temperature at two different pressures are illustrated in Figure 4.15, indicating that at very high hydrostatic/confining pressure (e.g $P = 100$MPa), the freezing point $T_f$ is depressed.

![Figure 4.15: Liquid saturation curves $\chi_L$ over freezing temperature at two hydrostatic pressures $P = 0, 100$MPa with indicated bulk freezing points (orange lines)](image)
4.5. CONSIDERATION OF PRESSURE MELTING

4.5.2 Evaluation of strength prediction on pressure melting

After consideration of the pressure and temperature dependency on liquid saturation, the new predicted macroscopic strength envelopes (black curves) of the undrained partially frozen soil at porosity $\phi = 0.99$, i.e. the ice-water mixture with pressure melting for three different temperatures ($T = -6, -11, -16^\circ C$) are compared in Figure 4.16 with the experimental data collected by Fish and Zaretsky (1997) from the tests conducted by Gagnon and Gammon (1995), Jones (1982), Rist and Murrell (1994). Therein, the two parameters controlling the shape of the liquid saturation curve are chosen as: $\Delta T_{ch} = 4^\circ C$ and $m = 0.5$, in order to best fit the prediction with the experimental results. Note that for the ice-water mixture these values would be unrealistic, since under atmospheric pressure phase transition occurs at the bulk freezing point $T_{f0}$. However, due to the assumption of matrix-inclusion morphology for this mixture, such values can still be reasonable. At a fixed temperature, the pressure melting of ice is represented with strength weakening at high hydrostatic pressure. When all ice is gradually melted into water with increasing hydrostatic pressure, the shear strength drops finally down to zero. As temperature decreases, the composite is fairly strengthened due to the formation of ice during freezing.

The new predicted macroscopic strength envelopes of the undrained partially frozen soil at porosity $\phi = 0.5$, i.e. the solid/crystal-liquid composite with pressure melting are shown in Figure 4.17 with complex shapes (black curves) for three different temperatures ($T = -6, -11, -16^\circ C$). With decreasing temperature, the strength envelop varies from the unfrozen state (blue line) to fully frozen state (green line). At low hydrostatic pressure, the strength of fully frozen soil dominates. As the pressure increases to certain extent, pressure melting of ice takes place, accompanied by weakening in the soil strength. Gradually when all ice is melted into water, the soil approaches the strength property of its unfrozen state. This prediction shows qualitatively a good agreement with the observed failure envelop of frozen soil according to Andersland and Ladanyi (2004) (see Figure 4.18).

Figure 4.16: Comparison of the predicted pressure-melting-incorporated macroscopic strength criteria (black curves) of undrained partially frozen soil at porosity $\phi = 0.99$, i.e. ice-water mixture at different temperatures ($T = -6, -11, -16^\circ C$) with the experimental data (see Fish & Zaretsky 1997)
Figure 4.17: Predicted pressure-melting-incorporated macroscopic strength criteria $J_{	ext{II}, \text{d}}^{\text{hom}}$ (black curve) of partially frozen soil at porosity $\phi = 0.5$ for different temperatures ($T = -6, -11, -16^\circ C$).

Figure 4.18: Schematic representation of the entire failure envelop for frozen Ottawa sand (taken from Andersland & Ladanyi 2004).
4.6 Validation of strength prediction

Although the predicted macroscopic strength criteria for frozen soils have been qualitatively evaluated and discussed at the end of each homogenization step, a more reliable validation procedure is conducted in this section in order to explore quantitatively the prediction quality of the proposed two-step strength upscaling strategies. To this end, the predicted strength properties for fully and partially frozen fine sand are compared systematically with experimental results, with focus on the investigation of the influences of porosity and liquid saturation degree on the predicted strength criteria.

4.6.1 Fully frozen Ottawa sand: Influence of porosity

In the first validation example, the influence of porosity, i.e. ice volume fraction of a sand-ice composite, on the macroscopic strength properties of the sand-ice composite is investigated. The validation procedure is divided into two steps.

In the first step, the strength properties of the matrix phase composed of sand particles (i.e. the cohesion $c_S$ and the frictional coefficient $M_S$), which are unable to be measured experimentally, are to be determined by means of a back-analysis with given strength properties of drained Ottawa silica sand obtained experimentally by Lee and Seed (1967) as listed in Table 4.2. The fine sand can be considered as a composite of sand particles intermixed with liquid water under drained condition, or equivalently a Drucker-Prager-type solid intermixed with pore voids. Hence, provided the macroscopic strength criterion of a sand-pore composite, i.e. $F^{II}_{hom}[\chi_L = 1] = F^*_{sand}$, the strength criterion of the matrix phase composed of fine sand particles $F_S$ can be determined through a back-analysis for given porosity $\phi = 0.37$ as illustrated in Figure 4.19a.

<table>
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<th>Parameters</th>
<th>Symbols</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
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<tr>
<td>Porosity</td>
<td>$\phi$</td>
<td>0.37</td>
<td>–</td>
</tr>
<tr>
<td>Cohesion</td>
<td>$c_{sand}$</td>
<td>0</td>
<td>MPa</td>
</tr>
<tr>
<td>Friction coefficient</td>
<td>$M^*_sand$</td>
<td>tan[31°] ≈ 0.6</td>
<td>–</td>
</tr>
</tbody>
</table>

In the second step, the obtained strength properties $c_S$ and $M_S$ characterizing the strength criterion $F_S$, together with the strength properties of polycrystalline ice (i.e. $c_C$ and $M_C$ characterizing...
a Drucker-Prager-type $\mathcal{F}_C$ considering only low confining pressure) obtained by Jones (1982) at $-11.8\,^\circ C$ and strain rate $\dot{\varepsilon} = 5.4 \times 10^{-6}\, s^{-1}$, are used as inputs for generating the prediction of the porosity-dependent strength properties of frozen fine sand as illustrated in Figure 4.19b. In Figure 4.20, the predicted porosity-dependent unconfined compressive strength $q_{uc}$ for the sand-ice composite is compared with experimental data obtained by Baker (1979), Goughnour and Andersland (1968) for frozen fine sand at $-12\,^\circ C$ and at strain rate $\dot{\varepsilon} = 2.2 \times 10^{-6}\, s^{-1}$. Herein, the unconfined compressive strength $q_{uc}$ is related to the strength properties $c_{hom}^I$ and $M_{hom}^I$ via

$$q_{uc} = \frac{c_{hom}^I}{1 - M_{hom}^I/\sqrt{3}},$$

(4.70)

in case of a Drucker-Prager-type strength criterion.

![Figure 4.19: a) Step 1: Determination of strength criterion of the matrix of fine sand particles $\mathcal{F}_S$ by means of a back-analysis with given strength properties $\mathcal{F}_s^{\ast}$ on fine sand from Bolton (1986); b) Step 2: Prediction of the macroscopic strength criterion for frozen fine sand $\mathcal{F}_{I\, hom}$ at any given porosity $\phi$.](image)

For the sand-ice composite, as already mentioned in Subsection 4.2.3, the predicted unconfined compressive strength $q_{uc}$ in Figure 4.20 varies between the two limit cases (i.e. pure sand particles and pure ice), when the porosity (i.e. inclusion volume fraction) changes from 0 to 1. It is interesting to note, that the predicted $q_{uc}$ conforms qualitatively well with the experimental results obtained by Baker (1979), Goughnour and Andersland (1968) with focus on the study of the influence of sand concentration on the strength of sand-ice mixture. They both indicate that a higher strength of the sand-ice composite, than that of pure ice, is obtained at high porosity (or low concentration of soil grains), due to the synergistic strengthening effects between the sand particle and crystal ice that prevent the collapse of the structure (namely structural hindrance) as explained in Kaplan (1971). With decreasing porosity (or increasing sand particle concentration), the homogenized cohesion reaches its maximum value due to the establishment of particle contact, and then drops rapidly toward that for pure sand. However, due to the lack of experimental data for ice of $-12\,^\circ C$ at the same strain rate of $2.2 \times 10^{-6}\, s^{-1}$ as for the investigated frozen Ottawa sand reported in Baker (1979), Goughnour and Andersland (1968), an overestimation of the unconfined compres-
4.6. VALIDATION OF STRENGTH PREDICTION

Figure 4.20: Comparison of the predicted unconfined compressive strength $q_{uc}$ over porosity $\phi$ with experimental data of frozen fine sand at $-12^\circ$C and strain rate $\dot{\varepsilon} = 2.2 \times 10^{-6} \text{s}^{-1}$ (BAKER 1979, GOUGHNOUR AND ANDERSLAND 1968)

sive strength $q_{uc}$ in the limit case of pure ice ($\phi = 1$) is observed since the strength properties of ice as listed in Table 4.2 are taken from JONES (1982) at a higher strain rate.

4.6.2 Partially frozen Ottawa sand: Influence of liquid saturation

In the second validation example, the influence of liquid saturation on the macroscopic strength properties of partially frozen sand is investigated, while the porosity is assumed unchanged. To avoid confusion, the same type of Ottawa silica sand as described in Subsection 4.6.1 is considered. In other words, the strength properties of the sand particles (i.e. $c_S$ and $M_S$) determined from the previous validation example can be directly used. Besides, the strength properties of polycrystalline ice (i.e. $c_C$ and $M_C$) obtained by JONES (1982) at $-11.8^\circ$C and strain rate $\dot{\varepsilon} = 5.5 \times 10^{-5} \text{s}^{-1}$ are taken as the other input parameters (see Table 4.3).

Table 4.3: Strength and material properties involved in Figure 4.21

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<th>Parameters</th>
<th>Symbols</th>
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<td>Porosity</td>
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<td>Cohesion</td>
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<td>MPa</td>
</tr>
<tr>
<td></td>
<td>$c_C$</td>
<td>3.1$^a$</td>
<td>MPa</td>
</tr>
<tr>
<td>Friction coefficient</td>
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</tr>
<tr>
<td></td>
<td>$M_C$</td>
<td>0.12$^a$</td>
<td>–</td>
</tr>
<tr>
<td>shear modulus ratio</td>
<td>$r_g$</td>
<td>3.8/37.5</td>
<td>–</td>
</tr>
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</table>

$^a$ Experimental data for polycrystalline ice at $-11.8^\circ$C and strain rate $\dot{\varepsilon} = 5.5 \times 10^{-5} \text{s}^{-1}$ from JONES (1982)
In Figure 4.21, the predicted strength criteria of partially frozen fine sand with porosity $\phi = 0.37$ at different liquid saturation degrees $\chi_L = 1, 0.45, 0.03$ are compared with the experimental results obtained by ALKIRE AND ANDERSLAND (1973) for Ottawa silica sand at $-12^\circ C$ and strain rate $\dot{\varepsilon} = 4.33 \times 10^{-5} \text{ s}^{-1}$. In Table 4.4, the corresponding numerical values of the predicted strength properties correlate well with the experimental results regarding to the overall tendency, despite a slight overestimation in both cohesion and frictional coefficient, resulting possibly from the choice of ice strength properties at a higher strain rate. It is interesting to note, that the frictional coefficient at high ice saturation (i.e. low liquid saturation) is smaller than that for unfrozen sand. This is explained in ALKIRE AND ANDERSLAND (1973) by the role of ice in the composite. Because ice interferes with the inter-granular contact between sand particles, the full development of frictional resistance is hence limited. On the other hand, with increasing ice saturation (i.e. decreasing liquid saturation), the cohesion becomes much larger than that of the unfrozen sand, and approaches eventually a value higher than that of pure ice in the fully frozen state, due to the synergistic strengthening effects between the sand particles and ice as already explained in the previous validation.

![Figure 4.21](image)

**Figure 4.21:** Comparison of the strength criteria of partially frozen fine sand with porosity $\phi = 0.37$ at different liquid saturation degrees ($\chi_L = 1, 0.45, 0.03$): a) obtained by experimental tests on Ottawa silica sand at $-12^\circ C$ and strain rate $\dot{\varepsilon} = 4.33 \times 10^{-5} \text{ s}^{-1}$ (ALKIRE AND ANDERSLAND 1973); and b) predicted based upon the proposed strength upscaling strategy.

