On Solar Wind Magnetic Fluctuations and Their Influence on the Transport of Charged Particles in the Heliosphere

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Chapter 1

Introduction

Since antiquity mankind has always been fascinated by events appearing as bright optical phenomena in the sky. They originate from different objects located within the solar system or farther out at galactic distances. The most prominent long-known phenomena are the solar corona at the time of a total solar eclipse, comets whose tails trail away from the Sun and, at larger scales, stars undergoing core collapse and repelling explosively their own atmospheres. They appear as the most spectacular events observable, namely as supernovae.

During a total solar eclipse, when a narrow strip of the Earth’s surface is shielded completely by the Moon from the disk of the Sun, the corona appears crown-like around the shade of the Moon. It was uncertain until the middle of the 20th century whether the corona is a solar phenomenon or related to the Moon or represents an artifact produced by the Earth’s atmosphere. The answer to this question was provided by Grotrian [1939] and Edlén [1942]. Based on studies of iron emission lines, they suggested that the surface of the Sun is surrounded by a hot tenuous gas having a temperature of million degrees Kelvin and a state of high ionization. In view of such high temperatures and in order to explain the orientation of cometary tails, Biermann [1951] predicted a corpuscular outflow from the Sun. Using a theoretical description, Chapman [1957] showed that this flow indeed extends out into interplanetary space and is not confined to the solar surface. Parker [1958a] corrected and improved the Chapman-model and described the corona as a thermally driven hot magnetized gas expanding continuously into interplanetary space with a speed of several hundred km s$^{-1}$. The expression “solar wind” was born. The existence of this supersonic flow was first verified in 1962 by observations made with instruments on the Mariner 2 spacecraft (Neugebauer and Snyder, 1962).

The interstellar bubble filled with the solar wind is referred to as the heliosphere. This specific astrosphere is dominated by the magnetized, spinning star called Sun whose magnetic field is frozen into the solar wind expanding radially to outer regions of this comparatively modest cavity surrounding the Sun. In addition to the solar wind matter, this bubble also contains different populations of energetic particles playing their individual roles on the “stage” called heliosphere. These particles interact with small-scale fluctuations embedded in the background solar wind medium and, therefore, are subjected to a variety of processes caused by solar activity varying in time and the solar wind convection changing in heliocentric distance. In view of the particle scattering by fluctuations combined with solar wind variations in space and time, the particle energy spectra reveal several modifications, also referred to as solar modulation. Chapter 2 gives a brief overview of the heliosphere, solar modulation and several particle populations considered in this thesis.

To describe the ionized state of a gas such as the solar wind, the chemists Tonks and Langmuir [1929] coined the expression “plasma”. A plasma contains so many free charged particles that their collective forces influence the properties of the ionized medium in a characteristic manner, leading to a variety of electromagnetic waves propagating through this plasma. If a background magnetic field is present a certain wave mode arises, the so-called Alfvén wave named after its discoverer Alfvén [1942]. In heliospheric physics and astrophysics, it is commonly assumed that such a magnetized plasma is free of collisions, i.e. a plasma particle is not affected by other constituents present in its
closer neighborhood. The plasmas in both interplanetary and interstellar space are examples for this. Because the present thesis is based on the plasma wave approach, appendix C summarizes several aspects of the standard theory of plasma waves.

Under the long-standing assumption that such plasma waves represent small-scale fluctuations with which energetic charged particles interact, the question how these particles propagate in a region filled with a magnetized plasma and its associated turbulence is crucial in both heliospheric physics and astrophysics. Chapter 3 introduces appropriate particle transport equations taking into account the properties of the underlying plasma wave turbulence. Based on the relativistic Vlasov equation and quasi-linear theory, the fundamental Fokker-Planck equation is deduced. The latter equation involves interactions of charged particles with plasma waves via so-called Fokker-Planck coefficients. These coefficients depend on the statistical properties of the background medium, the underlying plasma wave turbulence and the particle properties. Considering only the fastest diffusion processes, the particle phase space distribution quickly approaches a quasi-isotropic state. The Fokker-Planck equation then reduces to the so-called diffusion-convection transport equation taking into account not only diffusive particle transport along and normal to the background magnetic field as well as particle drifts (via a tensor with five elements), but also adiabatic deceleration and momentum diffusion (via two scalars). The seven particle transport parameters are given as functionals depending on the Fokker-Planck coefficients and, therefore, on the properties of the plasma wave turbulence.

The Fokker-Planck coefficients describe wave-particle interactions and represent a fundamental key for understanding particle diffusion in a turbulent magnetized plasma. Chapter 4 offers a thorough insight into their mathematical structure. There, all coefficients are treated as general as possible and new representations are deduced generalizing those available so far. For an isotropic turbulence, several restrictive cases are derived. Among those, the limit of a slab turbulence is considered, in which plasma waves propagate only either forward or backward along the background magnetic field. It is shown that particle diffusion normal to the background magnetic field is suppressed for this slab geometry.

Fokker-Planck coefficients represent not only the scattering of charged particles by plasma waves, they also include the variation of the turbulence intensity in both the spatial configuration as well as wavenumber space. The wavenumber evolution of the turbulence magnetic intensity is the focus of chapter 5. Based on plasma wave damping rates obtained with a linear Vlasov code and numerical solutions of the governing wave transport equation, it is demonstrated that plasma wave damping is not the only physical key to explain an often observed intermediate-frequency feature of solar wind magnetic fluctuation spectra. Taking into account an additional influence of plasma wave dispersion, a model is constructed which may represent the contributions of magnetosonic/whistler mode dispersion to solar wind magnetic turbulence spectra and, furthermore, may explain the specific feature observed at intermediate frequencies.

Calculations of several particle transport parameters are presented in chapter 6. For the first time, the transport parameter for particle diffusion normal to the background magnetic field is derived for an isotropic turbulence by using the Fokker-Planck approach combined with the plasma wave viewpoint. It is argued that the inclusion of a finite turbulence correlation length and a turbulence spectrum with broken power laws may be of importance for understanding perpendicular particle transport in a turbulent magnetized plasma. This is demonstrated by a comparison of the new perpendicular transport parameter with simulation results obtained previously by Giacalone and Jokipii [1999].

Transport parameters for particle drifts are derived for both an isotropic and a slab turbulence by evaluation of the corresponding Fokker-Planck coefficients. Based on the different dimensionalities of the turbulence geometries, detailed calculations show that no drift occurs for an isotropic turbulence. The particle drift is determined by the net polarization of the plasma wave turbulence
with slab geometry. It is demonstrated that this new drift term is related to the $E \times B$ drift via the electric field of the plasma wave turbulence. Comparable calculations are not known so far. Additionally, the transport parameters for parallel particle diffusion, adiabatic deceleration and diffusion in momentum space are recalculated for a slab turbulence and earlier results are confirmed.

**Chapter 7** introduces a first application of the diffusion-convection equation and its associated transport parameters derived in chapter 6. Stochastic acceleration of pick-up ions is the crucial point of chapter 7. The governing transport equation is solved analytically in terms of an arbitrary particle source function. The new solution significantly generalizes the two “classical” solutions used during the last three decades. Numerical calculations demonstrate the sensitive dependence of the pick-up ion distribution function on parameters of the underlying turbulence. The new semi-analytical approach renders the opportunity to check on modeling results obtained purely numerically with more complicated computer codes and, furthermore, allows to compare observations and theory in future studies.

A second application of the particle transport equation is presented in **chapter 8**. There, solar modulation of anomalous and galactic cosmic rays is discussed. The appropriate equation of transport is solved analytically. The new solution generalizes earlier more limited solutions presented in the context of solar modulation. Numerical calculations yields energy spectra of anomalous and galactic cosmic rays depending sensitively on the underlying turbulence physics. Besides an easy way to check more complicated computer codes developed for the investigation of solar modulation, the new solution enables one to compare theory and observations in forthcoming studies.

In **chapter 9** a summary of the main results and conclusions is given. Furthermore, it provides an outlook on future studies which, presumably, will further generalize the results presented in this thesis.
Chapter 1 Introduction
Chapter 2

Particle Populations in the Heliosphere

Whether with or without a peculiar motion relative to the ambient interstellar medium, wind-driving stars form bubbles into the surrounding matter. The extent and the structure of such an interstellar bubble, also referred to as an astrosphere, depend on a variety of physical parameters. These are the magnitude of the peculiar motion, the composition and velocity distribution of the (magnetized) stellar wind plasma and the total pressure of the ambient medium. The latter can be considered as a mixture consisting of fractionally ionized thermal gas and a nonthermal particle population representing cosmic rays with high kinetic energies that penetrate, above a certain energy threshold, the astrosphere. In addition to the pressure contributions resulting from these thermal and nonthermal particles, a pressure of the interstellar magnetic field has to be included. The competition of the interstellar and stellar pressures including the intrinsic peculiar motion creates, in general, an asymmetric cavity around the star.