**Table 4.4:** Comparison of the strength properties obtained in Figure 4.21

<table>
<thead>
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<th>Parameters</th>
<th>Symbols</th>
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<th>Predicted</th>
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<td>Cohesion</td>
<td>$c_{\chi L=1}$</td>
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<td>0</td>
<td>MPa</td>
</tr>
<tr>
<td></td>
<td>$c_{\chi L=0.45}$</td>
<td>1.26</td>
<td>2.2</td>
<td>MPa</td>
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<td>$c_{\chi L=0.03}$</td>
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<td>3.17</td>
<td>MPa</td>
</tr>
<tr>
<td>Friction coefficient</td>
<td>$M_{\chi L=1}$</td>
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<td>0.6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$M_{\chi L=0.45}$</td>
<td>0.51</td>
<td>0.55</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$M_{\chi L=0.03}$</td>
<td>0.49</td>
<td>0.59</td>
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Chapter 5

Extension to poroplasticity

In this chapter, the picture is extended from poroelasticity to poroplasticity following *COUSsy* (2004). Within the extended framework, a new mechanical constitutive model for freezing soils is proposed by adopting first the Clay and Sand Model (CASM) developed by *Yu* (1998) for the reference unfrozen state, and then extending it to freezing state in a similar way as in the enhanced Barcelona Basic Model (*Nishimura et al.* 2009). In particular, as an important feature of the proposed model, the homogenized strength properties obtained from the previous chapter is incorporated for the constitutive description of the freezing state.

5.1 Poroplastic behavior of soils

In order to address the mechanical behavior of soils involving irreversible strains and porosity changes as well as hardening effects, this chapter begins with the extension of poroelasticity to poroplasticity following *COUSsy* (2004) by applying the general theory of plasticity in soil mechanics. Since the thermal part remains unchanged during this extension, it will be ignored temporally for simplification in this section.

5.1.1 Plastic strain and plastic porosity

In contrast to poroelasticity, in order to capture the irreversible character of plasticity, internal variables – the plastic strains $\varepsilon_{ij}^{pl}$ and the plastic LAGRANGIAN porosity $\phi^{pl}$ – have to be added. Hence, the current strain tensor and the current LAGRANGIAN porosity are redefined as

$$
\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl}, \quad \phi = \phi_0 + \phi^{el} + \phi^{pl},
$$

(5.1)

where $\bullet^{el}$ is the previously considered reversible or elastic part, and $\bullet^{pl}$ represents the new added plastic part. The volumetric plastic strain $\varepsilon^{pl} = \varepsilon_{ii}^{pl}$ undergone by the soil skeleton is contributed by the plastic change in porosity together with the volumetric plastic strain of the solid grains:

$$
\varepsilon^{pl} = \phi^{pl} + (1 - \phi_0) \epsilon_S^{pl}.
$$

(5.2)
However, as in soil mechanics the compressibility of the solid grains is almost negligible compared to that of the soil skeleton, Eq. (5.2) is hence reduced as

\[ \varepsilon_{pl} = \phi_{pl}, \quad \text{since} \quad \varepsilon_S \ll \varepsilon_{pl}. \]  

(5.3)

Consistently the afore-defined Biot coefficient and tangent modulus in Eq. (2.19) are simplified:

\[ b = 1 \quad \text{and} \quad \frac{1}{N} = 0, \quad \text{since} \quad k_S \ll K. \]  

(5.4)

### 5.1.2 Ideal poroplasticity

Including the irreversible deformation and neglecting the thermal part, the positiveness of the skeleton dissipation takes the form:

\[ \sigma_{ij} \, d\varepsilon_{ij} + \bar{p} \, d\phi - d\Psi_S \geq 0 \]  

(5.5)

When considering only reversible or poroelastic evolutions, the values of internal plastic variables \( \varepsilon_{pl}^{ij} \) and \( \phi_{pl} \) remain the same and no dissipation is occurred, which leads to the expression of the free energy of the skeleton \( \Psi_S \) that accounts for the energy which can be eventually recovered in mechanical form:

\[ d\Psi_S = \sigma_{ij} \, d\varepsilon_{ij}^{el} + \bar{p} \, d\phi_{el} \Rightarrow \Psi_S = \Psi_S[\varepsilon_{ij} - \varepsilon_{pl}^{ij}, \phi - \phi_{pl}] \]  

(5.6)

Proceeding similarly as in Subsection 2.3.1 and considering the former assumptions (5.3) and (5.4), the skeleton state equations of linear isotropic poroelastoplasticity are derived as

\[ \sigma_{ij} = (K - \frac{2}{3} \, G) \, (\varepsilon - \varepsilon_{pl}) \, \delta_{ij} + 2 \, G \, (\varepsilon_{ij} - \varepsilon_{pl}^{ij}) - \bar{p} \, \delta_{ij}, \]  

\[ \phi - \phi_{pl} = \phi_0 + (\varepsilon - \varepsilon_{pl}), \quad \text{or} \quad \phi = \phi_0 + \varepsilon. \]  

(5.7)

(5.8)

Substitution of (5.6) into (5.5) gives the dissipated energy related to the skeleton plastic evolution

\[ \delta W_{pl} = \sigma_{ij} \, d\varepsilon_{ij}^{pl} + \bar{p} \, d\phi_{pl} \geq 0. \]  

(5.9)

Adopting the heuristic assumption (5.3) leads to

\[ \delta W_{pl} = \sigma'_{ij} \, d\varepsilon_{ij}^{pl}, \]  

(5.10)

where the effective stress defined as

\[ \sigma'_{ij} = \sigma_{ij} + \bar{p} \, \delta_{ij}, \]  

(5.11)

is found to be the unique force driving the plastic strain, since for granular material such as soil, the deformation is almost completely determined by changes of the concentrated forces in the contact points of the soil particles (see Figure 5.1). Eq. (5.11) is a statement of the principle of effective stress as first introduced by Terzaghi (1943), which forms the most important fundamental concept in soil mechanics. Accordingly, the plasticity in soils is deduced to be caused mainly by the irreversible
5.1. POROPLASTIC BEHAVIOR OF SOILS

Figure 5.1: Illustration of the effective stress concept in an isotropic loading case (taken from VERRUIJT (2012))

sliding of the contacting soil particles forming the solid skeleton. For the description of the material behavior, the effective stress and the plastic strain are decomposed into volumetric and deviatoric parts

\[ p' = -\frac{1}{3} \sigma'_{ii} \quad \text{and} \quad q = \sqrt{\frac{3}{2} (\sigma'_{ij} + p' \delta_{ij}) (\sigma'_{ji} + p' \delta_{ji})} \; ; \quad (5.12) \]

\[ \varepsilon^{pl}_p = -\varepsilon^{pl} \quad \text{and} \quad \varepsilon^{pl}_q = \sqrt{\frac{2}{3} (\varepsilon^{pl}_{ij} + \frac{1}{3} \varepsilon^{pl}_p \delta_{ij}) (\varepsilon^{pl}_{ji} + \frac{1}{3} \varepsilon^{pl}_p \delta_{ji})} \; , \quad (5.13) \]

using the naming convention in WOOD (1990) and defining compressive stresses and contractive strains to be positive. Alternatively, (5.10) can be written as

\[ \delta W^{pl} = p' \, d\varepsilon^{pl}_p + q \, d\varepsilon^{pl}_q \quad (5.14) \]

The yield criterion of the usual isotropic poroplastic materials is generally well captured by involving only these two stress invariants

\[ f[p', q] \leq 0 \; . \quad (5.15) \]

The irreversible mechanism that governs the evolution of plastic flow is defined by the flow rule, which describes the direction of the plastic strains by the derivation of the plastic potential \( g \) with respect to the current stress state:

\[ d\varepsilon^{pl}_p = d\lambda \, \frac{\partial g}{\partial p'} \; ; \quad d\varepsilon^{pl}_q = d\lambda \, \frac{\partial g}{\partial q} \; , \quad (5.16) \]

in case of a non-associated potential, where \( d\lambda \) is the so-called plastic multiplier that scales the intensity of plastic strain increments \( (d\varepsilon^{pl}_p, \; d\varepsilon^{pl}_q) \) and satisfies the KUHN-TUCKER relations

\[ d\lambda \geq 0 \; , \quad f \leq 0 \; , \quad d\lambda \cdot f = 0 \; ; \quad d\lambda \cdot df = 0 \; . \quad (5.17) \]

If \( g = f \) then the flow rule is said to be associated; otherwise, it is called non-associated. Due to the positiveness of the plastic multiplier \( d\lambda \), the positiveness of the plastic work finally requires \( g \) to satisfy:

\[ p' \, \frac{\partial g}{\partial p'} + q \, \frac{\partial g}{\partial q} \geq 0 \; . \quad (5.18) \]
5.1.3 Hardening poroplasticity

Taking into account the hardening effects, the free energy of the skeleton is extended in the form

$$\Psi_S = W_S[\varepsilon_{ij} - \varepsilon_{ij}^{pl}, \phi - \phi^{pl}] + U[h], \quad (5.19)$$

where $W_S$ stands for the aforementioned reversible part of the free energy, and $U$ represents the current trapped free energy resulting from the hardening state variable $h$ (COUSSY 2004). Accordingly, the positiveness of the dissipated energy can be written as

$$\delta W^{pl} - dU = p' d\varepsilon_{p}^{pl} + q d\varepsilon_{q}^{pl} + \zeta dh \geq 0, \quad (5.20)$$

where $\zeta$ is the hardening force consistently associated with the hardening variable $h$ according to the state equation

$$\zeta = -\frac{\partial U}{\partial h}. \quad (5.21)$$

The yield criterion for hardening poroplasticity reads

$$f[p', q, \zeta] \leq 0, \quad (5.22)$$

and the non-associated flow rule is extended to

$$d\varepsilon_{p}^{pl} = d\lambda \frac{\partial g}{\partial p'}; \quad d\varepsilon_{q}^{pl} = d\lambda \frac{\partial g}{\partial q}; \quad dh = d\lambda \frac{\partial g}{\partial \zeta}. \quad (5.23)$$

Reconsideration of the consistency condition of plastic loading leads to

$$df = d\zeta f + \frac{\partial f}{\partial \zeta} d\zeta = 0, \quad \text{with} \quad d\zeta f = \frac{\partial f}{\partial p'} dp' + \frac{\partial f}{\partial q} dq. \quad (5.24)$$

The hardening modulus $H$ is hence derived as

$$H = \frac{d\zeta f}{d\lambda} = -\frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial h} \frac{dh}{d\lambda} = -\frac{\partial f}{\partial \zeta} \frac{\partial^2 U}{\partial \zeta^2} \frac{\partial g}{\partial \zeta}. \quad (5.25)$$

5.2 Proposed mechanical constitutive model for freezing soils

Within the extended poroplastic formulation, a large class of constitutive models have been developed for the description of elastoplastic behavior of soils in the past decades. A detailed review on the usual poroplastic models has been presented in Subsection 1.2.2. In this section, a novel constitutive model for freezing soils is proposed by adopting the CASM (Yu 1998) for the unfrozen state, and the enhanced BBM (GENS 2010, NISHIMURA ET AL. 2009) together with the homogenized strength criteria obtained from the previous chapter for the freezing state (ZHOU AND MESCHKE 2014a). To be distinguished, this newly-proposed elastoplastic mechanical constitutive model with failure criterion upscaled through strength homogenization is named as “extended BBM” in this thesis.
5.2. PROPOSED MECHANICAL CONSTITUTIVE MODEL FOR FREEZING SOILS

5.2.1 Unfrozen state: Clay and Sand Model

Single effective stress

In the unfrozen state, the pore space of soil is fully occupied by liquid water. Therefore, a single stress that governs the skeleton deformation and strength can be used to formulate the stress-strain behavior of the soil skeleton. This is the aforementioned effective stress, which represents the equivalent inter-particle stresses that causing the soil skeleton to deform, and is considered as the foundation stone of modern soil mechanics. Following von Terzaghi (1936) regenerates the aforementioned Terzaghi’s effective stress $\sigma'$ in Eq. (5.11) for the unfrozen state:

$$\sigma'_{ij} = \sigma_{ij} + \bar{p} \delta_{ij}, \quad \text{with} \quad \bar{p} = p_L.$$  \hspace{1cm} (5.26)

As the liquid pressure does not affect the strength and deformation of the soil skeleton, the constitutive model in soil mechanics describes mainly a relationship between the effective stress $\sigma'$ and the total strain $\varepsilon$. Within this effective-stress-based framework, the CASM (Yu 1998) is able to capture the overall behavior of clay and sand observed under both drained and undrained loading conditions. In the drained case, i.e. the fluid is free to flow in or out of the soil sample, a change of total stress does not produce a change in pore pressure, whereas in the undrained case, i.e. the fluid is not permitted to flow in or out, a change of total stress can generate a change in pore pressure.

Yield surface and critical state line

As a most important component of an elastoplastic model, a yield surface marks the boundary of the domain of elastically attainable states of stress. Following the state parameter concept, the yield criterion of the CASM is given based upon the relation between the stress ratio and state parameter (Yu 1998):

$$f[p', q, p'_0] = \left( \frac{q}{M_\theta p'} \right)^{nn} + \frac{1}{\ln[rr]} \ln \left[ \frac{p'}{p'_0} \right] \leq 0.$$  \hspace{1cm} (5.27)

Herein, the effective consolidation pressure $p'_0$ represents the hardening variable of the CASM and indicates the maximum volumetric stress to which the soil has been subjected during the past plastic loadings. The shape parameter $nn$ and the spacing ratio $rr$ are additional parameters for the control of the shape of the yield surface (see Figure 5.2). It is interesting to note that the original Cam-clay yield surface is a special case of the CASM yield locus and can be recovered by setting $nn = 1$ and $rr = e$, while the modified Cam-clay yield surface can also be approximated by choosing $nn = 2$ and $rr = 1.5 - 2.0$.

As reported experimentally in Benz (2006), a circular shape of the failure surface in the $\Pi$-plane would greatly overestimate the strengths of both clay and sand, whereas a Mohr-Coulomb criterion would always underestimates the strengths (see Figure 5.3b). Therefore, the lode dependency is considered in the CASM in order to obtain an appropriate criterion lying somewhere between these two criteria. For that, the slope of the Critical State Line (CSL) $M_\theta$ is modified as function of the
LODE’s angle $\vartheta$: 

$$M_\vartheta = M_{\text{max}} \left( \frac{2 \alpha^4}{1 + \alpha^4 + (1 - \alpha^4) \sin[3 \vartheta]} \right)^{1/4}, \quad \text{with} \quad \alpha = \frac{3 - \sin[\Phi_{\text{CSL}}]}{3 + \sin[\Phi_{\text{CSL}}]} \quad (5.28)$$

where $M_{\text{max}}$ is the slope of the CSL under triaxial compression, and $\Phi_{\text{CSL}}$ is the friction angle at the critical state. Besides, the LODE’s angle $\vartheta$ is defined as

$$\vartheta = -\frac{1}{3} \arcsin \left( \frac{27}{2} J_3 \right). \quad (5.29)$$

with $J_3 = \det[\sigma'_{ij} + p' \delta_{ij}]$ being the third principal invariant of the deviatoric part of the effective stress $\sigma'$.

**Elastic behavior**

Within the yield surface, changes of stress are accompanied by purely elastic or recoverable deformations. The incremental relationship between the strain and stress of the CASM can be summarized as

$$\begin{bmatrix} d\varepsilon_{el_p} \\ d\varepsilon_{el_q} \end{bmatrix} = \begin{bmatrix} 1/K & 0 \\ 0 & 1/(3 G) \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}. \quad (5.30)$$

Herein, the effective bulk modulus $K$ is assumed to be proportional to the mean effective pressure $p'$ as in the standard CCM:

$$K = \begin{cases} v \frac{p' \kappa}{\kappa}, & \text{if} \quad p' > 1 \text{kPa} \\ v/\kappa, & \text{if} \quad p' \leq 1 \text{kPa} \end{cases} \quad (5.31)$$

with $v = 1 + e = 1/(1 - \phi)$ being the specific volume, and $e$ being the void ratio. The shear modulus $G$ is then defined as:

$$G = \frac{3 (1 - 2 \nu)}{2 (1 + \nu)} K. \quad (5.32)$$
with \( \nu \) being the Poisson coefficient. As shown in Figure 5.4, the volumetric stress response of the CASM along the Unloading-ReLoading Line (URL) is a straight line in the \( \ln[p'] - \varepsilon_p \) space. Alternatively the elastic stress-strain response of the CASM can be expressed in index notation as:

\[
d\sigma'_{ij} = C^e_{ijkl} \, d\varepsilon^e_{kl}, \quad \text{with} \quad C^e_{ijkl} = K \delta_{ij} \delta_{kl} + 2 G (I_{ijkl} - \frac{1}{3} \delta_{ij} \delta_{kl}),
\]

(5.33)

where \( C^e_{ijkl} \) is the effective elastic tangent modulus, and \( I \) is the symmetric projection tensor computed as: \( I_{ijkl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \).

**Figure 5.3:** Yield surface of the CASM: a) in principal stress space and in the \( p' - q \) plane (taken from NAGEL AND MESCHKE (2010)); b) in the \( \Pi \)-plane compared to the Mohr-Coulomb failure criterion and experimental results (BENZ 2006)

**Figure 5.4:** Volumetric effective stress against volumetric strain for isotropic compression of the CASM
Hardening law

The hardening law describes the dependency of the size of the yield surface on the plastic strains. The CASM is an isotropic volumetric plastic strain hardening model in which the dependency of the hardening on the deviatoric plastic strains \( \varepsilon_q^{pl} \) is not observed. Instead, the size of the yield surface is controlled by the a history variable, i.e. the preconsolidation pressure \( p'^0 \), which depends only on the plastic volumetric strain \( \varepsilon_p^{pl} \). As a result, the incremental form of the hardening law is expressed as:

\[
dp' = \frac{v}{\lambda - \kappa} \ dp' \ dz_p^{pl},
\]  

(5.34)

where \( \lambda \) is the slope of the isotropic Normal Consolidation Line (iso-NCL) in Figure 5.4.

Plastic potential and flow rule

Following Yu (1998), Rowe’s stress-dilatancy relation (ROWE 1962) is integrated into the CASM to give the following expression for the plastic potential:

\[
g[p', q] = 3 M_0 \ln \left( \frac{p'}{C} \right) + (3 + 2 M_0) \ln \left( \frac{2 q}{p'} + 3 \right) - (3 - M_0) \ln \left( 3 - \frac{q}{p'} \right),
\]  

(5.35)

where \( C \) is the size parameter that can be determined easily from the condition that the plastic potential surface passes through the current state, i.e. by solving Eq. (5.35) for the given stress state \((p', q)\). Note that since the plastic potential is not identical to the yield surface, the adopted plastic flow rule in the CASM is non-associated and is formulated as

\[
\frac{\partial g}{\partial p'} \ dp' + \frac{\partial g}{\partial q} \ dq \geq 0.
\]  

(5.36)

Similar as the standard Cam-clay model, the CASM exhibits plastic contraction (\( dz_p^{pl} > 0 \)) for a shear load at high stress ratio \( p'/q > M_0 \) (wet clay or loose sand), and plastic dilatancy (\( dz_p^{pl} < 0 \)) for \( p'/q < M_0 \) (dry clay or dense sand), whereas for \( p'/q = M_0 \) the plastic evolution occurs at constant volume (see Figure 5.5). This is a unique behavior for both clay and sand.

Figure 5.5: Yield surface and corresponding plastic potential with direction of plastic strain increments
**Model parameters**

In the unfrozen state, the adopted CASM requires in total seven material constants, whose common values are listed in Table 5.1.

<table>
<thead>
<tr>
<th>Properties</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu$</td>
<td>$0.15 - 0.35$</td>
<td>–</td>
</tr>
<tr>
<td>Swelling index (slope of URL)</td>
<td>$\kappa$</td>
<td>$0.01 - 0.06$ (clay)</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\sim 0.005$ (sand)</td>
<td>–</td>
</tr>
<tr>
<td>Compression index (slope of iso-NCL)</td>
<td>$\lambda$</td>
<td>$0.1 - 0.2$ (clay)</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.01 - 0.05$ (sand)</td>
<td>–</td>
</tr>
<tr>
<td>Friction coefficient (slope of CSL)</td>
<td>$M_{\text{max}}$</td>
<td>$0.8 - 1.0$ (clay)</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1.1 - 1.4$ (sand)</td>
<td>–</td>
</tr>
<tr>
<td>Friction angle at critical state</td>
<td>$\Phi_{\text{CSL}}$</td>
<td>$&lt; 48.59^\circ$</td>
<td>$\circ$</td>
</tr>
<tr>
<td>Shape parameter</td>
<td>$nn$</td>
<td>$1.0 - 5.0$</td>
<td>–</td>
</tr>
<tr>
<td>Spacing ratio</td>
<td>$rr$</td>
<td>$1.5 - 3.0$ (clay)</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td></td>
<td>much higher (sand)</td>
<td>–</td>
</tr>
</tbody>
</table>

5.2.2 Freezing state: Extended BBM based upon strength upscaling

**Two stress variables**

In the freezing state, the pore space is partially occupied by liquid water and partially by crystal ice. Due to the presence of the ice phase that is capable of carrying not only isotropic (both tensile and compressive) but also shear stress, the use of effective stress as a single stress variable, i.e. considering the ice phase with only isotropic loading capacity similar to the water phase, is certainly not an appropriate way to describe the full plastic material response of freezing soils, such as the relatively high short-term shear strength observed for ice-rich loose frozen soils as mentioned in NISHIMURA ET AL. (2009). Therefore, in order to capture the experimentally observed mechanical behavior, the BISHOP’s effective stress concept for three-phase continua (BISHOP AND BLIGHT 1963) expressed as

$$
\sigma'_{ij} = \sigma_{ij} + \bar{p} \delta_{ij}, \quad \text{with} \quad \bar{p} = \chi p_L + (1 - \chi) p_C, \quad (5.37)
$$

if inferring the BBM for unsaturated soils (ALONSO ET AL. 1990), has to be combined with another variable to take into account the mechanical contribution of the ice phase, for instance, on the development of the yield surface and the failure criterion, i.e. CSL. Besides, in Eq. (5.37), the parameter $\chi$ is called BISHOP parameter which is a material function depending on the liquid saturation degree $\chi_L$ but hard to measure for practical problems. Hence, it is commonly assumed that it equals to liquid saturation degree $\chi = \chi_L$ in geotechnical engineering, which provides a good estimate for a wide range of soils (FREDLUND AND RAHARDJO 1993). Consequently, the former definition of the effective stress in Eq. (5.11) can be again retrieved.
In the enhanced BBM (GENS 2010, NISHIMURA ET AL. 2009), a new stress variable representing the (capillary) suction, i.e. the crystal-liquid pressure difference, is used by noticing the close analogy between the physics of frozen-saturated and unfrozen-unsaturated soils. Alternatively, in the proposed extended BBM in this work, noticing the linear relationship between the suction and the temperature as derived in Eq. (2.27), the temperature \( T \) during freezing \( (T < T_f) \) is considered as the second independent variable, instead of the suction.

It is also important to note that in the enhanced BBM, instead of using the BISHOP’s effective stress as defined in Eq. (5.37), a net stress representing the external confinement is introduced:

\[
\sigma'_{ij} = \sigma_{ij} + \bar{p} \delta_{ij}, \quad \text{with} \quad \bar{p} = \begin{cases} p_L, & \text{if } T \geq T_f; \\ p_C, & \text{if } T < T_f. \end{cases}
\]  
(5.38)

The above definition of the net stress in Eq. (5.38) emphasizes the mechanical role of the ice phase as crystals in the pore space and hence disregards that of the liquid water, whereas in Eq. (5.37) the ice phase is assumed as a pore fluid like the gas phase by inferring the concept of the BBM for unsaturated soils. From a physical point of view, the net stress defined in Eq. (5.38) would better reflect the crystal feature of the ice phase and hence is adopted in the proposed extended BBM.

**Failure criterion based upon strength upscaling**

The adoption of the second stress variable in the constitutive model leads to a development of the failure criterion, i.e. the CSL if a linear criterion is assumed, during freezing. In the enhanced BBM, the dependence of shear strength of frozen soil on porosity and temperature are captured by means of an empirical function based upon experimental tests on frozen Ottawa sand at \(-7^\circ C\) and strain rate of \(4.4 \times 10^{-4} \text{s}^{-1}\). However, evidently, the strength properties of freezing soils are significantly influenced by the states of temperature and applied strain rate as well as the soil types, which indicates, that such an empirical function alone represents only the strength properties of a certain type of soil in one specific state. In order to capture the full strength development for different soils in various states, a great many test results on different frozen soils at different temperatures and strain rates have to be considered, which are certainly too costly and time-consuming to be realistic.