2.1 The Heliosphere

It was Dessler [1967] who first introduced the expression “heliosphere”, referring with this term to a comparatively modest interstellar bubble filled with the solar wind plasma emitted continuously from the Sun inside this cavity. Figure 2.1 gives a schematic view illustrating the shape and structure of the heliosphere determined by the solar wind and the relative motion of the Sun with respect to the Local InterStellar Medium (LISM). In the rest frame of the solar system the LISM can be interpreted as an interstellar wind. The balance of the total LISM pressure with the ram pressure of the radially expanding solar wind leads to a heliospheric configuration with a dimension of several hundred astronomical units (AU) and a specific arrangement of shocks resulting from the collision of the LISM and the solar wind. The adaptation of the solar wind to the surrounding LISM takes place in the heliospheric interface, also called Baranov-Parker interface, a region defined by the heliopause and the heliospheric termination shock. In contrast to the heliospheric and the bow shock, the heliopause is a contact discontinuity or separatrix of the system, defining a surface that separates the interstellar and the solar plasma from each other and through which no mass flow of charged particles takes place. The heliopause can, therefore, be considered as the outer boundary of the heliosphere. The separation and adaptation is illustrated in figure 2.1 by the curved arrows in the inner and outer heliosheath. The latter defines the region between the heliopause and the bow shock and depends on the dynamical state of the LISM. The existence of the bow shock is, nowadays, still uncertain because its formation depends on properties of the LISM in which in-situ observations are not yet possible. Only crude estimates are available leading to the uncertainties regarding the bow shock and the nature of the interstellar side of the heliosheath. The production and the characteristic features of both the bow as well as the heliospheric shock depend on the physical and geometrical properties of the plasma flows in which they are “embedded”. In contrast to the heliopause as a separating contact discontinuity, the bow shock and the heliospheric shock are surfaces reflecting discontinuous transitions from supersonic to subsonic flow speeds of the
Chapter 2 Particle Populations in the Heliosphere

Figure 2.1: A simplified illustration of the heliosphere reflecting the configuration of distinguished shocks, due to the relative motion of basic plasma contributions representing charged particles of the solar wind and of the LISM, as seen in the rest frame of the Sun.

corresponding plasma flows, i.e. the solar wind or the stream of the LISM, respectively. The deceleration from super- to subsonic speeds is accompanied by corresponding increases in pressure and temperature of the plasma flows. Using for the solar wind proton number density, $n_{sw,p} \sim 7 \text{ cm}^{-3}$, the Heliospheric Magnetic Field (HMF), $B_{HMF} \sim 5 \text{ nT}$, and the solar wind temperature, $T_{sw} \sim 10^5 \text{ K}$, at the Earth’s orbit, one obtains for the Alfvén speed $v_A = B_{HMF}/\sqrt{4\pi n_{sw,p}m_p}$ and the sound speed $v_s = \sqrt{k_B T_{sw}/m_p}$ values of $v_A \approx v_s \sim 40 \text{ km s}^{-1}$, exhibiting a supersonic and Alfvénic nature of the solar wind at 1 AU expanding into the heliosphere with approximately $400 \sim 800 \text{ km s}^{-1}$. From figure 2.1 it is clear that the geometry of the heliosphere is, in general, asymmetric. Because the two plasma flows collide head-on in the apex faced region and the heliopause separates the flows from each other, a heliotail arises at the antapex faced side of the heliosphere where the two flows stream almost parallel to each other.

2.2 Particle Populations

The heliosphere described above can be considered as a stage for different populations of energetic particles subjected to various processes at sites throughout the heliosphere. They originate from the central part of the heliosphere, i.e. from the Sun, via solar flares in the low corona, from shock waves driving massive clouds, which are eruptively ejected from the Sun, or from so-called Corotating Interaction Regions (CIRs)\(^1\). They come from planetary magnetospheres interacting with the solar wind plasma, from interplanetary traveling shocks and cometary tails. They come from the outer edge of the heliospheric cavity, namely the heliospheric termination shock. Having high kinetic energies, a few particles also come from regions far away from the heliosphere. These particles are emitted from galactic and extragalactic sources and, subsequently, travel through interstellar and intergalactic space before they reach the cavity surrounding the Sun. The way of playing their individual roles on the stage called heliosphere depends, in general, on their origins and characteristic interactions with the surrounding medium, i.e. the solar wind plasma, in which these energetic particles are embedded. Such particles are detectable directly via in-situ observations on board of spacecraft moving through the interplanetary medium and indirectly by groundbased detectors on Earth. In the latter case, the energies of the particles have to be sufficiently high, so that they

\(^1\)CIRs are, with the Sun corotating, regions in the expanding solar wind plasma, in which an overtaking of slow by fast solar wind streams takes place, resulting in forward and backward propagating shocks being able to accelerate charged particles to higher energies.
can reach Earth’s orbit after their passage through the solar wind plasma. When entering Earth’s atmosphere the particles interact with atoms in the upper layers leading, via nuclear processes and chain reactions, to air showers consisting of subsequently produced particles and radiation. In order to illustrate the magnitudes of the energy of several particle populations, figure 2.2 displays a simplified sketch of energy spectra of several important populations such as (extra)galactic cosmic rays (GCRs) and anomalous cosmic rays (ACRs).

2.2.1 Cosmic Rays

It was the Austrian physicist Victor Hess who first discovered in the year 1912 a by then unknown energetic particle component in Earth’s atmosphere, which later was to become the secondary component of cosmic rays. Based on measurements made with an iron chamber on a manned balloon, Hess noticed an increasing particle radiation level with increasing balloon altitude, leading to his famous suggestion that a radiation of great power penetrates Earth’s atmosphere from above. This was in contrast to what was expected from its supposed terrestrial origin. Following Harwit [1981], the discovery of the extraterrestrial radiation by Hess can be considered as the beginning of cosmic ray physics. During following balloon flights, Hess ruled out the Sun as a possible source for the increased ionization in the lower layer of the atmosphere. He denoted this flux with the German expression “Höhenstrahlung”, referring with it to the observed radiation related to the, nowadays, well-known air showers. The detection of such air showers enables one to derive conclusions with regard to the nature of the primary component entering Earth’s atmosphere from above. This population of cosmic rays represents particles causing, via nuclear processes, those air showers.

From a modern point of view, the population of the primary component of cosmic rays can roughly be distinguished into several components: galactic and extragalactic cosmic rays (GCRs) originating from outside of the heliosphere, solar cosmic rays or solar energetic particles (SEPs) stemming from the inner heliosphere, and anomalous cosmic rays (ACRs) produced in the outer heliosphere. The population of GCRs consists always of completely ionized atoms covering a huge kinetic energy interval from the keV to the TeV nucleon$^{-1}$ range and hitting the Earth with a rate of about 1000 particles m$^{-2}$ s$^{-1}$. Figure 2.3 shows a composite of typical measurements representing differential energy spectra for the GCR particle species H, He, C+O and Fe as observed at the Earth’s orbit. The GCR population consists mainly of protons and α-particles and of the corresponding amount.

![Figure 2.2: Illustration of energy spectra of solar wind particles, anomalous and galactic cosmic rays as well as particles originating from solar flares and CIRs.](image-url)
of electrons. Heavier ions are usually observed in much smaller numbers. Particles with kinetic energies above a few GeV nucleon\(^{-1}\) can penetrate the heliosphere without having interactions with small-scale fluctuations carried by the solar wind plasma, whereas particles with lower kinetic energies are subjected to the solar wind plasma turbulence. Such a turbulence can be described by electromagnetic plasma wave modes superposed to the heliospheric background magnetic field originating from the Sun. Due to the interactions of these plasma waves with less energetic particles, the latter diffuse in phase space, i.e. in their spatial coordinate space and in their momentum or, equivalently, energy space. Regarding the energy spectra of such low energetic GCRs, the scattering on the magnetic field irregularities leads, together with the convective character of the magnetized solar wind plasma and a changing solar activity, to significant modifications or modulations in space and time.

Modulation in Space

When reaching the heliosphere, the energy spectrum of GCRs at relatively high energies is determined by their initial interstellar spectrum. The latter can be roughly approximated by a power law in energy, i.e. \(E^{-s}\), where the spectral index is given by \(s \sim 2.6\) (compare in figure 2.3 the spectra for energies above \(\sim 1\) GeV nucleon\(^{-1}\)). Having energies below several times GeV nucleon\(^{-1}\), the particles feel the small-scale turbulence in the solar wind and experience, due to the wave-particle interactions, the process of adiabatic deceleration (or cooling). The latter is a consequence of the divergence of the expanding solar wind flow. In other words, the principal shape of modulated cosmic ray spectra results from the diverging solar wind and its turbulence, causing the GCR flux to turn its negative slope in energy over into a positive slope, i.e. to decrease with decreasing kinetic energy, and, furthermore, with decreasing heliocentric distance.

Modulation in Time

Below the threshold of a few GeV, GCR energy spectra also show a strong dependence on solar activity changing quasi-periodically in time with a period of about 22 years. The temporal modulation of spectra is anticorrelated to solar activity. This anticorrelation can be visualized with time profiles of the sunspot numbers indicating the level of solar activity. When solar activity is
low, indicated by a small number of sunspots, the GCR flux level is relatively high and vice versa: with increasing solar activity, the flux of the GCR species decreases to lower levels. Furthermore, the maximum of each energy spectrum shifts towards higher energies. Both features can be easily understood by considering again the small-scale turbulences in the solar wind and their interactions with the particles. During a phase of low solar activity, the solar magnetic field can approximately be described by a dipole component at low heliographic latitudes. With increasing activity, the configuration of the solar background magnetic field will be changed by a break up of the dipole symmetry. This presumably leads to a higher turbulence level throughout the heliosphere. Because the possibility of wave-particle interactions is higher for increasing electromagnetic turbulence intensities, the scattering of the particles increases and their mean free paths decrease. Then, spatial diffusion is less efficient than during quiet-time phases of Sun’s activity, leading to the observed decrease in GCR fluxes. Moreover, higher turbulence levels shift the onset of the particle cooling to higher energies, resulting in the observed shifts of the maximum peaks to higher energies.

2.2.2 The Anomalous Component of Cosmic Rays

The anomalous component of cosmic rays was discovered in the early 1970s by investigating quiet-time spectra of different GCR elements at their low energy regimes. In contrast to expected strictly monotonically decreasing cosmic ray intensities as functions in kinetic energy, Garcia-Munoz et al. [1973] found considerable enhancements in cosmic ray fluxes at energies around $10 \sim 80 \text{ MeV per nucleon}$. Two years later Garcia-Munoz et al. [1975] pointed out that their observed enhancements consisted completely of $^4\text{He}$. Almost at the same time Hovestadt et al. [1973] observed unexpected enhancements of oxygen, $^{16}\text{O}$, at energies around $1 \sim 9 \text{ MeV nucleon}^{-1}$ in cosmic ray spectra. Soon after, McDonald et al. [1974] confirmed these curious abundances and characterized these fluxes, which were observed as humps in the energy spectra of GCRs (see figure 2.4), as anomalous, referring with this expression to the uncertainty regarding the unknown flux enhancements.