In contrast with the above-mentioned phenomenological approach, a better and efficient way to characterize the failure criterion development of soils during freezing would be established on the basis of the knowledge of their microstructure, i.e. the volume fractions and strength properties of constituent phases. To this end, the macroscopic strength properties of drained partially frozen soil (i.e. the homogenized cohesion \(c_{\text{II, hom}}^{d}\) and friction coefficient \(M_{\text{II, hom}}^{d}\)) obtained through a two-step strength upscaling from the previous chapter in Section 4.3 are incorporated into the proposed extended BBM. As illustrated in Figure 5.6, each freezing state is related to a certain temperature during freezing \( (T < T_f) \), resulting from the assumption of a purely temperature-dependent liquid saturation function \(\chi_L[T]\). As inputs for the strength upscaling model, the porosity and liquid saturation are directly provided based on the current state, whereas the strength properties of the soil particles \( (c_S \text{ and } M_S) \), and the crystal ice \( (c_C \text{ and } M_C) \) have to be determined based upon the experimental results, respectively, on different types of drained soils and on pure ice at different
5.2. PROPOSED MECHANICAL CONSTITUTIVE MODEL FOR FREEZING SOILS

temperature and strain rate (see Section 4.6). Nevertheless, such data is relatively easier to access
than that of frozen soils in view of the accomplished experimental contributions.

---

\[ q = c_T + M_{\theta_T} p', \]

(5.39)

where \( c_T \) is the intersection of the CSL with \( q \)-axis dependent on the freezing temperature \( T_f - T \), and \( M_{\theta_T} \) is the slope of the CSL dependent on the LODE’s angle \( \vartheta \) and the freezing temperature. Applying the homogenization results obtained from last Chapter in Section 4.3, \( c_T \) and \( M_{\theta_T} \) are defined as:

\[ c_T = c_{H,d}^{\text{hom}}, \quad \text{and} \quad M_{\theta_T} = M_{H,d}^{\text{hom}} \left( \frac{2 \alpha^4}{1 + \alpha^4 + (1 - \alpha^4) \sin[3 \vartheta]} \right)^{1/4}, \]

(5.40)

where \( c_{H,d}^{\text{hom}} \) and \( M_{H,d}^{\text{hom}} \) are the homogenized cohesion and friction coefficient for drained partially frozen soils, both of which are porosity- and temperature-dependent.

As shown in Figure 5.7b, at certain freezing temperature, e.g. when \( T = -6 ^\circ C \), the variation of porosity from 0 to 1 results in a transition of pure soil grains (dry sand or clay) to ice-water mixture. The homogenized cohesion \( c_{\text{hom}}^{H,d} \) obtains a little higher value at high porosity or relatively low concentration of soil grains, than that of ice-water mixture, due to structural hindrance (see
definition in Section 4.2 of Chapter 4). With decreasing porosity or increasing grain concentration, $c_{\text{hom}}^{II,d}$ reaches its maximum value due to the establishment of particle contact, and then drops rapidly toward that for pure soil grains. At extremely low temperature, as concluded in Section 4.2, this predicted porosity-dependent cohesion conforms qualitatively well with the study conducted by Goughnour and Andersland (1968) on the influence of sand concentration on the strength of sand-ice mixture, as well as the test results reported by Kaplan (1971). The homogenized friction coefficient $M_{\text{hom}}^{II,d}$ decreases monotonically with increasing porosity until it reaches the critical porosity $\phi_{\text{crit}}$, where a transition of the homogenized strength envelop from hyperbolic to elliptic occurs due to void weakening effect for large volume fraction of liquid water $\phi_L = \phi \chi_L [T]$. However, soils conditioned in situations beyond such high critical porosity are quite rare in the application of AGF, and hence are disregarded in what follows. On the other hand, at certain fixed porosity, the homogenized cohesion $c_{\text{hom}}^{II,d}$ increases monotonically with decreasing temperature due to ice strengthening, whereas the homogenized friction coefficient $M_{\text{hom}}^{II,d}$ varies only slightly during freezing if only soils whose porosities are smaller than the above-mentioned critical value are considered.

\[ f(p', q, T) = \left( \frac{q}{M_{\phi_T} (p' + S_T)} \right)^n + \frac{1}{\ln|\tau|} \ln \left[ \frac{p' + S_T}{p_{0_T} + S_T} \right] \leq 0, \quad (5.41) \]
Table 5.2: Material parameters for Figure 5.7

<table>
<thead>
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<th>Properties</th>
<th>Symbol</th>
<th>Value</th>
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<td>Characteristic cooling</td>
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<td>Pore size distribution parameter</td>
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<td>Parameter controlling stiffness growth rate</td>
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<tr>
<td>Preconsolidation:</td>
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<td>MPa</td>
</tr>
</tbody>
</table>

with

$$S_T = \frac{c_T}{M_\phi T} \quad \text{and} \quad p'_0(T) = \frac{p_0}{p_{atm}} \left( \frac{\lambda_T}{\kappa - \lambda_T} \right),$$

(5.42)

representing, respectively, the homogenized hydrostatic tensile strength and the effective consolidation pressure at temperature $T$. Herein, $p'_0$ and $p_{atm}$ are, respectively, the preconsolidation pressure in the unfrozen state, and the reference atmospheric pressure ($p_{atm} = 0.1$ MPa), while $\lambda_T$ is the temperature-dependent slope of iso-NCL in the freezing state expressed as:

$$\lambda_T = \lambda \left( (1 - r) \exp\left[ -\beta S_f(T_f - T) \right] + r \right),$$

(5.43)

following similar definition as presented in the BBM (ALONSO ET AL. 1990) and the enhanced BBM (NISHIMURA ET AL. 2009). Eq. (5.43) describes the increase in the compressive stiffness of freezing soil with decreasing temperature, where $r$ is a constant related to the maximum compressive stiffness, and $\beta$ is a parameter controls the increasing rate of the compressive stiffness with freezing temperature.

The corresponding yield surfaces in the $p' - q$ plane and their projections in the $p' - T$ plane are illustrated in Figure 5.7a-c for two different porosities ($\phi = 0.3$ and 0.6) at three different temperatures ($T = 0, -1, -6^\circ$C). With decreasing temperature, the yield surfaces expand in both compressive (right) and tensile (left) directions due to, respectively, enhanced particle interlocking and ice strengthening. For high-porosity (ice-rich) soils, this freezing-induced expansion of the yield surface towards the left direction becomes more significant in Figure 5.7c than that for low-porosity (ice-poor) soils in Figure 5.7a.

**Plastic potential**

Based upon the above-proposed CSL, the plastic potential reads:

$$g[p', q, T] = 3 M_\phi T \ln \left[ \frac{p' + S_T}{C_T} \right] + (3 + 2 M_\phi T) \ln \left[ \frac{2 q}{p' + S_T} + 3 \right]$$

$$- (3 - M_\phi T) \ln \left[ 3 - \frac{q}{p' + S_T} \right],$$

(5.44)

where $C_T$ is the size parameter relative to the given state ($p', q, T$). Accordingly, the non-associated flow rule remains the same form as in Eq. (5.36).
Hardening law

For freezing state at temperature $T$, the incremental form of the hardening law is reformulated as:

$$\frac{d\sigma}{\sigma_0'} = \frac{v}{\lambda_T} \frac{\sigma_0'}{\kappa} \frac{d\varepsilon}{\varepsilon_0'},$$

(5.45)

where $\lambda_T$ is the slope of the isotropic Normal Consolidation Line (iso-NCL) at temperature $T$.

Elastic behavior

In the freezing state, the elastic deformation can occur in both tensile and compressive directions due to the presence of ice phase. However, the expression for the effective bulk modulus $K$ remains unchanged as for unfrozen state in Eq. (5.31), since for both tensile and very small compressive load (i.e. $p' \leq 1$ kPa), $K$ is assumed to be independent of the volumetric stress $p'$ by defining $K = v/\kappa$. Besides, it is worth to note that, similar as the enhanced BBM (NISHIMURA ET AL. 2009), this proposed model deactivates some features of the original BBM (SHENG ET AL. 2004), such as the dependency of the URL slope $\kappa$ on temperature (or suction in the BBM), the temperature-induced elastic volumetric changes.

Model parameters

Taking into account the strength development of soils during freezing, in the proposed extended BBM, the model parameters would be increased to a number of 15 as illustrated in Figure 5.6. Their common values are listed in Table 5.3.

5.3 Model verification

The above presented mechanical constitutive model of freezing soils and its implementation are evaluated by means of four benchmark examples. The first three of these benchmarks - an isotropic compression test, and two triaxial compression tests under drained and undrained conditions - are used to verify the implementation of the CASM in the unfrozen state, whereas in the last series of tests - drained freezing followed by drained triaxial compression has been performed for the evaluation of the extended BBM in different freezing states. In addition to the verification of the algorithmic formulation, the presented simulation results are also used to discuss some of the major characteristics of soil behavior obtained by application of the implemented constitutive model.

5.3.1 Unfrozen test: drained isotropic compression

In the first validation test, a purely volumetric stress state, i.e. isotropic compression, is applied to a soil sample at fixed temperature $T = 1$ °C under drained condition by fixing liquid water pressure $p_L$ to 0 Pa. All involved material parameters for this isotropic compression test are listed in Table 5.4. The simulation is conducted as follows: The soil example is first compressed isotropically from 4 kPa at point A to 100 kPa at point C with load increment $\Delta p' = 0.2$ kPa, unloaded back to 4 kPa at point D, and then reloaded to 200 kPa at point F as shown in Figure 5.8a. Along the loading path
Table 5.3: Parameters for the proposed extended BBM

<table>
<thead>
<tr>
<th>Properties</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Parameters inherited from the CASM</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu$</td>
<td>0.15 – 0.35</td>
<td>–</td>
</tr>
<tr>
<td>Swelling index (slope of URL)</td>
<td>$\kappa$</td>
<td>0.01 – 0.06</td>
<td>(clay)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\sim$ 0.005</td>
<td>(sand)</td>
</tr>
<tr>
<td>Compression index (slope of iso-NCL)</td>
<td>$\lambda$</td>
<td>0.1 – 0.2</td>
<td>(clay)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.01 – 0.05</td>
<td>(sand)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.1 – 1.4</td>
<td>(sand)</td>
</tr>
<tr>
<td>Friction angle at critical state</td>
<td>$\Phi_{\text{CSL}}$</td>
<td>$&lt; 48.59^\circ$</td>
<td>–</td>
</tr>
<tr>
<td>Shape parameter</td>
<td>$\eta$</td>
<td>1.0 – 5.0</td>
<td>–</td>
</tr>
<tr>
<td>Spacing ratio</td>
<td>$\upsilon$</td>
<td>1.5 – 3.0</td>
<td>(clay)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>much higher</td>
<td>(sand)</td>
</tr>
<tr>
<td><strong>Parameters inherited from the BBM</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Parameter defining max. compressive stiffness</td>
<td>$r$</td>
<td>0.5 – 1</td>
<td>–</td>
</tr>
<tr>
<td>Parameter controlling stiffness growth rate</td>
<td>$\beta$</td>
<td>0 – 0.5</td>
<td>–</td>
</tr>
<tr>
<td><strong>Parameters for strength upscaling</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Porosity</td>
<td>$\phi$</td>
<td>0 – 1</td>
<td>–</td>
</tr>
<tr>
<td>Liquid saturation</td>
<td>$\chi_L$</td>
<td>0 – 1</td>
<td>–</td>
</tr>
<tr>
<td>Cohesion</td>
<td>$c_S$</td>
<td>0</td>
<td>(sand)</td>
</tr>
<tr>
<td></td>
<td>$c_C$</td>
<td>$&gt; 0$</td>
<td>–</td>
</tr>
<tr>
<td>Friction coefficient</td>
<td>$M_S$</td>
<td>0.65 – 0.9</td>
<td>(sand)</td>
</tr>
<tr>
<td></td>
<td>$M_C$</td>
<td>0.01 – 0.4</td>
<td>–</td>
</tr>
<tr>
<td>shear modulus ratio</td>
<td>$r_g$</td>
<td>0.1 – 0.2</td>
<td>–</td>
</tr>
</tbody>
</table>

AC, the soil sample first undergoes an elastic deformation with a slope of the graph in Figure 5.8b equal to the swelling index $\kappa$. After passing the preconsolidation pressure $p'_0 = 50$ kPa at point B, plastic deformation occurs with volumetric hardening following the path in Figure 5.8b with slope equal to the compression index $\lambda$, resulting in an increase of the preconsolidation pressure to 100 kPa. Whereas the unloading process is associated with an elastic load path of the same slope $\kappa$, the reloading process leads first to an elastic compression from point D to E, and then a plastic deformation with volumetric hardening once the previously reached volumetric pressure $p'_0 = 100$ kPa is exceeded at point E. Eventually, the preconsolidation pressure reaches 200 kPa at point F.