Only one year after the discovery of these anomalous fluxes Fisk et al. [1974] made the first step to explain these enhancements at low energies. They proposed a scenario in which interstellar neutral atoms enter the heliosphere from the outside. They suggested that elements embedded in the local interstellar matter, which have a first ionization potential (FIP) below that of hydrogen at $13.6 \text{ eV}$, would be ionized while atoms like He, N, O and Ne would be neutral resulting from their higher FIPs. Because of the motion of the solar system through the interstellar medium, the neutral elements can easily flow into the heliospheric cavity, whereas the charged particles are effectively excluded from the heliosphere by the magnetic field configurations. When neutral atoms approach the nearby environment of the Sun, they can become singly ionized in the interplanetary medium. These freshly ionized particles, which later were to become the so-called pick-up ions (PUIs), experience a stochastic acceleration from keV nucleon$^{-1}$ up to a range of MeV nucleon$^{-1}$ energies. Within the framework of this model it was not possible to explain the absence of expected intensity maxima of these accelerated ions around 30 AU. The second and decisive step was made by Pesses et al. [1981]. Based on the model by Fisk et al. [1974], they suggested that the main acceleration of these ions takes place at the heliospheric shock, resulting from the Fermi-I or diffusive shock acceleration process described briefly in chapter 1. When reaching the heliospheric shock, an uncertain fraction of the PUI population is subjected to this diffusive shock acceleration. After a sufficient energization of these PUIs to energies in the MeV nucleon$^{-1}$ range, some of them will escape the shock region as the anomalous component of cosmic rays and propagate back, against the expanding solar wind flow, to the inner heliosphere where they are observed as the energy enhancements in the spectra of GCRs. Such flux enhancements measured aboard the spacecraft Voyager 1 (V1) and Voyager 2 (V2) can be seen in figures 2.4(a) and 2.4(b).
Chapter 2 Particle Populations in the Heliosphere

(a) Helium spectra

(b) Hydrogen spectra

Figure 2.4: Energy spectra of ACRs and GCRs as observed with Voyager 1 (V1) and 2 (V2). The left panel shows a composition of helium spectra for the time periods 1985 (open symbols) and 1987 (filled symbols), as presented by Christian et al. [1988]. The right panel shows enhancements of anomalous hydrogen at low energies. The dashed line represents the strictly monotonically decreasing hydrogen intensity for the case that the spectra would purely consist of GCR hydrogen (taken from Christian et al. [1995]).

Beside the anomalous fluxes of singly ionized helium and oxygen, many other ACR species, like N, Ne and Ar, were found during the last decades. A crucial test of this scenario was the finding of anomalous hydrogen fluxes by Christian et al. [1988,1995] and McDonald et al. [1995]. Because PUIs are singly ionized, one would expect that the corresponding anomalous component would have the same charge state as the PUIs. In the following years, the ionization states of several ACR species at low and medium energies around $8 - 10$ MeV nucleon$^{-1}$ have been measured on the SAMPEX spacecraft, confirming a charge state of unity as it was expected if the ACRs represent accelerated and singly-ionized PUIs (see, e.g., Klecker et al., 1995, 1998). In the case of anomalous oxygen, however, components of doubly-ionized ACR ions were found at higher energies above 20 MeV nucleon$^{-1}$ (see Jokipii, 1996, Mewaldt et al., 1996; Klecker et al., 1998). These ions are presumably produced at the heliospheric shock, because their acceleration to higher energies requires more time and increases the chance for further ionization. In contrast to ACRs, GCRs are always fully stripped of their electrons. The charge state of GCRs heavier than hydrogen is, therefore, higher than that for the same ACR element, i.e. $q_{\text{GCR}} > q_{\text{ACR}} = 1$. So, a more efficient spatial diffusion of an ACR particle, compared to the same GCR particle having the same energy, is expected. Considering the charge state of hydrogen compared to heavier PUI elements, it is obvious that this is a special one, because $q_{\text{GCR}} = q_{\text{ACR}}$. The low flux enhancements of anomalous hydrogen at low energies, shown in figure 2.4(b), are rather an effect of the heliospheric termination shock than an effect resulting from spatial diffusion, indicating that the conversion of pick-up protons to the corresponding hydrogen abundances of ACRs is not very efficient for this lightest and most abundant element.
2.2 Particle Populations

2.2.3 Pick-Up Ions

It was already mentioned in the previous section that the freshly produced ions, providing for the seed of the anomalous cosmic ray component, are, nowadays, referred to as pick-up ions (PUIs). The existence of PUIs, as a consequence of an inflow of interstellar neutral atoms of the LISM into the heliosphere and their subsequent ionization in the vicinity of the Sun, was already predicted in a number of publications about 30 years ago (see, e.g., Fahr, 1968; Blum and Fahr, 1970; Fahr, 1971; Holzer, 1972). Indeed, first evidence of inflowing interstellar neutral hydrogen and helium was found in measurements of solar ultraviolet resonance lines backscattered by these neutral elements, resulting in estimates of the velocity, temperature and number density of these neutral gas components (Thomas and Krassa, 1971; Bertaux and Blamont, 1971; Weller and Meier, 1974).

For the direct detection of the corresponding ions, time-of-flight detectors were necessary, which can measure the ratios of mass and energy relative to the particle charge as well as particle’s mass separately with an extremely low background. The first observations of these PUIs, having typical energies in the keV nucleon$^{-1}$ range, date back to 1985 when Möbius et al. [1985] detected suprathermal interstellar pick-up He$^+$ by using the plasma analyzer SULEICA (Suprathermal Energy Ionic Charge Analyser) on the IRM (Ion Release Module) on board the earth-bound satellite AMPTE (Active Magnetosphere Particle Tracer Explorer), allowing first quantitative analyses of this particle population. More detailed measurements were performed later with the time-of-flight SWICS (Solar Wind Composition Spectrometer) instrument aboard the deep space probe Ulysses. Using the SWICS instrument, Gloeckler et al. [1993] and Geiss et al. [1994a, 1994b, 1995] reported the first direct detection of pick-up hydrogen and heavier PUIs such as carbon ($^{12}$C$^+$), nitrogen ($^{14}$N$^+$), oxygen ($^{16}$O$^+$) and neon ($^{20}$Ne$^+$). Based on their observations they estimated, by calculating the corresponding PUI fluxes, the rate of ionization and the abundances of these interstellar neutral elements. The detections made with SWICS supplement those which were made by using the AMPTE spacecraft at the Earth orbit. Furthermore, it appears that a check of older data obtained by instruments on board the three deep space probes Pioneer 10, Voyager 1 and 2 reveals additional in-situ observations of the PUI population (Decker et al., 1995; Intriligator et al., 1996; Mihalov and Gazis, 1998).

Figure 2.5 shows representative observations of pick-up hydrogen and helium obtained with SWICS on Ulysses. The left panel shows phase space distributions measured in the undisturbed solar wind at solar minimum, whereas the right panel presents velocity distributions in a disturbed solar wind region close to a CIR. In particular the left panel of figure 2.5 reflects the major features of PUI velocity distributions. The most important properties can be understood by considering neutral atoms traveling through the expanding solar wind plasma, independent of what their origin is. The neutral atoms penetrate the inner heliosphere with a low number density and propagate close enough to the Sun, where, predominantly, solar radiation and solar wind particles act to ionize an uncertain fraction of these neutral atoms. Under undisturbed solar wind conditions the mean free path for collisions between solar wind and neutral particles is very long so that the rate of collisions is very low and, therefore, the ionization process is mainly determined by ionizing solar photons. When such a neutral atom suddenly becomes ionized, it has to respond to the electromagnetic fields embedded in the supersonic solar wind plasma, so it is subjected to the local electric induction forces. In a reference frame moving with the solar wind, the response motion can be illustrated by a helical trajectory of the freshly produced ion with respect to the local heliospheric magnetic background field. This is frozen, due to the high electrical conductivity, into the expanding solar wind plasma. During this process, the particle gyrates with its initial perpendicular speed component along the magnetic background field and retains its initial parallel speed component. Since the freshly generated ions are coupled to the magnetic field, i.e. they are picked up, the PUIs convect with the streaming solar wind towards the outer heliosphere. Because of the expansion of the solar wind the particles lose energy as they are convected outwards.
Considering a parcel of the solar wind, it will pick-up new ions and cools the older ions during the convection, leading to a filled-in sphere in the phase space of the particles, appearing as distinctive isotropic velocity distributions with, more or less, sharp cut-offs at twice the solar wind speed (see, e.g., Vasyliunas and Siscoe, 1976).

Figure 2.5: Pick-up hydrogen and helium phase space distributions in the spacecraft frame observed in 1991 by the SWICS instrument on board the deep space probe Ulysses. Left panel: velocity distributions in the quiet and undisturbed solar wind at about 4.8 AU (taken from Gloeckler et al. [1993]). Right panel: velocity distributions in the disturbed solar wind at 4.5 AU (taken from Gloeckler et al. [1994]), where the perturbation is due to a nearby CIR.

Considering now the right panel of figure 2.5 in more detail and comparing it with the left panel, one can relate the observations to the scenario described above. Furthermore, one can recognize another interesting feature, namely the so-called high energy tails of the velocity distributions (Gloeckler et al., 1994; Zurbuchen, 2000). This characteristic had not been seen in the case of pick-up helium detected by the Earth-bounded satellite AMPTE. The extended tails, which had long been suspected to occur (Fisk, 1976a, 1976b; Isenberg, 1987), can be considered as signatures of a local energization of PUIs within the solar wind plasma by electromagnetic turbulences via the stochastic second-order Fermi process or, from a more localized point of view, at CIRs by the first-order Fermi acceleration process, i.e. diffusive shock acceleration. It is still unclear whether the energization results from the Fermi-I process, as studied by Jokipii and Giacalone [1996], Giacalone et al. [1997] and Giacalone and Jokipii [1997], or from the Fermi-II process, considered by Fisk [1976a], Isenberg [1987], Chalov et al. [1995,1997] and Fichtner et al. [1996]. Gloeckler et al. [1994] suggested a combination of both which was modeled by Dworsky and Fahr [2000].