5.3.2 Unfrozen tests: drained triaxial compression at different OCRs

In this validation test, two soil samples at fixed temperature $T = 1^\circ$C with preconsolidation pressure $p'_0 = 150$ kPa are assumed initially at an isotropic compressive stress state $p'_i = 60$ kPa at point O as shown in Figure 5.9. Both samples have the same material properties as listed in Table 5.5. In order to study the drained triaxial compressive behavior of soils at difference overconsolidation ratios (OCRs = $p'_0/p'$), one of samples is loaded isotropically under drained condition to 100 kPa (OCR = 1.5) at point B and prepared as a lightly overconsolidated soil, whereas the other is unloaded.
Table 5.4: Model validation with regard to isotropic compression test in the unfrozen state: Material parameters

<table>
<thead>
<tr>
<th>Properties</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Porosity</td>
<td>$\phi_0$</td>
<td>0.3</td>
<td>-</td>
</tr>
<tr>
<td>Characteristic cooling</td>
<td>$\Delta T_{ch}$</td>
<td>2</td>
<td>°C</td>
</tr>
<tr>
<td>Pore size distribution parameter</td>
<td>$m$</td>
<td>0.7</td>
<td>-</td>
</tr>
<tr>
<td>Swelling index</td>
<td>$\kappa$</td>
<td>0.07</td>
<td>-</td>
</tr>
<tr>
<td>Compression index</td>
<td>$\lambda$</td>
<td>0.02</td>
<td>-</td>
</tr>
<tr>
<td>Preconsolidation</td>
<td>$p'_0$</td>
<td>50</td>
<td>kPa</td>
</tr>
</tbody>
</table>

Figure 5.8: Model validation with regard to isotropic compression test at $T = 1$°C: a) Evolution of volumetric stress (blue) and preconsolidation pressure (red), and b) volumetric strain $\varepsilon_p$ over volumetric stress $\ln[p]$ to 20 kPa ($OCR = 7.5$) at point D and prepared as a heavily overconsolidated soil.

For the lightly overconsolidated sample, the isotropic loading OA and the subsequent drained triaxial compression AB are purely elastic processes, since the corresponding stress paths are inside the initial yield surface (see Figure 5.9b) and lying on the unloading-reloading curve (see Figure 5.9d). After passing the initial yield point B, plastic strains start to develop and plastic hardening occurs with volumetric contraction (negative volumetric strain) until the critical state (point C) is reached, in which deviatoric strain increases without any further change in stress. During the drained triaxial compression, the lightly overconsolidated shows a steady increase in deviatoric stress $q$ with increasing deviatoric strain $\varepsilon_q$ (see Figure 5.9a) and a steady decrease in volumetric strain $\varepsilon_p$ (Figure 5.9c).

For the heavily overconsolidated sample, the isotropic unloading OD and the subsequent drained triaxial compression DE are again purely elastic processes. However, differing from the behavior described above, in order to reach the critical state, plastic softening EF occurs instead with volu-
metric expansion. Besides, during the drained triaxial compression, the heavily overconsolidated soil shows a rise in deviatoric stress $q$ to a peak followed by a drop with continued shearing, and an initial decrease in volume followed by a volumetric expansion.

The predicted features of the drained triaxial compression response of lightly and heavily overconsolidated soils as described above are important behavior observed in real soils (see Wood (1990) for more details), except that real soil has unsurprisingly smoothed off the sharp peak at point E predicted by the model in Figure 5.9a for heavily overconsolidated case.

### 5.3.3 Unfrozen tests: undrained triaxial compression at different OCRs

In this validation test, in order to study the undrained triaxial compressive behavior of soils at difference OCRs, the same lightly and heavily overconsolidated soil samples as described in the previous test (see Subsection 5.3.2) are used. However, as this triaxial compression test is conducted under undrained condition, liquid water in the pore space is not permitted to flow into or out of the samples and thus the volume occupied by the soil structure is assumed to be constant (namely isochoric) as indicated in Figure 5.10d. In order to cope, positive or negative liquid pressure is hence generated. According to the definition in Eq. (5.26), the liquid pressure at each stage can be deduced from the horizontal difference between the total stress path (TSP) and the effective stress path (ESP) in the $p' - q$ plane as shown in Figure 5.10.

For the lightly overconsolidated sample, the first phase of the undrained triaxial compression test is purely elastic AB with no change in effective stress, that is, only associated with elastic deviatoric strain (see Figure 5.10b) and with changes in liquid pressure equal to the change in total stress (see Figure 5.10b). When yielding begins at point B, plastic deviatoric strain develops along BC with slight hardening, as the yield surface expands when the ESP turns left with decrease in effective stress towards the critical state C (see Figure 5.10b). Meanwhile, the liquid pressure remains positive with a steady increase.

For the heavily overconsolidated sample, the first loading phase DE is again only associated with elastic deviatoric strain and with changes in liquid pressure equal to the change in total stress. However, when yielding begins at point E, plastic deviatoric strain increases along EF with plastic

---

**Table 5.5: Model validation with regard to triaxial compression tests in the unfrozen state: Material parameters**

<table>
<thead>
<tr>
<th>Properties</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Porosity</td>
<td>$\phi_0$</td>
<td>0.35</td>
<td>–</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu$</td>
<td>0.3</td>
<td>–</td>
</tr>
<tr>
<td>Shape parameter</td>
<td>$nn$</td>
<td>1.5</td>
<td>–</td>
</tr>
<tr>
<td>Spacing ratio</td>
<td>$rr$</td>
<td>2</td>
<td>–</td>
</tr>
<tr>
<td>Slope of CSL</td>
<td>$M$</td>
<td>1.0</td>
<td>–</td>
</tr>
<tr>
<td>Swelling index</td>
<td>$\kappa$</td>
<td>0.07</td>
<td>–</td>
</tr>
<tr>
<td>Compression index</td>
<td>$\lambda$</td>
<td>0.02</td>
<td>–</td>
</tr>
<tr>
<td>Preconsolidation</td>
<td>$p'_0$</td>
<td>150</td>
<td>kPa</td>
</tr>
<tr>
<td>Prestress</td>
<td>$p'_i$</td>
<td>60</td>
<td>kPa</td>
</tr>
</tbody>
</table>
softening, as the yield surface shrinks when the ESP turns right with increase in effective stress towards the critical state F. Meanwhile, the liquid pressure reduces from its maximum value obtained at the initial yield point E, and eventually becomes negative when approaching the critical state F.

The predicted features of the undrained triaxial compression response of lightly and heavily overconsolidated soils as described above are similar to the observed behavior in real soils (see Wood (1990) for more details).
Figure 5.10: Model validation with regard to undrained triaxial compression tests at $T = 1{}^\circ\text{C}$ on lightly (blue curve) and heavily (green curve) overconsolidated soils: a) deviatoric stress $q$ over deviatoric strain $\varepsilon_q$; b) effective and total stress paths in $p' - q$ plane with yield surface and CSL; c) volumetric strain $\varepsilon_p$ over deviatoric strain $\varepsilon_q$; and b) volumetric strain $\varepsilon_p$ over volumetric effective stress $p'$

5.3.4 Freezing tests: drained freezing and subsequent triaxial compression

After validating the soil response in the unfrozen state ($T = 1{}^\circ\text{C}$) by means of isotropic and triaxial compression tests, a series of triaxial compression tests is conducted to investigate the soil behavior in the freezing state. To this end, three soil samples initially at temperature $T = 1{}^\circ\text{C}$ with preconsolidation pressure $p'_0 = 1\text{ MPa}$ are assumed at an isotropic compressive stress state $p'_i = 0.5\text{ MPa (OCR = 2)}$ at point $O$ as shown in Figure 5.11. All samples have the same material properties as listed in Table 5.6. It is important to note, that the volumetric effective stress $p'$ can become zero or negative during freezing, representing a state in which soil particles are allowed to separate – no
contacting force, as the ice phase alone is capable of carrying the applied load. In order to cope
with the elastic deformation in both tensile and compressive directions, the effective bulk modulus
\( K \) (5.31) is proposed to be independent of the volumetric effective stress \( p' \): \( K = v/\kappa \), in these
tests with the assumption of an extremely small value of the swelling index \( \kappa \). However, in real soils
the elastic volumetric behavior is neither linear, as just assumed, nor logarithmically linear, as in the
CASM adopted for the unfrozen state.

Table 5.6: Model validation with regard to triaxial compression test in the freezing state: Material parameters

<table>
<thead>
<tr>
<th>Properties</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Porosity</td>
<td>( \phi_0 )</td>
<td>0.35</td>
<td>–</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>( \nu )</td>
<td>0.3</td>
<td>–</td>
</tr>
<tr>
<td>Shape parameter</td>
<td>( \eta )</td>
<td>2</td>
<td>–</td>
</tr>
<tr>
<td>Spacing ratio</td>
<td>( r_r )</td>
<td>2</td>
<td>–</td>
</tr>
<tr>
<td>Swelling index</td>
<td>( \kappa )</td>
<td>0.0002</td>
<td>–</td>
</tr>
<tr>
<td>Compression index</td>
<td>( \lambda )</td>
<td>0.05</td>
<td>–</td>
</tr>
<tr>
<td>Parameter defining max. compressive stiffness</td>
<td>( r )</td>
<td>0.8</td>
<td>–</td>
</tr>
<tr>
<td>Parameter controlling stiffness growth rate</td>
<td>( \beta )</td>
<td>0.2</td>
<td>–</td>
</tr>
<tr>
<td>Preconsolidation</td>
<td>( p'_0 )</td>
<td>1</td>
<td>MPa</td>
</tr>
<tr>
<td>Prestress</td>
<td>( p'_i )</td>
<td>0.5</td>
<td>MPa</td>
</tr>
</tbody>
</table>

Each simulation test consists of two phases: drained freezing and subsequent drained trial com-
pression. In the first phase, the temperature of the three samples is reduced to \(-1, -3, -10 \degree C\),
respectively, under drained conditions. Taking the soil sample being cooled to \(-3 \degree C\) as an example,
during this freezing process from point O to A, the initial yield surface of \( 1 \degree C \) expands in both
compressive (right) and tensile (left) directions to that of \(-3 \degree C\) (see Figure 5.11b, red curve) due to,
respectively, particle interlocking and ice strengthening as explained in Subsection 5.2.2. Volume
expansion occurs \((\varepsilon_p > 0)\) along the URL from point O to A with the volumetric effective stress
\( p' \) dropping from positive to negative. In the second phase, the temperature is kept unchanged,
whereas a gradually-increased vertical compressive load is applied at the top surface of the sample.
Within the expanded yield surface, the soil sample undergoes first a purely elastic deformation AB,
accompanied by volumetric contraction along URL with partially recovering volumetric effective
stress. Once the yield point B is passed, plastic hardening occurs with more significant volumetric
contraction until reaching the critical state at point C, where shear deformation continues infinitely
without any further change in stresses or volume.
Figure 5.11: Model validation with regard to drained triaxial compression tests for different freezing states 
($T = −1, −3, −10 °C$): a) deviatoric stress $q$ over deviatoric strain $\varepsilon_q$; b) stress paths in $p'−q$ plane 
with yield surface and CSL; c) liquid pressure $p_L$ over deviatoric strain $\varepsilon_q$; and d) volumetric strain $\varepsilon_p$ over 
volumetric effective stress $p'$; e) temperature $T$ over deviatoric strain $\varepsilon_q$; f) temperature $T$ over volumetric 
effective stress $p'$.
Chapter 6

Simulation examples

Having established a comprehensive freezing soil model within the framework of thermo-poro-plasticity, in this chapter, the performance and applicability of the proposed model are demonstrated by two simulation examples. In the first simulation a one-dimensional soil freezing test is conducted with focus on the exploration of the development of ground freezing with frost accretion, while in the second simulation a case study on artificial ground freezing (AGF) processes during tunneling is re-analyzed for extended investigation on ground heave prediction, influence of seepage flow on the formation of desired frost arch, and optimization of freeze pipe arrangement.

6.1 One-dimensional soil freezing

6.1.1 Model setup

In the first simulation example, a one-dimensional soil freezing test is conducted in order to investigate the development of the freezing process under one-dimensional heat flux and the accompanying frost heave resulting from the micro-cryo-suction mechanism (see Chapter 2). To this end, a fully saturated cuboidal soil sample with a height of 1 m and a cross-section of $0.2 \times 0.2 \text{ m}^2$ at an initial temperature $T_i = 2 \degree \text{C}$, is subjected to freezing in the axial direction from the top downwards (see Figure 6.1a). At the top surface, a convection-type boundary condition is assigned, which relates the normal heat flux $q^*$ to the difference of the current surface temperature $T$ and the surrounding air temperature $T_{\infty} = -8 \degree \text{C}$:

$$q^* = h_{\text{conv}} A (T - T_{\infty}), \quad (6.1)$$

where $h_{\text{conv}} = 50 \text{ W/(m}^2\text{K)}$ is the convection heat transfer coefficient, and $A$ is the top surface area through which convection takes place. The temperature of the bottom surface is fixed at 2 \degree C, and all the side surfaces are adiabatic. Water can only be supplied from the bottom by considering drained bottom surface with a reference pressure $p_L = 0 \text{ Pa}$ and all rest surfaces to be imperme-
able. Mechanically only the bottom surface is fixed. The involved material parameters are listed in Table 6.1.

\[
T_\infty = -8 ^\circ C
\]
\[
T_i = 2 ^\circ C
\]
\[
0.2 \, m
\]
\[
1 \, m
\]

\textbf{Figure 6.1:} 1D soil freezing: a) Model setup; b) Development of frozen fringe during freezing

During one-dimensional soil freezing there are three distinct zones existing in the soil layer: the passive frozen zone, the active unfrozen zone and the frozen fringe. The frozen fringe, defined first by Miller (1972), is the zone between the growing ice lens which is not incorporated in this paper as explained in Subsection 2.1.1, and the frost front where the warmest pore ice exists (see Figure 6.1b).

\subsection*{6.1.2 Development of ground freezing with frost accretion}

Figure 6.2 shows the obtained simulation results of this one-dimensional freezing analysis at selected time instants: 1 hour, 10, 50, and 200 days. Based upon temperature variation on time, the simulation procedure can be divided into transient phase and steady phase.

Firstly, in the transient phase, freezing starts at the top surface and moves gradually downwards until the temperature distribution is stabilized at the end of transient freezing. Meanwhile, the frost front propagates from the top surface towards its steady state location. On one hand, as ice has smaller density than water, volume expansion occurs during freezing accompanied by an increase in porosity and an excess of liquid water expelled from the freezing site. On the other hand, due to the latent heat released during phase transition, liquid pressure at the frozen fringe drops quickly from zero to negative, resulting in the so-called cryo-suction (see Subsection 2.1.1) that drives the water in the unfrozen zone towards the freezing site. Based upon the above-mentioned two effects, in the early stage of freezing for instance after 1 hour, liquid water tries to flow in both directions:
a downward flux expelling water from the freezing site resulting from the liquid-crystal density difference, and an upward flux drawing water to the freezing site due to the cryo-suction. However, since the former becomes more dominant than the latter during the freezing process, only upward flux is observed as a result of the counteraction.

After 10 days, the thermal steady state has been reached and the stabilized temperature profile along the depth shows a nonlinear distribution dependent on the ice saturation profile as the thermal conductivity of ice is larger than that of water. A frozen fringe is developed at a depth between 0.3 and 0.86 m. As more and more water from the drained bottom surface is drawn upwards, frost accretes in the frozen fringe with a continuous increase in porosity. However, as time goes on, this increase is slowed down since the water pressure profile flattens. Moreover, this accumulation is limited at the upper side of the frozen fringe because of the reduction in permeability with increasing ice saturation. The predicted frost accretion can be considered as an initialization for the formation of an ice lens, and is hence an essential ingredient of the frost heave phenomenon observed for
frost-susceptible soils such as silty clay (TABER 1929, ZHU ET AL. 2000). Nevertheless, due to the assumptions of a purely temperature-dependent liquid saturation and a zero ice flow in this thesis, the formation of ice lens or alternating layers of ice lenses, recognized as another important ingredient of frost heave, can not be reproduced.

![Figure 6.2: 1D soil freezing: Evolution of different properties along the depth at selected time instants](image)

### 6.1.3 Comparative study of four tests

In order to obtain an intuitive understanding of the causes or preconditions for the observed frost accretion in the frozen fringe, a series of four comparison tests with different material parameters or boundary condition is conducted. Test 0 is the original simulation test as described in the previous
The obtained deformation and ice saturation profiles of the soil samples in the four tests after 200 days are compared in Figure 6.4 and the corresponding distributions of the porosity, water pressure and water flow along the depth are presented in Figure 6.3. In Test 0, as explained already, two counteractive water fluxes exist in the frozen fringe with the upward suction prevailing. With drained bottom surface, the water can be supplied continuously from external to the freezing site. In Test 1, two counteractive fluxes exist as well in the frozen fringe. However, with a undrained bottom surface, the uptake of water is restricted resulting in a limited increase in porosity or frost accretion because there is no external supply available. In Test 2, only the upward suction exists due to the fictitious setting of equal density between water and ice ($\rho_L = \rho_C$). Surprisingly, a significant frost accretion is still observed with merely a minor difference with Test 0. On the contrary, by assigning a zero latent heat ($S_f = 0$), the suction no longer exists during the freezing process and the water is expelled from the freezing site downwards as a result of the liquid-crystal density difference. Moreover, since the bottom is drained, water flows out from the sample and hence no frost accretion occurs.

Based upon the aforementioned comparison results, it can be concluded that the latent heat released during the phase transition from water to ice is an essential prerequisite for the frost accretion in the frozen fringe, whereas the volume expansion resulting from the liquid-crystal density difference turns out to be relatively insignificant. Besides, drained bottom surface in the unfrozen zone acts as a continuous external water supply for the ongoing frost accretion. In an undrained system, frost accretion occurs as well however with less heave.

6.2 Numerical simulation of Artificial Ground Freezing

6.2.1 Model setup

In the second simulation example, the validated computational model for soil freezing is applied to the numerical simulation of AGF for the temporary ground support during tunneling. To this end, a case study of AGF performed by ZIEGLER AND BAIER (2007) is re-analyzed numerically by the proposed model, and only monotonic freezing is investigated due to the assumption of non-hysteretic phase transition (see Subsection 2.3.3). To obtain frozen soil with high bearing capacity and impermeability, freeze pipes with a fixed temperature of $-35 \, ^\circ\text{C}$ and a diameter of 0.2 m are installed horizontally in a soil layer initially at an in situ ground temperature of $13.45 \, ^\circ\text{C}$, which rests on a rigid impervious base (depth: 10 m, width: 38 m) (see Figure 6.5). The inner surfaces, where the freeze pipes are located, remain immobile and undrained. According to the geotechnical requirements, eventually a circular frozen arch with a thickness of $\approx 1.5 \, \text{m}$ is desired. The involved material parameters are listed in Table 6.2, where the values of $\Delta T_{ch}$ and $m$ are adapted according to the empirical unfrozen water content function given by ZIEGLER ET AL. (2009). Besides, as the
Figure 6.3: 1D soil freezing: Comparison of the distribution of selected properties along the depth after 200 days for the four tests.

Figure 6.4: 1D soil freezing: Comparison of the deformed soil samples with ice saturation profiles (Deformation ×10) after 200 days for the four tests.

The proposed mechanical constitutive model for the soil skeleton is based upon empirical evidence, the thermal contraction effect of solid gains is hence ignored by assigning $\alpha_S = 0$. 
6.2. NUMERICAL SIMULATION OF ARTIFICIAL GROUND FREEZING

6.2.2 Absence of horizontal seepage flow

First, the numerical simulation is carried out by considering that initially there is no horizontal seepage flow across the soil layer. This simulation contains 2040 hexahedra elements and in total 4344 nodes. Instead of applying a horizontal pressure gradient, a prescribed liquid pressure \( p_L = 0 \) Pa is applied at both left and right boundaries, resulting in a drained soil layer. As in the proposed model the film water exists even at extremely low temperature, a criterion of ice saturation \( \chi_{C_{\text{crit}}} = 0.95 \), equivalent to temperature \( T_{\text{crit}} = -3.59^\circ C \), is used to terminate the simulation. In other words, once all points along the frost wall that defines the boundaries of the frost arch obtain an ice saturation above its critical value 95%, the desired frozen arch is said to be achieved.

The resulting ice saturation profiles, i.e. the formation of the frozen wall, and deformations of the cross-section at different time instants (Day 1 and Day 9) are presented in Figure 6.6. In the very early stages, the soil freezes around the freeze pipes forming frozen soil columns. As time goes on, the columns start touching each other and eventually a circular frozen arch is generated with the desired thickness after 9 days.

The temporal evolutions of the vertical disarrangement at three selected points (A, B & C) on the ground surface and the ice saturation at four selected point (E, F, G & H) on the desired frost wall are plotted in Figure 6.7. Differing from the strain analysis test in Subsection 2.5.3, the settlement observed in the early stage result from thermal contraction disappears in this simulation due to the assumption of \( \alpha_S = 0 \). In the beginning, the ground surface starts to heave first at the center (point A) and then propagates to the side (point C). With more and more liquid water transforming into crystal ice accompanied with volume expansion, ground heave at all three observation points (A, B and C) increases to a large extent. As analyzed in the one-dimensional freezing test (see Section 6.1), frost accretion arising from the micro-cryo-suction effect plays also a crucial role in the ground heave, even within such a relatively short period. In Figure 6.7b, the ice saturation evolution curves conform that after 9 days the frost arch has been achieved. Besides, the points on the inner wall (F & H) reach first the critical value than the outer (E & G) due to the arch geometry.


Table 6.2: Numerical simulation of AGF: Material parameters

<table>
<thead>
<tr>
<th>Properties</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Porosity</td>
<td>$\phi_0$</td>
<td>0.41</td>
<td>–</td>
</tr>
<tr>
<td>Bulk freezing temperature</td>
<td>$T_f$</td>
<td>273</td>
<td>K</td>
</tr>
<tr>
<td>Freezing entropy</td>
<td>$S_f$</td>
<td>1.2</td>
<td>MPa</td>
</tr>
<tr>
<td>Characteristic cooling</td>
<td>$\Delta T_{ch}$</td>
<td>1</td>
<td>°C</td>
</tr>
<tr>
<td>Pore size distribution parameter</td>
<td>$m$</td>
<td>0.7</td>
<td>–</td>
</tr>
<tr>
<td>Initial mass density</td>
<td>$\rho_S$</td>
<td>2650</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td></td>
<td>$\rho_L$</td>
<td>1000</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td></td>
<td>$\rho_C$</td>
<td>917</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>Bulk modulus</td>
<td>$k_S$</td>
<td>50</td>
<td>GPa</td>
</tr>
<tr>
<td></td>
<td>$k_L$</td>
<td>2.2</td>
<td>GPa</td>
</tr>
<tr>
<td></td>
<td>$k_C$</td>
<td>8.6</td>
<td>GPa</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>$g_S$</td>
<td>37.5</td>
<td>GPa</td>
</tr>
<tr>
<td></td>
<td>$g_L$</td>
<td>1</td>
<td>MPa</td>
</tr>
<tr>
<td></td>
<td>$g_C$</td>
<td>3.4</td>
<td>GPa</td>
</tr>
<tr>
<td>Heat capacity</td>
<td>$c_S$</td>
<td>900</td>
<td>J/(kg K)</td>
</tr>
<tr>
<td></td>
<td>$c_L$</td>
<td>4180</td>
<td>J/(kg K)</td>
</tr>
<tr>
<td></td>
<td>$c_C$</td>
<td>2100</td>
<td>J/(kg K)</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>$\lambda_S$</td>
<td>1.8</td>
<td>W/(m K)</td>
</tr>
<tr>
<td></td>
<td>$\lambda_L$</td>
<td>0.56</td>
<td>W/(m K)</td>
</tr>
<tr>
<td></td>
<td>$\lambda_C$</td>
<td>2.24</td>
<td>W/(m K)</td>
</tr>
<tr>
<td>Thermal dilation</td>
<td>$\alpha_S$</td>
<td>0</td>
<td>K$^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\alpha_L$</td>
<td>$95.43 \times 10^{-6}$</td>
<td>K$^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\alpha_C$</td>
<td>$51.67 \times 10^{-6}$</td>
<td>K$^{-1}$</td>
</tr>
<tr>
<td>Intrinsic permeability</td>
<td>$\kappa_0$</td>
<td>$4.4 \times 10^{-11}$</td>
<td>m$^2$</td>
</tr>
<tr>
<td>Reference viscosity (at $T_f$)</td>
<td>$\eta_0$</td>
<td>$1.79 \times 10^{-3}$</td>
<td>Pa s</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu$</td>
<td>0.3</td>
<td>–</td>
</tr>
<tr>
<td>Shape parameter</td>
<td>$nn$</td>
<td>2</td>
<td>–</td>
</tr>
<tr>
<td>Spacing ratio</td>
<td>$rr$</td>
<td>2</td>
<td>–</td>
</tr>
<tr>
<td>Swelling index</td>
<td>$\kappa$</td>
<td>0.0002</td>
<td>–</td>
</tr>
<tr>
<td>Compression index</td>
<td>$\lambda$</td>
<td>0.02</td>
<td>–</td>
</tr>
<tr>
<td>Parameter defining max. compressive stiffness</td>
<td>$r$</td>
<td>0.8</td>
<td>–</td>
</tr>
<tr>
<td>Parameter controlling stiffness growth rate</td>
<td>$\beta$</td>
<td>0.2</td>
<td>–</td>
</tr>
<tr>
<td>Preconsolidation</td>
<td>$p'_0$</td>
<td>1</td>
<td>MPa</td>
</tr>
<tr>
<td>Prestress</td>
<td>$p'_i$</td>
<td>0.5</td>
<td>MPa</td>
</tr>
</tbody>
</table>
Figure 6.6: Numerical simulation of AGF: Ice saturation profiles over the deformed cross-section after 1 and 9 days.