Regarding the isotropy of PUI velocity distributions, more recently performed analyses of measurements obtained with SULEICA and SWICS pointed out, that the velocity distributions might be anisotropic in some regions of the heliosphere, e.g. at high heliographic latitudes (Gloeckler et al., 1995), or under certain interplanetary magnetic field conditions (Möbius et al., 1998). Based on this, several authors (e.g. Isenberg, 1997; Isenberg and Lee, 1998; Schwadron, 1998; Chalov and Fahr, 1998; le Roux et al., 2001) revisited the theoretical research regarding such anisotropies and reviewed, thereby, the theoretical discussions of PUI behavior in phase space.
2.3 Summary

In this chapter an overview is given of the heliosphere, solar modulation and several particle populations detectable within this particular astrosphere. The configuration of this comparatively modest interstellar bubble filled with the solar wind plasma is a consequence of the competition of the local interstellar medium and the solar wind pressure. The outward expanding solar wind plasma affects the transport of charged particles within the heliosphere significantly, since charged particles interact with the small-scale fluctuations and the large-scale background magnetic field carried by the solar plasma plasma. The appropriate mathematical approach is the subject of the following chapter.
Chapter 3

Equations of Transport

The influence of a plasma turbulence on the transport of charged particles forms a crucial point in a wide variety of studies performed during the last decades. Processes providing for a turbulence in a plasma exist in a rich profusion so that both the interplanetary and interstellar space plasma indeed exhibits a turbulent character (see, e.g., Denskat et al., 1983; Tu et al., 1989, 1990; Marsch and Tu, 1990; Hajivassiliou, 1992; Goldstein et al., 1995). In general, it is assumed and widely accepted that these electromagnetic fluctuations represent wave modes of which the plasma turbulence consists.

The basis of almost all transport theories is a diffusion equation describing the evolution of particles in both the spatial configuration as well as momentum space. Under the assumption that the fluctuating fields vary only slightly in space and time, this diffusion equation can be derived on the basis of the relativistic Vlasov equation combined with the quasi-linear theory. The result of the quasi-linear derivation is the so-called Fokker-Planck equation (see, e.g., Kennel and Engelmann, 1966; Hall and Sturrock, 1967; Schlickeiser, 2002). It allows to investigate the diffusive particle transport in turbulent magnetized plasmas in detail. The particle diffusion in phase space is attributed to interactions of these particles with plasma wave modes. Wave-particle interactions enter the Fokker-Planck equation via the so-called Fokker-Planck coefficients. They depend solely on the statistical properties of the underlying plasma wave turbulence and the background medium. Due to the presence of the waves and the associated particle scattering, the propagation of the particles is rather a random walk than a straight-line motion or, if a background magnetic field is present, a simple gyration.

The Fokker-Planck equation with all its 25 Fokker-Planck coefficients is very complicated, no general solution of it is known or likely to be obtained. Consequently, simplifications are introduced which allow a further treatment of the Fokker-Planck equation. Usually, it is assumed that the fastest wave-particle interactions are diffusion in the particle’s gyrophase and pitch angle. Then, the particle phase space distribution function approaches quickly a quasi-isotropic state. In this case the initial phase space distribution can be split into an isotropic and anisotropic term and the standard diffusion approximation can be applied to the pitch angle and gyrophase averaged Fokker-Planck equation (see, e.g., Jokipii, 1966; Hasselmann and Wibberenz, 1968, 1970; Skilling, 1975; Schlickeiser, 2002). The result is the diffusion-convection transport equation describing the evolution of the isotropic contribution of a particle distribution in phase space. This equation contains the most important processes determining sensitively the particle transport in a turbulent plasma. These processes are diffusion in spatial configuration space, diffusion in momentum space as well as adiabatic deceleration/acceleration, due to the plasma flow. The first is given by a tensor, while the second and third processes are determined by scalars. The elements of the tensor as well as the two scalars are transport parameters characterized by the Fokker-Planck coefficients and, therefore, by the properties of the underlying plasma wave turbulence. The diffusion-convection equation is the fundamental transport equation of charged particles and finds extensive applications in heliospheric as well as astrophysical problems.


3.1 The Vlasov Equation in Guiding Center Coordinates

The phase space distribution function \( f_\alpha \) of a non-thermal particle species \( \alpha \) can be described by the collisionless Boltzmann equation, sometimes also referred to as Vlasov’s equation,

\[
\frac{\partial f_\alpha}{\partial t} + \mathbf{x} \cdot \nabla_x f_\alpha + \mathbf{p} \cdot \nabla_p f_\alpha = S_\alpha(x, p, t) \tag{3.1}
\]

Here, \( x \) and \( p \) are the position and the momentum of a charged particle at time \( t \), respectively. On the right-hand side of equation (3.1), the function \( S_\alpha \) allows to include an injection or loss of particles. It represents sources and/or sinks of the non-thermal population \( \alpha \). The approach used here is the so-called test particle approach. It enables one to investigate the diffusive transport of a certain particle species in a plasma. The diffusive character of the particle transport results from interactions of these particles with electromagnetic fluctuations which are assumed to be given. Quite opposite to the test particle concept is the so-called test wave approach. It allows to investigate the properties of plasma wave modes, such as dispersion relations and polarization states, for the case that the distribution functions \( f_\alpha \) of the thermal plasma constituents are given.

In other words, the problem is turned around. Since an understanding of the fundamental transport equation is not possible without a knowledge of plasma wave properties, appendix C provides for a brief overview of the standard theory of plasma waves.

In both the test particle as well as the test wave approach, the phase space trajectories of non-thermal particles and plasma components result from Lorentz as well as electric forces. Within the framework of the test wave approach, the plasma constituents are effected by microscopically electromagnetic fields, due to neighboring particles within the Debye sphere of the plasma particle (see appendix C). The presence of these rapidly fluctuating micro-scale fields leads to the appearance of a “collision term” on the right-hand side of the Boltzmann equation (C.4). There, the total electromagnetic fields enter the calculations by the non-relativistic equations of motion (C.2). In the test particle approach, the number density of the non-thermal population is mostly orders of magnitudes smaller than the number density of the background plasma so that microscopic fluctuations are negligible, only the source function \( S_\alpha \) appears on the right-hand side of the collisionless Boltzmann equation (3.1). In contrast to the non-relativistic plasma constituents considered in chapter C, the particle population propagating through the plasma turbulence is allowed to be of relativistic nature. The appropriate equations of motion then read

\[
\mathbf{x} = v = \frac{\mathbf{p}}{(\gamma m_\alpha)} \quad \text{and} \quad \dot{\mathbf{p}} = q_\alpha \left[ \mathbf{E}(x, t) + \frac{\mathbf{p} \times \mathbf{B}(x, t)}{\gamma m_\alpha c} \right] \tag{3.2}
\]

where \( q_\alpha \) and \( m_\alpha \) denote the particle charge and mass, respectively, while \( \gamma = 1/\sqrt{1 - v^2/c^2} \) is the Lorentz factor. The total electromagnetic field can be written as a superposition of the large-scale, uniform field components \( \mathbf{B}_0 \) and \( \mathbf{E}_0 \) and of the small-scale plasma turbulence contributions \( \delta \mathbf{B}(x, t) \) and \( \delta \mathbf{E}(x, t) \). This means \( \mathbf{B}(x, t) = \mathbf{B}_0 + \delta \mathbf{B}(x, t) \) and \( \mathbf{E}(x, t) = \mathbf{E}_0 + \delta \mathbf{E}(x, t) \). Then, the right-hand side of equation (3.2) reads

\[
\dot{\mathbf{p}} = q_\alpha \mathbf{E}_0 + q_\alpha \delta \mathbf{E}(x, t) + \frac{\Omega_\alpha}{\mathbf{B}_0} \mathbf{p} \times \mathbf{B}_0 + \frac{\Omega_\alpha}{\mathbf{B}_0} \mathbf{p} \times \delta \mathbf{B}(x, t) \tag{3.3}
\]

Here, \( \Omega_\alpha = \frac{q_\alpha \mathbf{B}_0}{(\gamma m_\alpha c)} \) is the relativistic gyrofrequency of a particle of the species \( \alpha \) in the uniform field \( \mathbf{B}_0 \). Due to the high conductivity of the plasma, any large-scale electric fields are neglected (see appendix C). Furthermore, the uniform magnetic field is aligned along the \( z \)-axis, i.e. \( \mathbf{B}_0 = B_0 \mathbf{e}_z \). In order to simplify the calculations, a helical description for the fluctuating electromagnetic field is commonly introduced. The Cartesian components normal to the background
magnetic field are then expressed by left- and right-handed contributions, so that
\[
\delta B_{L,R} = \frac{1}{\sqrt{2}} (\delta B_x \pm i \delta B_y), \quad \delta B_\parallel = \delta B_z
\] (3.4a)
\[
\delta E_{L,R} = \frac{1}{\sqrt{2}} (\delta E_x \pm i \delta E_y), \quad \delta E_\parallel = \delta E_z
\] (3.4b)

Within the framework of the quasi-linear test particle approach, a fundamental task is to determine the temporal evolution of the particle momentum, the three spatial coordinates, the pitch angle and the gyrophase of the particle. To do so, it is convenient to use spherical coordinates in momentum space
\[
p_x = p \cos \phi \sqrt{1 - \mu^2}, \quad p_y = p \sin \phi \sqrt{1 - \mu^2}, \quad p_z = p \mu
\] (3.5)
where $\phi$ and $\mu$ are the gyrophase and the cosine of the pitch angle, respectively. Furthermore, one usually introduces guiding center coordinates defined by
\[
R = (X,Y,Z) = x + \frac{v \times e_z}{\Omega_\alpha}
\] (3.6)