Figure 6.7: Numerical simulation of AGF: Temporal evolutions of a) vertical disarrangement $u_Z$ of point A, B & C at the ground surface, and b) ice saturation $\chi_C$ of point E, F, G & H at the desired frost wall.
6.2.3 Presence of horizontal seepage flow

In this subsection, the influence of horizontal seepage flow on the formation of a frozen arch wall is investigated numerically by means of the proposed computational model. Horizontal seepage flow is simulated by applying a constant positive liquid pressure on the left boundary and zero pressure on the right boundary. This simulation contains 1968 hexahedra elements and in total 4192 nodes. According to the model geometry (Figure 6.5) and the material parameters (Table 6.2) employed in this case study, a horizontal seepage flow of 1.0 m/d is equivalent to assigning \( p_L = 17.9028 \) kPa on the left boundary. Figure 6.8 compares the simulation results for different values of seepage flows at different time instants. Since flowing water provides a continuous source of heat, the freezing process is considerably affected and the formation of a closed frozen arch around the tunnel profile is delayed under a relatively high seepage flow. To generate an arch of frozen ground with a thickness of \( \sim 1.5 \) m to support the excavated tunnel cross section, the required duration for the four investigated scenarios for the seepage flow \( v_{IL} = 0, 0.5 \) and 1.0 m/d are computed as 9, 24 and 140 days, respectively, whereas for the case of \( v_{IL} = 1.5 \) m/d the desired frozen arch can not be achieved even after 200 days. This is attributed to the fact, that a state of thermal equilibrium has been reached in this system and hence the soil stops freezing.

The distribution of Darcy’s velocity for the case of seepage flow \( v_{IL} = 1.0 \) m/d at different time instants is shown in Figure 6.9. Initially, water flow is almost homogeneously passing through the entire cross section. As the frozen soil columns grow (e.g. after 2 days), the flow velocity increases considerably within the gaps between the freezing pipes, which inhibits the formation of a closed arch by delaying a connection between adjacent frozen piles. After 10 days, only two open gaps are remaining. Through these gaps a relatively large amount of liquid water is flowing outwards, driven by the excess of liquid pressure caused by volume expansion during freezing (see Figure 6.8, middle column). Once the frozen arch is closed (e.g. after 20 days), there is no more water flow within the interior of the frozen arch and, consequently, the impact of seepage flow on the temperature evolution is significantly reduced. From then on, the frozen arch grows much faster towards the inwards direction than outwards. This explains also how the groundwater is controlled by means of ground freezing. The desired thickness of \( \sim 1.5 \) m is reached, however, after 140 days, due to the fact that the remaining formation of the frost body at the downstream side takes about half of the total time.

6.2.4 Optimization of freeze pipe arrangement

As can be concluded from the previous subsection, a large seepage flow, e.g. \( v_{IL} = 1.5 \) m/d, will considerably delay or even prevent the formation of a closed frozen arch around the tunnel profile during freezing process. Following SCHULTZ ET AL. (2008), the critical seepage velocity is approximately 2 m/d for brine freezing and 4 \( \sim 6 \) m/d for liquid nitrogen (LN2) freezing (see definitions of these two freezing techniques in Chapter 1). When a critical seepage value is passed, a state of thermal equilibrium is reached in which the soil will stop freezing and the closure of the frozen arch cannot be achieved. It is evident, that an even distribution of the freeze pipes is not the optimal solution in case of presence of seepage flow. Therefore, an optimization algorithm has been used to
6.2. NUMERICAL SIMULATION OF ARTIFICIAL GROUND FREEZING

<table>
<thead>
<tr>
<th>( v_L )</th>
<th>Day 2</th>
<th>Day 5</th>
<th>Day 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 m/d</td>
<td>![Image]</td>
<td>![Image]</td>
<td>![Image]</td>
</tr>
<tr>
<td>0.5 m/d</td>
<td>![Image]</td>
<td>![Image]</td>
<td>![Image]</td>
</tr>
<tr>
<td>1.0 m/d</td>
<td>![Image]</td>
<td>![Image]</td>
<td>![Image]</td>
</tr>
<tr>
<td>1.5 m/d</td>
<td>![Image]</td>
<td>![Image]</td>
<td>![Image]</td>
</tr>
</tbody>
</table>

**Figure 6.8:** Numerical simulation of AGF: Influence of seepage flow on formation of the frozen arch

**Figure 6.9:** Numerical simulation of AGF: Distribution of Darcy's velocity over the undeformed cross-section at different time instants for seepage flow \( v_L = 1.0 \) m/d

improve the freezing efficiency by searching for the optimal location of freeze pipes depending on the direction and magnitude of the present groundwater flow.

To this end, an optimized model based upon the FE model of freezing soils developed in this thesis, has been established by means of a cooperation with my colleague M.Sc. Ahmed Marwan during my research. It adopts the Ant Colony Optimization method – a probabilistic technique which aims to search the optimal path in a graph by mimicking the behavior of the ants seeking a path between their colony and a source of food. The optimization algorithm proposes two distribution functions (one for the radial distance between freeze pipes and the central point, and the other for the spacing between the freeze pipes), and then searches for optimum parameterization for the proposed functions. At the current stage, the number of freeze pipes remains the same as in the original simulation model which has been presented in the previous section, while the seepage flow is set to 1.0 m/d as an initial condition for the optimization algorithm.
In order to investigate the improvement on freezing efficiency, a comparative study is conducted among the original simulation model (denoted as “Model Original”), our developed optimized model (denoted as “Model Opt-SD”), and two optimized models (denoted as ”Model Opt-Baier-1” and “Model Opt-Baier-2”) proposed in the literature of AULBACH ET AL. (2009), ZIEGLER ET AL. (2009). The arrangements of freeze pipes in these four models are compared in Figure 6.10. In the Model Opt-Baier-1, two additional freeze pipes are placed evenly and still radially among the original freeze pipes at the upstream side (i.e. the left half part), while in the Model Opt-Baier-2, two additional
freeze pipes are located in the upstream direction ahead of the original freeze pipes in order to cause pre-cooling. By contrast, in the Model Opt-SD, the number of the freeze pipes remains the same as in the Model Original. However, the location of the freeze pipes is highly adapted to the present seepage flow (see Figure 6.10d) as a result of the proposed optimization scheme.

Figure 6.11 compares the formation of the frozen arch between the original model and the three optimized models at seepage flow $v_L = 1.0$ m/d after 5 and 10 days, and at the day when the frost body has reached the required thickness. All of the three optimized models reduce significantly the total freezing time, indicating that a flow-adapted arrangement of the freeze pipes can result in a better freezing efficiency. Especially in the Model Opt-Baier-2 and Model Opt-SD, a sharp improvement of the freezing efficiency has been observed by dropping the freezing time from 140 days for Model Original to, respectively, 14 and 17 days. Without increasing the number of freeze pipes, i.e., no additional cost on cooling energy, the performance demonstrated by our Model Opt-SD is already proud.

The required freezing time for obtaining the desired thickness of the four models is compared at different seepage flows ($v_L = 0.5$, 1.0 and 1.5 m/d) in Table 6.3. The Model Opt-Baier-2 maintains the lead in reducing the freezing time for all cases of seepage flows, especially for the case of $v_L = 1.5$ m/d, where the formation of required frost arch is completed after only 22 days, whereas in the Model Opt-Baier-1 and Opt-SD, the closure can still not be achieved.

<table>
<thead>
<tr>
<th>Model</th>
<th>Day 5</th>
<th>Day 10</th>
<th>Day of completion of closed arch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original (140 d)</td>
<td><img src="image1" alt="Image" /></td>
<td><img src="image2" alt="Image" /></td>
<td><img src="image3" alt="Image" /></td>
</tr>
<tr>
<td>Opt-Baier-1 (30 d)</td>
<td><img src="image4" alt="Image" /></td>
<td><img src="image5" alt="Image" /></td>
<td><img src="image6" alt="Image" /></td>
</tr>
<tr>
<td>Opt-Baier-2 (14 d)</td>
<td><img src="image7" alt="Image" /></td>
<td><img src="image8" alt="Image" /></td>
<td><img src="image9" alt="Image" /></td>
</tr>
<tr>
<td>Opt-SD (17 d)</td>
<td><img src="image10" alt="Image" /></td>
<td><img src="image11" alt="Image" /></td>
<td><img src="image12" alt="Image" /></td>
</tr>
</tbody>
</table>

**Figure 6.11:** Numerical simulation of AGF: Comparison of the formation of the frozen arch for the four models at seepage flow $v_L = 1$ m/d
Table 6.3: Numerical simulation of AGF: Comparison of the required freezing time for obtaining frozen arch with desired thickness of 1.5 m for the four models at different seepage flows

<table>
<thead>
<tr>
<th>Model</th>
<th>$v_L = 0.5$ m/d</th>
<th>$v_L = 1.0$ m/d</th>
<th>$v_L = 1.5$ m/d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td>24 d</td>
<td>140 d</td>
<td>&gt; 200 d</td>
</tr>
<tr>
<td>Opt-Baier-1</td>
<td>16 d</td>
<td>30 d</td>
<td>&gt; 200 d</td>
</tr>
<tr>
<td>Opt-Baier-2</td>
<td>13 d</td>
<td>14 d</td>
<td>22 d</td>
</tr>
<tr>
<td>Opt-SD</td>
<td>16 d</td>
<td>17 d</td>
<td>&gt; 200 d</td>
</tr>
</tbody>
</table>
7.1 Summary and concluding remarks

7.1.1 Three-phase FE model within thermoporoelasticity

A three-phase FE model for freezing soils consisting of solid grains, pore water and pore ice was presented for the description of the coupled THM behavior of water-saturated soft soils subjected to frost action. A basic model was first formulated within the frameworks of thermo-poroelasticity (COUSSEY 2005) and premelting dynamics (WETTLAUFER AND WORSTER 2006) (see also MESCHKE ET AL. (2011), ZHOU AND MESCHKE (2011a,b, 2012, 2013)). By choosing solid displacement $u$, the liquid water pressure $p_L$ and the temperature $T$ of the mixture as primary field variables, three corresponding physical laws were set up as the governing balance equations for the FE formulation as an IBVP, within which the interaction terms represent the related THM couplings. It was demonstrated by three validation tests in Section 2.5, that the basic model is capable to reproduce the fundamental interacting mechanisms occurring during the freezing process, such as the phase transition and associated latent heat effect, the liquid transport within the pores and associated mechanical deformation of the solid skeleton. Additionally, the contribution of the micro-cryo-suction mechanism to the frost heave phenomenon was included, whereas most recent studies consider only the contribution due to the volume expansion of water transforming into ice. However, with an assumption of nondissipative phase transition, the capillary hysteresis observed during freeze-thaw cycles was not incorporated.