The introduction of guiding center coordinates simplifies further calculations, since the actual position of a particle, gyrorotating with respect to the ambient magnetic field, is of less interest. Making use of equations (3.3) through (3.6), one obtains for the variations in time the following expressions:
\[
\dot{p} = g_p = q_\alpha \left[ \mu \delta E_\parallel + \sqrt{(1 - \mu^2)/2} \left( e^{-i\phi} \delta E_L + e^{i\phi} \delta E_R \right) \right]
\] (3.7a)
\[
\dot{\mu} = g_\mu = \frac{\Omega_\alpha}{B_0} \left[ \frac{c}{v} (1 - \mu^2) \delta E_\parallel + i \sqrt{(1 - \mu^2)/2} \left( e^{i\phi} \left( \delta B_R + \mu \frac{C}{v} \delta E_R \right) - e^{-i\phi} \left( \delta B_L - \mu \frac{C}{v} \delta E_L \right) \right) \right]
\] (3.7b)
\[
\dot{\phi} = g_\phi = -\Omega_\alpha \frac{\delta B_\parallel}{B_0} + \frac{\Omega_\alpha}{\sqrt{2(1 - \mu^2)B_0}} \left[ e^{i\phi} \left( \delta B_R + \mu \frac{C}{v} \delta E_R \right) - e^{-i\phi} \left( \delta B_L - \mu \frac{C}{v} \delta E_L \right) \right]
\] (3.7c)
\[
\dot{X} = g_X = -v \cos \phi \sqrt{1 - \mu^2} \frac{\delta B_\parallel}{B_0} + \frac{c}{\sqrt{2}B_0} \left[ \delta E_R - \delta E_L - \mu \frac{C}{v} (\delta B_R + \delta B_L) \right]
\] (3.7d)
\[
\dot{Y} = g_Y = -v \sin \phi \sqrt{1 - \mu^2} \frac{\delta B_\parallel}{B_0} - \frac{c}{\sqrt{2}B_0} \left[ \delta E_R + \delta E_L + \mu \frac{C}{v} (\delta B_L - \delta B_R) \right]
\] (3.7e)
\[
\dot{Z} = g_Z = 0
\] (3.7f)

By making use of equations (3.5), (3.6) and (3.7a) through (3.7f), the collisionless Boltzmann equation (3.1) can be transformed to the coordinates
\[
x_\sigma = (p, \mu, \phi, X, Y, Z)
\] (3.8)

The transformation yields the Vlasov equation in guiding center coordinates. It reads
\[
\frac{\partial f_\alpha}{\partial t} + v_\mu \frac{\partial f_\alpha}{\partial Z} - \Omega_\alpha \frac{\partial f_\alpha}{\partial \phi} = S_\alpha(x, p, t) - \frac{1}{p^2} \frac{\partial}{\partial x_\sigma} \left( p^2 g_{x_\sigma} f_\alpha \right)
\] (3.9)

Here, the fluctuating force fields $g_{x_\sigma}$, appearing in the second term of the right-hand side, are given by the expressions (3.7a)-(3.7f).
3.2 The Fokker-Planck Equation

Since the fluctuating force fields \( g_{x,\sigma} \), equations (3.7a) through (3.7f), act irregularly on \( f_{\alpha} \), an ensemble of distribution functions is commonly considered. This has the advantage that statistical properties of \( g_{x,\sigma} \) are included into an expectation value of \( f_{\alpha} \). For the following considerations, the average over all members of the ensemble is introduced by \( < f_{\alpha} > \). Furthermore, the function \( f_{\alpha} \) is split into a sum of two contributions denoted with \( f_{\alpha,0} \) and \( f_{\alpha,1} \), where \( |f_{\alpha,1}| \ll |f_{\alpha,0}| \). Since fluctuations are irregularly, the ensemble average with respect to \( f_{\alpha,0} \) vanishes. Taking the ensemble average of the Vlasov equation in guiding center coordinates, equation (3.9), one arrives at

\[
\frac{\partial < f_{\alpha,0} >}{\partial t} + v_\mu \frac{\partial < f_{\alpha,0} >}{\partial Z} - \Omega_\alpha \frac{\partial < f_{\alpha,0} >}{\partial \phi} = S_\alpha(x, p, t) - \frac{1}{p^2} \frac{\partial}{\partial x_\sigma} \left( < p^2 g_{x,\sigma} f_{\alpha,1} > \right) \tag{3.10}
\]

Subtraction of the averaged Vlasov equation (3.10) from its initial version (3.9) results in the following differential equation:

\[
\frac{\partial f_{\alpha,1}}{\partial t} + v_\mu \frac{\partial f_{\alpha,1}}{\partial Z} - \Omega_\alpha \frac{\partial f_{\alpha,1}}{\partial \phi} \simeq -g_{x,\sigma} \frac{\partial < f_{\alpha,0} >}{\partial x_\sigma} \tag{3.11}
\]

This can be solved by using the method of characteristics, yielding

\[
f_{\alpha,1}(t) = f_{\alpha,1}(t_0) - \int_{t_0}^{t} d\xi \left[ g_{x,\sigma}(x_\nu, \xi) \frac{\partial}{\partial x_\sigma} < f_{\alpha,0} > \right] \tag{3.12}
\]

The expression in brackets has to be calculated for an unperturbed particle orbit in the uniform magnetic field, i.e. along the characteristics. For the derivation of equation (3.11), it was assumed that the phase space density of the particles is uncorrelated to the fluctuating force fields so that \( < f_{\alpha,1} g_{x,\sigma} > = 0 \). Furthermore, the calculation is restricted to small fluctuations. This means, a time scale exists for which the force fields \( g_{x,\sigma} \) affect the particle distribution significantly small so that the variation of \( f_{\alpha,1} \) remains much smaller than \( < f_{\alpha,0} > \).

In order to provide for an equation which is more appropriate for the mathematical treatment of the diffusive transport of charged particles, one usually substitutes (3.12) into the ensemble averaged equation (3.10). Furthermore, one assumes a correlation time scale \( t_c \) beyond which the electromagnetic irregularities fall, at a time \( t - s > t_c \), to negligible magnitudes. Since time scales much longer than the correlation time scale \( t_c \) are considered, the calculations are restricted to stationary fluctuations. With these conditions the average of (3.10) over all members of the ensemble can be rearranged to get a diffusion equation. This fundamental transport equation, which involves second-order correlation functions for the fluctuating force fields \( g_{x,\sigma} \), is the so-called Fokker-Planck equation

\[
\frac{\partial < f_{\alpha,0} >}{\partial t} + v_\mu \frac{\partial < f_{\alpha,0} >}{\partial Z} - \Omega_\alpha \frac{\partial < f_{\alpha,0} >}{\partial \phi} = S_\alpha(x, p, t) + \frac{1}{p^2} \frac{\partial}{\partial x_\sigma} \left[ p^2 D_{x_\mu x_\nu} \frac{\partial < f_{\alpha,0} >}{\partial x_\nu} \right] \tag{3.13}
\]

where the Fokker-Planck coefficients

\[
D_{x_\mu x_\nu} = \Re \int_0^\infty ds < \bar{g}_{x,\sigma}(t) \bar{g}_{x,\sigma}^*(t + s) > \tag{3.14}
\]

define interactions of non-thermal particles with the turbulence via a homogeneous integral, also referred to as Taylor-Green-Kubo formula. The fluctuating force fields \( \bar{g}_{x,\sigma}(t) \) have to be calculated along the unperturbed particle orbit. In order to calculate the fundamental Fokker-Planck coefficients (3.14), equations (3.7a) through (3.7f) have to be determined for unperturbed particle orbits. This is the topic of appendix A.
3.3 The Diffusion-Convection Transport Equation

The Fokker-Planck equation (3.13) enables one to consider a wide variety of physical scenarios and their associated underlying physical processes in which particles interact with a plasma turbulence. However, solving the Fokker-Planck equation with the full set of Fokker-Planck coefficients (which have to be calculated for a certain turbulence model) is very complicated. Therefore, it is common to consider the Fokker-Planck equation on the basis of the so-called diffusion approximation.

3.3.1 The Diffusion Approximation

The basic assumption of the diffusion approximation is the following: the particle phase space distribution \( f_\alpha \) adjusts very rapidly to a quasi-equilibrium state due to pitch angle diffusion. Writing out the interaction term in equation (3.13), one would obtain 25 terms describing interactions of particles with electromagnetic irregularities via their corresponding 25 Fokker-Planck coefficients. Restricting the considerations to the fastest interaction processes, namely diffusion in the pitch angle and diffusion in the gyrophase, a standard analysis can be used (see, e.g., Jokipii, 1966; Schlickeiser, 2002). For this, the ensemble averaged particle distribution function is split into a quasi-isotropic contribution \( f_{\alpha,i} \) and an anisotropic part \( f_{\alpha,a} \), i.e.

\[
<f_{\alpha,0}> = f_{\alpha,i} + f_{\alpha,a} \tag{3.15}
\]

where

\[
f_{\alpha,i} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-1}^1 d\mu <f_{\alpha,0}> \quad \text{and} \quad 0 = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-1}^1 d\mu f_{\alpha,a} \tag{3.16}
\]

The diffusion limit only applies if the particle density varies slowly in time and space and, furthermore, if the particles have time to adjust locally to a quasi-isotropic equilibrium, so that \( f_{\alpha,a} \ll f_{\alpha,i} \). The derivation of the diffusion-convection transport equation is very similar to the derivation of the Fokker-Planck equation (3.13) derived above. Averaging the Fokker-Planck equation (3.13) with respect to \( \mu \) and \( \phi \) yields

\[
\frac{\partial f_{\alpha,i}}{\partial t} - S_{\alpha}(x, p, t) + \nabla \cdot \int_0^{2\pi} d\phi \int_{-1}^1 d\mu f_{\alpha,a}
\]

\[
= \frac{1}{4\pi p^2} \frac{\partial}{\partial p} \left[ 2\pi \int_0^{2\pi} d\phi \int_{-1}^1 d\mu p^2 \left( D_{pp} \frac{\partial f_{\alpha,i}}{\partial p} + D_{pX} \frac{\partial f_{\alpha,i}}{\partial X} + D_{pY} \frac{\partial f_{\alpha,i}}{\partial Y} \right) \right]
\]

\[
+ \frac{1}{4\pi} \frac{\partial}{\partial X} \left[ 2\pi \int_0^{2\pi} d\phi \int_{-1}^1 d\mu \left( D_{Xp} \frac{\partial f_{\alpha,i}}{\partial p} + D_{XX} \frac{\partial f_{\alpha,i}}{\partial X} + D_{XY} \frac{\partial f_{\alpha,i}}{\partial Y} \right) \right]
\]

\[
+ \frac{1}{4\pi} \frac{\partial}{\partial Y} \left[ 2\pi \int_0^{2\pi} d\phi \int_{-1}^1 d\mu \left( D_{Yp} \frac{\partial f_{\alpha,i}}{\partial p} + D_{XY} \frac{\partial f_{\alpha,i}}{\partial X} + D_{YY} \frac{\partial f_{\alpha,i}}{\partial Y} \right) \right]
\]

\[
+ \frac{1}{4\pi p^2} \frac{\partial}{\partial x_n} \cdot \left[ \int_0^{2\pi} d\phi \int_{-1}^1 d\mu p^2 \left( D_{nm} \frac{\partial f_{\alpha,a}}{\partial x_m} \right) \right]
\]