7.1.2 Strength homogenization of matrix-inclusion composites

For the prediction of the temperature- and porosity-dependent strength properties of freezing soils, a novel multi-scale strength homogenization procedure has been proposed. As a reference scheme for each step, a robust strength homogenization methodology for a two-phase nonlinear matrix-inclusion composite was first presented by extending the LCC approach proposed by ORTEGA ET AL. (2011) within the yield design framework SALENCEN (1990). The presented reference
scheme allows for predictions of the homogenized strength properties of composite materials for different assumptions concerning the individual strength properties of the cohesive-frictional matrix material and the inclusions. The matrix material may be either represented by a Drucker-Prager-type (hyperbolic) or an elliptical strength criterion, which are typically adopted for geological and cement-based materials. As the inclusion phase either air voids, representing a composite with air pores, pores filled with a fluid, representing a water-saturated porous composite, cohesive-frictional materials of the same type as the matrix phase, but with different strength properties, representing a composite reinforced by aggregates, are considered. In addition, as a limit case, also composites reinforced by rigid inclusions are considered. The LCC methodology predicts the macroscopic strength of a nonlinear composite by means of determining the dissipation potential from an optimally chosen linear thermo-elastic comparison composite with a similar underlying microstructure, which is used to evaluate limit states of macroscopic stresses in the framework of the yield design theory. An efficient algorithm was proposed to generate the macroscopic dissipation potential, which allows to establish the macroscopic strength functional in terms of stress invariants in an explicit format for all investigated cases, including complex combinations of matrix and inclusions characterized by Drucker-Prager-type as well by elliptic strength criteria. The generated strength criteria are also of a hyperbolic or elliptical format. This analytical format allows a straightforward generalization of the proposed procedure to a multi-scale strength homogenization strategy for complex hierarchical composites.

This reference strength homogenization scheme has been applied to different classes of composites (see also ZHOU AND MESCHKE (2014b)). The numerical applications have shown, that the generated strength envelopes of the composite material correctly predicts the limit cases with regards to volume fractions of the inclusions at $n = 0$ and $n = 1$. A Drucker-Prager-type matrix with air voids yields a strength criterion which shows a transition from a hyperbolic to an elliptic characteristics with increasing volume fraction, finally collapsing to a point as $n \to 1$. A matrix with an elliptic strength criterion with air voids leads to an elliptical macroscopic strength envelope which shrinks with increasing void ratio and also degenerates to a point as $n \to 1$. Composites with a matrix material and solid inclusions both being characterized by either a hyperbolic (Drucker-Prager-type) or a elliptic criterion (i.e. a criterion, which predicts strength limit in both hydrostatic tension and compression) show, at a macroscopic level, a smooth transition between the strength envelopes of both phases. The model predictions for selected classes of composites were compared with results from a 1-D thought model (DORMIEUX ET AL. 2006), experiments on sand specimens with gravel inclusions (PEDRO 2004) and on leached and unleached cement specimens (HEUKAMP ET AL. 2003, LEMARCHAND ET AL. 2002), respectively, as well as with an alternative strength homogenization method – the modified secant model (MAGHOUSS ET AL. 2009). A very good correlation between model predictions for the homogenized strengths characteristics for unleached and leached mortar was obtained. However, a considerable overprediction of the friction coefficient were found for the upscaling of composites reinforced with rigid inclusions. It should be noted, however, that in contrast to the modified secant model by MAGHOUSS ET AL. (2009), the proposed model predicts the correct asymptotic limit for the friction coefficient ($M \to \infty$) as $n \to 1$. This overprediction of the friction coefficient is attributed partially to the fact, that, as a consequence of
the principle of maximum plastic dissipation inherent to the upper bound theorem within the yield design theory, the LCC-based upscaling method is not able to capture non-associated plastic flow observed for many geological materials. In addition, it is concluded from the comparison with the modified secant model proposed by MAGHOUS ET AL. (2009), that the differences may also arise from the linear averaging rule applied for the definition of the effective strain rate in the proposed model.

7.1.3 Strength upscaling for freezing soils

On the basis of the validated reference scheme, in the proposed two-step strength upscaling strategies, the solid particle phase and the crystal ice phase are assumed to be characterized by two different Drucker-Prager strength criteria and the liquid water phase has, either zero strength capacity under drained condition, or zero shear strength capacity under undrained condition. Through the first homogenization step, the influence of porosity on the predicted strength properties for fully frozen soil has shown qualitatively a good correlation with the test results reported by KAPLAR (1971), where a cohesion larger than that of pure ice was obtained due to the synergistic strengthening between the solid grains and ice matrix, and gradual establishment of particle contact. In the second homogenization step for drained freezing soil, the predicted strength criterion would undergo a transition from hyperbolic (or Drucker-Prager-type) to elliptic for large liquid volume fraction, due to the void weakening, whereas for undrained freezing soil, it remains always hyperbolic (or Drucker-Prager-type) without hydrostatic strength limit in compression. In the complementary step for the undrained case, consideration of the ice pressure melting effect leads to good concordance between the strength prediction of the ice-water mixture with appropriate-chosen $\Delta T_{ch}$ and $m$, and the experimental data on ice strength collected by FISH AND ZARETSKY (1997).

Through model evaluation at the end of each homogenization step, the resultant strength prediction for the failure criterion of ice-water mixture is in excellent concordance with the experimental data provided two appropriate-chosen parameters that characterize the pore size and pore size distribution, while the prediction for undrained partially frozen soil shows qualitatively a good agreement with observed phenomena, such as strengthening of soil during freezing and weakening of soil during pressure melting. Moreover, the proposed strength upscaling strategies are validated further by comparing the predicted strength properties systematically with experimental results on fully and partially frozen Ottawa sand presented in ALKIRE AND ANDERSLAND (1973), BAKER (1979), GOUGHNOUR AND ANDERSLAND (1968), through which the dependencies of macroscopic strength properties of freezing soils on porosity and liquid saturation degree have been nicely represented, such as synergistic strengthening effects between the sand particle and crystal ice observed in sand-ice composites, and the frictional resistance depression due to the presence of ice by interfering with the inter-granular contact between sand particles.

7.1.4 Critical state elastoplastic mechanical constitutive model

Subsequently, the framework was extended from poroelasticity to poroplasticity and accordingly a new critical state elastoplastic mechanical constitutive model has been proposed, by adopting the
CASM for the reference unfrozen state, and successfully extending it to freezing states based on the concepts of the enhanced BBM. As a highlight of this thesis, the temperature- and porosity-dependent strength properties of freezing soils predicted through the two-step strength upscaling procedure were incorporated into the constitutive model. The verification of the algorithmic formulation and evaluation of the proposed constitutive model were conducted by four benchmark examples in both unfrozen and freezing states. By means of an isotropic compression test and two triaxial compression tests on unfrozen soils at different OCRs, important features observed in real soils (Wood 1990) have been predicted, such as plastic hardening followed by volumetric contraction for lightly overconsolidated soil, plastic softening followed by volumetric contraction for heavily overconsolidated soil, and infinite plastic shearing occurring without any volume change in the critical state. In the freezing tests, the dependence of shear strength on temperature has been demonstrated, together with the expansion of yield surfaces in both compressive and tensile directions due to, respectively, particle interlocking and ice strengthening.

7.1.5 Simulation examples

In the one-dimensional soil freezing test dealing with the development of ground freezing, the cryosuction process, identified as the driving force of the frost heave phenomenon (Taber 1929, Zhu et al. 2000), has been successfully represented with a continuous but slowed-down ice accumulation in the frozen fringe. This frost accretion initiates an ice lens formation, however, with assumptions of a purely temperature-dependent liquid saturation and a zero ice flow, the formation of ice lens or alternating layers of ice lenses can not be reproduced in the present work. Through a comparison study on four tests, it has been concluded that the latent heat released during freezing process is an essential prerequisite for the observed frost accretion, whereas the volume expansion, resulting from the density difference between water and ice, turns out to be relatively insignificant. Besides, drainage in the unfrozen zone provides a continuous external water supply for the ongoing frost accretion, whereas in an undrained system frost accretion occurs as well however with less heave.

In the application example concerned with AGF to provide temporary excavation support, it was shown, that the volume expansion due to phase transition and the cryo-suction effect contribute both to the frost heave observed on the ground surface. Besides, the seepage flow has a considerable influence on the formation of a closed and stable arch of frozen ground. The comparative study on the optimization of the freeze pipe arrangements indicates that a flow-adapted arrangement of the freeze pipes can result in a significant shorter time for the freezing process. For a project-specific freezing system provided the actual groundwater conditions at the tunnel site, the operating costs can be significantly reduced by investigating an optimized arrangements of freeze pipes.

7.2 Outlook

Although the presented three FE model of freezing soils is already capable of reproducing the coupled THM behavior involved in the ground freezing during tunnel construction for excavation sup-
port and groundwater control, there are some issues that require further development and investigation.

In reality, during freeze-thaw cycles hysteresis effect exits due to dissipative phase transition and capillary hysteresis. As a result, a hysteresis loop is generally observed in the liquid saturation curve (FABBRI ET AL. 2009). To take into account this hysteretic nature, the purely temperature-dependent liquid saturation function (2.33) has to be improved by adopting a hysteretic SWCC and considering dissipative phase transition (PHAM ET AL. 2005).

The formation and growth of ice lenses within the frozen fringe are interesting phenomena observed during soil freezing as a result of thermal regelation (see Subsection 2.1.1). As a result of the ice lens growth, the magnitude of the generated heave can be increased to a large extent (FOWLER AND KRANTZ 1994) and the resultant lateral nonuniformity can cause massive damage to the surface infrastructure. For an integrated description of the frost heave phenomenon, a nonzero ice flow with respect to the soil skeleton has to be incorporated in a future extension of the model.

In real soils, the unloading and reloading are neither linear nor logarithmically linear as presented in the present model. For a comprehensive description of the elastic volumetric behavior of unfrozen and freezing soils under both tensile and compressive loadings, more sophisticated functions are needed.

Due to the viscoelastic nature of ice and the complex interaction between the soil grains and pore phases (film water and crystal ice), frozen soil exhibits time-dependent behavior such as creep and relaxation. This time-dependency or rate-dependency can lead to a decrease in strength and stiffness from 40% to 60% of their initial values as stated in SCHULTZ ET AL. (2008). However, in the presented model, the rate-dependency of the constitutive behavior of frozen soils is not considered. A more realistic mechanical prediction requires an extension of the present work from an elasto-plastic to an elastic-viscoplastic constitutive model that is capable of reflecting the highly viscous constitutive behavior of frozen soils (see e.g. CUDMANI (2006), ORTH (1986)).
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Curriculum Vitae

Name: Meng-Meng Zhou

Birth: 1985.02.11

Nationality: Chinese

since 2008 Research Associate at the Institute for Structural Mechanics (Prof. Dr. techn. Günther Meschke), Ruhr University Bochum, Germany
since 2009 Fellow of the Research Departments of the Ruhr University Bochum, Germany

2008 Award for Excellent Performance as an International Student (DAAD-Prize 2008), Ruhr University Bochum, Germany
2007 - 2008 Student Assistant at the Institute for Structural Mechanics (Prof. Dr. techn. Günther Meschke), Ruhr University Bochum, Germany
2006 - 2008 Master of Science in Computational Engineering, Ruhr University Bochum, Germany
2004 - 2006 Master in Electromechanics option Automation, Group T Leuven Engineering College, Belgium

2003 - 2004 Candidate Industrial Engineer, Group T Leuven Engineering College, Belgium
2003 Award as the top student in the first Bachelor year of the Department of Mechanical & Electronic Control Engineering (First-Prize Scholarship), Beijing Jiaotong University, China
2002 - 2003 Bachelor of Science in Mechanical & Electronic Control Engineering, Beijing Jiaotong University, China

1999 - 2002 Pingyang No.1 Senior Middle School, Zhejiang, China
1997 - 1999 Kunyang No.2 Junior Middle School, Zhejiang, China
1996 - 1997 Wanquan Junior Middle School, Zhejiang, China
1991 - 1996 Wanquan Primary School, Zhejiang, China