(3.17)

Here, it is assumed that the anisotropic contribution \( f_{\alpha,a} \) is a periodic function with respect to \( \phi \), i.e. \( f_{\alpha,a}(\phi) = f_{\alpha,a}(\phi + 2\pi) \). Furthermore, the condition \( D_{\mu x_n} = 0 \) for \( |\mu| = 1 \) was used. Subtraction
of the averaged equation (3.17) from the initial Fokker-Planck equation (3.13) results in

$$
\nuZ \frac{\partial f_{\alpha,i}}{\partial Z} - \frac{\partial}{\partial \mu} \left[ D_{\mu \mu} \frac{\partial f_{\alpha,i}}{\partial \mu} \right] - \frac{\partial}{\partial \mu} \left[ D_{\mu \alpha} \frac{\partial f_{\alpha,i}}{\partial X} \right] - \frac{\partial}{\partial \mu} \left[ D_{\mu \phi} \frac{\partial f_{\alpha,i}}{\partial \phi} \right] 
+ \frac{\partial}{\partial \phi} \left[ D_{\phi \mu} \frac{\partial f_{\alpha,i}}{\partial \mu} \right] + \frac{\partial}{\partial \phi} \left[ D_{\phi \phi} \frac{\partial f_{\alpha,i}}{\partial \phi} \right] 
\approx \left[ \Omega_\alpha + \frac{\partial}{\partial \mu} D_{\mu \phi} \right] \frac{\partial f_{\alpha,a}}{\partial \phi} + \frac{\partial}{\partial \phi} \left[ D_{\phi \phi} \frac{\partial f_{\alpha,a}}{\partial \phi} \right] + \frac{\partial}{\partial \mu} \left[ D_{\mu \mu} \frac{\partial f_{\alpha,a}}{\partial \mu} \right]
$$

(3.18)

Making use of the \(\phi\)-periodicity of \(f_{\alpha,a}\), one can express \(f_{\alpha,a}\) as a Fourier series,

$$
f_{\alpha,a} = \sum_{m=-\infty}^{\infty} f_{\alpha,a}^{(m)} \exp(ime) \tag{3.19}
$$

with which one can derive an auxiliary equation. The latter is useful to determine the anisotropic part of the phase space density:

$$
\frac{\partial}{\partial \mu} \left[ D_{\mu \mu} \frac{\partial f_{\alpha,a}}{\partial \mu} + D_{\mu p} \frac{\partial f_{\alpha,i}}{\partial p} \right] = vZ \frac{\partial f_{\alpha,i}}{\partial Z} \tag{3.20}
$$

Integrating equation (3.20) with respect to the pitch angle \(\mu\), determining appropriate integration constants for which it is required that \(D_{\mu \mu} = D_{\mu p} = 0\) for \(|\mu| = 1\), and using the right-handed condition of (3.16), one derives for the anisotropic contribution the expression

$$
f_{\alpha,a} = \frac{vZ}{4} \frac{\partial f_{\alpha,a}}{\partial Z} \left[ \int_1^{-1} d\mu \frac{(1 - \mu^2)(1 - \mu)}{D_{\mu \mu}(\mu)} - 2 \int_1^{\mu} d\rho \frac{1 - \rho^2}{D_{\mu \mu}(\rho)} \right]
+ \frac{1}{2} \frac{\partial f_{\alpha,i}}{\partial p} \left[ \int_1^{-1} d\mu \frac{D_{\mu p}(\mu)}{D_{\mu \mu}(\mu)} - 2 \int_1^{\mu} d\rho \frac{D_{\mu p}(\rho)}{D_{\mu \mu}(\rho)} \right] \tag{3.21}
$$

Substitution of the anisotropy (3.21) into the averaged Fokker-Planck equation (3.17) yields, after collecting and rearranging terms, an isotropic transport equation describing the diffusive particle transport in a plasma at rest. In general, the plasma moves with respect to the observer, so that the bulk velocity \(\mathbf{V}\) has to be taken into account. Examples for such scenarios are the galactic wind and, more local, the solar wind plasma. In order to include the bulk speed of such a plasma, several approaches have been presented in the past. First, in the case that the plasma moves with a relativistic bulk speed, it is convenient to consider the diffusion approximation in a mixed comoving coordinate system (see, e.g., Kirk et al., 1988). There, the space coordinates are measured in the laboratory system, while the momentum coordinates of the particles are measured in the rest frame of the moving background plasma. For non-relativistic streaming plasmas, it is sufficient to apply a Galilean transformation\(^1\). Without going into details, the fundamental transport equation then reads as follows:

$$
\frac{\partial f_{\alpha,i}}{\partial t} - S_\alpha(x, p, t) = \frac{\partial}{\partial Z} \left[ \kappa_{\alpha \alpha} \frac{\partial f_{\alpha,i}}{\partial Z} \right] - \left[ \frac{1}{4p^2} \frac{\partial}{\partial p} \left( p^2 u_{a1} \right) \right] \frac{\partial f_{\alpha,i}}{\partial Z}
+ \frac{\partial}{\partial X} \left[ \kappa_{X \alpha} \frac{\partial f_{\alpha,i}}{\partial X} + \kappa_{X Y} \frac{\partial f_{\alpha,i}}{\partial Y} \right]
+ \frac{\partial}{\partial Y} \left[ \kappa_{Y \alpha} \frac{\partial f_{\alpha,i}}{\partial Y} + \kappa_{Y X} \frac{\partial f_{\alpha,i}}{\partial X} \right]
+ \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 a_2 \frac{\partial f_{\alpha,i}}{\partial p} \right) + \left[ \frac{p}{3} \frac{\partial \mathbf{V}}{\partial Z} + \frac{v}{4} \frac{\partial u_{a1}}{\partial Z} \right] \frac{\partial f_{\alpha,i}}{\partial p} \tag{3.22}
$$

\(^1\) Assuming a turbulence consisting of plasma wave modes, Schlickeiser [1989b] showed that the non-relativistic streaming of a plasma can also be taken into account by incorporating the bulk speed into the plasma wave dispersion relations.
Equation (3.22), which is also referred to as the diffusion-convection equation, describes the evolution of the quasi-isotropic pitch angle averaged phase space distribution function \( f_{\alpha,i} \). It contains diffusion and convection terms in both the spatial configuration as well as momentum space. The transport parameters, representing fundamental processes of importance for the diffusive transport of energetic particles, are given by pitch angle integrals over the Fokker-Planck coefficients (A.8) through (A.14):

\[
\kappa_{ZZ} = \frac{v^2}{8} \int_{1}^{1} d\mu \frac{(1 - \mu^2)^2}{D_{\mu\mu}(\mu)} \quad (3.23a)
\]

\[
\kappa_{lm} = \frac{1}{2} \int_{1}^{1} d\mu D_{lm}(\mu) \quad (l, m = X, Y) \quad (3.23b)
\]

\[
a_1 = \int_{1}^{1} d\mu \frac{(1 - \mu^2) D_{\mu\mu}(\mu)}{D_{\mu\mu}(\mu)} \quad (3.23c)
\]

\[
a_2 = \frac{1}{2} \int_{1}^{1} d\mu \left( D_{pp}(\mu) - \frac{D_{\mu\mu}(\mu)^2}{D_{\mu\mu}(\mu)} \right) \quad (3.23d)
\]

The spatial diffusion coefficient \( \kappa_{ZZ} = \kappa_{||} \) provides for the parallel particle diffusion with respect to the background magnetic field. The transport parameters \( \kappa_{lm} \) (with \( l, m = X, Y \)) represent spatial particle diffusion perpendicular to the ambient magnetic field \( (\kappa_{XY}, \kappa_{YX}) \) and, presumably, particle drifts \( (\kappa_{XY}, \kappa_{YX}) \). Since equation (3.22) holds in the mixed comoving system, the rate of adiabatic cooling due to the turbulence, i.e. \( a_1 \), occurs as a separated term from the adiabatic deceleration rate \( p(\partial V/\partial Z)/3 \) in the last term on the right-hand side of equation (3.22). The latter term results from the plasma bulk speed \( V \). Furthermore, it appears as a separated contribution in the expression describing spatial convection. This is the second term on the right-hand side of equation (3.22). The momentum diffusion coefficient, \( a_2 \), provides for the acceleration of particles, e.g. the energization of cosmic rays in the interstellar medium or the stochastic acceleration of PUIs within the solar wind plasma. This is also referred to as the second-order Fermi acceleration or Fermi-II process. Depending on the signs of the gradients in the momentum convection term and, furthermore, on the sign of \( a_1 \), adiabatic deceleration can switch into acceleration of particles. This is referred to as the first-order Fermi acceleration or Fermi-I process.

Under certain circumstances it can be necessary to take into account continuous or catastrophic momentum loss processes. Examples for this are synchrotron radiation and spallation, respectively. The time scale on which the loss mechanisms operate may be comparable to the diffusion and acceleration time scales, so that two additional terms have to be included into the diffusion-convection transport equation (3.22). Introducing for \( f_{\alpha,i} \) the notation \( f_{\alpha} \), equation (3.22) is expressible as

\[
\frac{\partial f_{\alpha}}{\partial t} - S_{\alpha}(x, p, t) = \nabla \cdot (K \cdot \nabla f_{\alpha}) - \left[ V + \frac{1}{4p^2} \frac{\partial}{\partial p} \left( p^2 a_1 \right) \right] \cdot \nabla f_{\alpha} + \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 a_2 \frac{\partial f_{\alpha}}{\partial p} \right) + \frac{2}{3} \nabla \cdot V + \frac{\nabla \cdot a_1}{4} \frac{\partial f_{\alpha}}{\partial p} - \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 \dot{\rho} f_{\alpha} \right) - \frac{f_{\alpha}}{T_{\epsilon}} \quad (3.24)
\]

The last two terms on the right-hand side are related to continuous and catastrophic loss processes, with \( \dot{\rho}(x, p) \) and \( T_{\epsilon}(x, p) \) being the continuous loss rate and the catastrophic loss time scale, respectively. The spatial diffusion coefficients (3.23a) and (3.23b) enter equation (3.24) via the spatial diffusion tensor \( K \). Equation (3.24) is the fundamental transport equation. It finds comprehensive applications in the physics of the heliosphere as well as in astrophysical problems such as the
acceleration of particles in accretion flows. Depending on the physical situation of interest, all transport parameters may arise and have to be determined by calculating the Fokker-Planck coefficients (A.8) through (A.14). All Fokker-Planck coefficients and, therefore, all transport parameters are solely determined by the statistical properties of the underlying turbulence. Since time scales of the diffusion and acceleration processes can differ quite significantly, one may neglect several transport parameters for certain physical scenarios. For instance, chapter 7 is related to stochastic acceleration of pick-up ions. There, all spatial diffusion coefficients are neglected so that Fermi-II acceleration is the dominant process. Therefore, the transport parameter \( q_2 \) has to be calculated for this scenario. In contrast to this, the heliospheric transport of GCRs and ACRs is considered in chapter 8. There, diffusion in spatial configuration space overwhelms stochastic acceleration within the interplanetary plasma so that \( q_2 \) is usually set to zero. Hence, to solve a simplified version of the diffusion-convection equation (3.22) all spatial diffusion coefficients have to be specified.

### 3.3.2 The Diffusion Tensor

The main emphasis of a variety of studies is the determination of the spatial diffusion tensor \( K \), entering the diffusion-convection equation (3.24) by the first term on the right-hand side. In the orthogonal coordinate system with one axis parallel to the ambient magnetic field \( B_0 = B_0 e_z \), the diffusion tensor \( K \) can be expressed as

\[
K = K_S + K_T + K_D = \begin{pmatrix}
    \kappa_{XX} & 0 & 0 \\
    0 & \kappa_{YY} & 0 \\
    0 & 0 & \kappa_\parallel
\end{pmatrix} + \begin{pmatrix}
    0 & \kappa_{XY} & 0 \\
    \kappa_{YX} & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
    0 & \kappa_0 & 0 \\
    \kappa_0 & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix}
\]

The first tensor, \( K_S \), refers to the parallel diffusion coefficient and the two directions perpendicular to the mean magnetic field \( B_0 \). Since the off-diagonal elements \( \kappa_{XY} \) and \( \kappa_{YX} \) of the second tensor, \( K_T \), are still undetermined, an interpretation of \( K_T \) cannot be given so far. However, it is supposed that they are related to drifts of charged particles, presumably due to the turbulence itself. If \( K_T \) represents drifts then it is expected that it has an antisymmetric structure with regard to its off-diagonal elements. In section 4.6 it will be shown that both elements of \( K_T \) satisfy the condition \( \kappa_{XY} = -\kappa_{YX} = \kappa_T \) with \( \kappa_T \) being a so far uncertain drift transport parameter. Then, it indeed reveals an antisymmetric structure. In section 6.2.1 it is shown that \( \kappa_T \) is a result of the well-known \( E \times B \) drift. The third tensor, \( K_D \), is attributed to curvature and gradient particle drifts in the large-scale averaged magnetic field. Note that the latter tensor has strictly to be distinguished from \( \kappa_T \). Initially, the inclusion of drift effects was first pioneered by Jokipii et al. [1977]. On the basis of a simplified version of equation (3.24), which is usually referred to as Parker’s equation (Parker, 1958b; 1965) for \( K_T = K_D = 0, a_1 = 0, a_2 = 0 \) and without loss terms, they recognized the importance of large-scale drifts of GCRs in the heliospheric background magnetic field. In order to take into account such drift effects they supplemented Parker’s equation by an appropriate term, an additional contribution to the convective bulk velocity \( V \). For an Archimedean spiral pattern of \( B_0 \), this contribution is given by the drift velocity \( V_D = cvp/(3q_0)|\nabla \times B_0|/B_0^2 \) (see, e.g., Parker, 1957; Fisk and Schwadron, 1995). It changes its sign with the reversal of the magnetic field polarity and particle charge. The terms including the drift velocity can, in principle, be separated from \( V \), so that the drift effects are expressed in terms of an additional diffusion tensor. The latter is nothing else than \( K_D \) in equation (3.25). The off-diagonal elements of \( K_D \) may then be written as

\[
\kappa_D = \frac{cvp}{3q_0B_0}
\]

The representation (3.23b), which gives the perpendicular diffusion coefficients \( \kappa_{XX} \) and \( \kappa_{YY} \) as well as the drift elements \( \kappa_{XY} \) and \( \kappa_{YX} \) as simple pitch angle averages of the Fokker-Planck coefficients (A.11) to (A.14), respectively, has to be considered carefully. Actually, three dimensional
spatial particle diffusion is expected to be strongly anisotropic. However, the transport equation (3.24) is valid only for a quasi-isotropic distribution function $f_\alpha$. Moreover, the periodicity of $f_\alpha$ in the particle’s gyrophase $\phi$, which is used for the derivation of equation (3.24), is questionable, since the gyro-orbit of the particle is not closed after one gyration. The assumed periodicity introduces strong restrictions in so far as the concept of perpendicular diffusion and drifts requires a phase space distribution $f_\alpha$ which is non-periodic in the gyrophase $\phi$. Processes such as perpendicular diffusion and particle drifts will, in generally, change the particle’s direction of motion. In other words, these mechanisms provide for a particle transport across magnetic field lines. In order to explain this cross field diffusion, three models have been presented so far during the last four decades, namely the hard-sphere scattering model, a quasi-linear approach and, more recently, a non-linear theory. These models have in common that the perpendicular diffusion coefficient, which is usually denoted with $\kappa_\perp$, and the antisymmetric term $\kappa_D$ can be expressed by formulas of the same form. In brief, both transport parameters are given by
\begin{align}
\kappa_\perp &= \frac{v R_L}{3} \frac{\Omega_\alpha \tau}{1 + (\Omega_\alpha \tau)^2}, \\
\kappa_D &= \frac{v R_L}{3} \frac{(\Omega_\alpha \tau)^2}{1 + (\Omega_\alpha \tau)^2},
\end{align}
where $R_L$, $\Omega_\alpha$ and $\tau$ denote the particle Larmor radius, the relativistic gyrofrequency and a time scale, respectively. The latter represents a rate at which the trajectory of a particle decorrelates from a helical orbit, due to the interactions of the particle with the plasma turbulence. Generally, it is expected that the interactions of a particle with the turbulence will lead to different decorrelation rates of the perpendicular and parallel particle’s velocities (see Bieber and Mattheaus, 1997). In this sense, the time scale $\tau$ is related to a decorrelation rate of the perpendicular velocity component of a particle. Introducing for $\kappa_{XX}$ and $\kappa_{YY}$ the standard notation $\kappa_{\perp,1}$ and $\kappa_{\perp,2}$, respectively, the perpendicular diffusion coefficient $\kappa_\perp$, given by equation (3.27), would correspond in $K_S$ to the case $\kappa_{\perp,1} = \kappa_{\perp,2}$, so that spatial diffusion is anisotropic only in parallel and perpendicular direction. However, it is generally expected that particle diffusion in both perpendicular directions has an anisotropic character too, so that $\kappa_{\perp,1} \neq \kappa_{\perp,2}$ (see, e.g., Jokipii and Kota, 1995; Potgieter, 1997). Although revealing identical forms for $\kappa_\perp$ and $\kappa_D$, all three models are strongly different with respect to the crucial quantity $\Omega_\alpha \tau$ and its physical interpretation.

Considering first the isotropic hard-sphere scattering model introduced by Gleeson [1969] and Forman and Gleeson [1975], it is assumed that a particle is scattered by hard-spheres, i.e. obstacles, during its gyro-orbit. For this case, it has been argued that the decorrelation time scale can be defined by $\lambda_\parallel = v \tau$ where $\lambda_\parallel = 3 \kappa_\parallel / v$ is the mean free path in parallel direction (e.g., Bieber, 1998; Giacalone and Jokipii, 1999). Thus, for the hard-sphere scattering model, the dimensionless parameter $\Omega_\alpha \tau$ can be cast into the form $\Omega_\alpha \tau = \lambda_\parallel / R_L$. With this, equation (3.27) yields
\begin{align}
\frac{\kappa_\perp}{\kappa_\parallel} &= \frac{1}{1 + \lambda_\parallel^2 / R_L^2}.
\end{align}
For high energy cosmic rays, the parallel mean free path is generally much larger than the particle Larmor radius, so that $\Omega_\alpha \tau \gg 1$. In other words, a particle executes many gyrations between two interactions with the turbulence. As a consequence of this weak scattering limit, the perpendicular diffusion coefficient $\kappa_\perp$ has been usually taken to be much smaller than $\kappa_\parallel$, i.e. $\kappa_\perp / \kappa_\parallel \ll 1$: a result which finds many applications in heliospheric physics. Considering $\kappa_D$, given by equation (3.28), the weak scattering limit leads to $\kappa_D \simeq cvp / (3q_0 B_0)$. This is commonly used in numerical modulation models (see, e.g., Potgieter and Moraal, 1985; Potgieter and le Roux, 1994; Burger and Hattingh, 1998).
Chapter 3 Equations of Transport

The second model, which has been developed by Forman et al. [1974], is based on quasi-linear theory. Here the crucial parameter $\Omega_{\alpha}\tau$ is given by the relation $\Omega_{\alpha}\tau = 2R_L B_0^2/(3l_\perp \delta B_0^2)$ where $l_\perp$ is the correlation length of the purely magnetic turbulence, while $\delta B_0^2$ is the intensity of the fluctuating field. In the weak scattering limit, i.e. $\Omega_{\alpha}\tau \gg 1$, equation (3.27) results for the perpendicular diffusion coefficient in

$$\kappa_\perp \simeq \frac{v_l c}{2} \left( \frac{\delta B_e}{B_0} \right)^2$$

(3.30)

whereas $\kappa_D$ is still given by the hard-sphere scattering result. In contrast to the hard-sphere approach and its associated particle scattering at obstacles, it has been argued that field line random walk (FLRW) is the most important process providing for perpendicular diffusion. Initially, the concept of FLRW was first introduced by Jokipii [1966] and Jokipii and Parker [1969]. They suggested that random fluctuations in the direction of the magnetic background field result in random displacements of the lines of force relative to each other. In other words, the field lines are braided in transverse direction, i.e. they execute a random walk in normal direction. Since an individual charged particle has the tendency to stay with a magnetic field line, this particle can move normal to the average magnetic field. The corresponding perpendicular field diffusion and, therefore, the transverse diffusion of charged particles can be expressed by a diffusion coefficient. The perpendicular deflection of the particles can then be expressed as a function of distance along the field (see, e.g., Giacalone and Jokipii, 1999). Since it has been indicated that slab geometry is only a minority component of the solar wind turbulence (see Matthaeus et al., 1990; Bieber et al., 1996), and since the quasi-linear approach is valid only for slab geometry as well as small-amplitude fluctuations and high energies, a third model has been introduced more recently.

The third model, introduced recently by Bieber and Matthaeus [1997], is developed on a non-linear approach. It leads to a perpendicular diffusion and drift coefficient given by (3.27) and (3.28), respectively. In contrast to the first two models, the crucial quantity $\Omega_{\alpha}\tau$ is here given by the expression $\Omega_{\alpha}\tau = 2R_L/(3D_\perp)$ where $D_\perp$ describes FLRW. This theory generalizes the quasi-linear approach in so far as it allows to include, via the non-linear coefficient $D_\perp$, more general turbulence geometries, e.g. a composition of a two-dimensional and a slab geometry (Bieber et al., 1996). In addition, this approach is not restricted to high energies of the particles. Since the non-linear approach by Bieber and Matthaeus [1997] was able to explain several observations, for example such as the rigidity dependence of $\kappa_\perp$ obtained from ACR spectra, it can be considered as the best existing theory so far.

Commonly, for the drift coefficient $\kappa_D$ the hard-sphere scattering result is assumed in modulation models, whereas empirical forms are usually used for $\kappa_\parallel$, $\kappa_\perp,1$ and $\kappa_\perp,2$. These empirical formulas are deduced by fitting data obtained by in-situ measurements made on several spacecraft. The dependences on energy and heliocentric distance of the parallel and perpendicular diffusion coefficients can then be extracted and estimated from such data fits.

In many studies the transport equation (3.24) and the diffusion tensor (3.25) are often used in a spherical polar coordinate system $(r, \theta, \varphi)$ where $r$, $\theta$ and $\varphi$ represent the heliocentric distance, the heliographic co-latitude and the longitude, respectively. Assuming that the large-scale heliospheric magnetic field is related to the rotating solar magnetic dipole, the spiral field geometry of $B_0$ can be represented by the Parker field (Parker, 1958a)

$$B_0 = B_E \left( \frac{r_E}{r} \right)^2 (\mathbf{e}_r - \tan \psi \mathbf{e}_\varphi) \quad \text{with} \quad \tan \psi = \frac{\Omega_{\odot}(r - r_{\odot})}{V} \sin \theta$$

(3.31)

where $B_E$ refers to the heliospheric magnetic field at Earth’s orbit $r_E$. $\Omega_{\odot} = 2.9 \cdot 10^{-6}$ s$^{-1}$ and $r_{\odot}$ are the angular speed and the radius of the Sun, respectively. The vectors $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ are unit vectors in the radial and the azimuthal direction, respectively. The opposite directions of the Parker
field in the northern and southern hemisphere are separated by the heliospheric current sheet. The latter one is roughly located at the heliospheric equatorial plane, i.e. \( \theta \approx \pi/2 \). Under the condition of an assumed Parker field geometry, the diffusion tensor \((3.25)\) in spherical coordinates reads

\[
K = \begin{pmatrix}
\sin \psi & 0 & -\cos \psi \\
0 & 1 & 0 \\
\cos \psi & 0 & \sin \psi
\end{pmatrix} \begin{pmatrix}
\kappa_{XX} & \kappa_{XY} - \kappa_{D} & 0 \\
\kappa_{XY} + \kappa_{D} & \kappa_{YY} & 0 \\
0 & 0 & \kappa_{||}
\end{pmatrix} \begin{pmatrix}
\sin \psi & 0 & \cos \psi \\
0 & 1 & 0 \\
-\cos \psi & 0 & \sin \psi
\end{pmatrix}
\]

(3.32)

For spherical symmetry, the diffusion tensor \((3.32)\) has to be incorporated into the correspondingly transformed diffusion-convection equation \((3.24)\). For \(\kappa_{XY} = \kappa_{YX} = 0\), equation \((3.24)\) was solved numerically, in combination with \((3.32)\), in a variety of studies performed in the past.

### 3.4 Summary and Conclusions

In this chapter the fundamental transport equations and their hierarchy are introduced. Based on the Vlasov equation in guiding center coordinates, it is demonstrated how the fundamental Fokker-Planck equation can be derived from quasi-linear theory. The interactions of non-thermal particles with the underlying turbulence enter this equation via the so-called Fokker-Planck coefficients. These coefficients are defined as homogeneous integrals along unperturbed particle orbits of the fluctuating force fields. The latter involve the irregular electromagnetic fields of the turbulence as second-order correlation functions. Upon using this integral representation and the fluctuating force terms, the derivations of the most important Fokker-Planck coefficients are shortly presented in appendix A.

Under the assumption that diffusion in the particle’s pitch angle and gyrophase are the fastest processes, the standard diffusion approximation is applied to the Fokker-Planck equation. Following the classical approach, the particle phase space distribution function is split into an isotropic and anisotropic part. The average of the Fokker-Planck equation with respect to the particle’s pitch angle and gyrophase results in the diffusion-convection transport equation. This equation describes the temporal evolution of the quasi-isotropic pitch angle averaged particle distribution function in both the spatial configuration as well as momentum space. Diffusion and convection in spatial and momentum space are then represented by transport parameters which are defined as pitch angle integrals of the Fokker-Planck coefficients. The parameters of transport enter the diffusion-convection equation in terms of a tensor and two scalars. The first plays a crucial role in cases where diffusion in configuration space becomes important. This spatial diffusion tensor represents, by its components, parallel and perpendicular diffusion of charged particles and, furthermore, effects resulting from drifts.

Generally, perpendicular diffusion and drifts change the particle’s direction of motion and, subsequently, the particle’s gyrophase. This means, the gyro-orbit of a particle is not closed after one gyration. In other words, the particle is offset with regard to its initial position and the condition for periodicity of the phase space distribution is not fulfilled. If the scattering of the particles at the turbulence occurs with sufficient randomness, the particle phase space distribution will become quasi-isotropic, and the distribution function can be split into an isotropic and anisotropic contribution. But, however, the assumed periodicity in the gyrophase is not justified anymore so that the anisotropic part of the distribution function cannot be expressed by a Fourier series. A more realistic approach should also involve, at least via an appropriate Fokker-Planck coefficient \(D_{\phi\phi}\), diffusion in the particle’s gyrophase. Presumably, this would lead to transport parameters revealing...
more complicated dependences on the Fokker-Planck coefficients. Of course, this requires another approach to the diffusion-convection transport equation and its associated (modified) transport parameters, but this is beyond the purview of this thesis.
Chapter 4
On Fokker-Planck Coefficients

The aim of this chapter is to investigate the Fokker-Planck coefficients (hereafter FPCs) introduced in section 3.2 in more detail. The FPCs are defined as time-integrals over second-order correlation functions. These functions are commonly subjected to standard perturbation methods. Then, all FPCs can be written in terms of Bessel functions of the first order. This is explained in detail in appendix A where the most important FPCs are presented. A further reduction of these coefficients can only be achieved if additional assumptions are made with regard to the electromagnetic correlation tensors (A.15a) through (A.15d). These tensors determine solely the nature of the underlying turbulence and, therefore, characterize sensitively all FPCs (A.8) to (A.14). The so-called plasma wave viewpoint is a popular approach to describe the turbulent nature of a background plasma. Electromagnetic fluctuations are then represented by superpositions of individual plasma wave modes. Appendix C establishes the most relevant transverse plasma waves and their important properties. For simplicity, it would be convenient to assume specific plasma wave dispersion relations from the beginning of the calculations. Then, all FPCs would reveal drastic simplifications and collapse to more simple expressions. Under certain circumstances, these expressions can also be evaluated analytically. This has been done in the past for the two most prominent cases. These are the dispersionless Alfvén wave, propagating parallel and antiparallel with respect to the background magnetic field (Schlickeiser, 1989a), and the fast magnetosonic wave propagating in a plane within wavenumber space (Schlickeiser and Miller, 1998; Lerche and Schlickeiser, 2001a). Both wave modes are considered in appendix C by using a kinetic as well as MHD approach.

In contrast to studies performed in the past, all FPCs are treated here from the most general point of view. Particularly, this means that neither a specific plasma wave dispersion relation nor a certain turbulence geometry is assumed. The helical description of the electromagnetic correlation tensors introduced in appendix A is dropped and the Bessel function representation of all FPCs is rearranged. This leads, after laborious calculations, to a new representation for each FPC. The new representations have the advantage that all FPCs are expressible, first, as a sum of three terms and, second, in terms of Cartesian coordinates. All FPCs are valid for an arbitrary turbulence geometry. The properties of the underlying turbulence enter each FPC by distinguished factors characterizing each individual sum contribution. For an isotropic turbulence, each FPC simplifies to a compact expression including an arbitrary plasma wave dispersion relation and, therefore, an arbitrary plasma wave damping function. For the latter case, it is pointed out that calculations performed before are incorrect if plasma wave damping becomes important. The new representations are considered for several limits. These are the cases for a vanishing wave damping, a strong restriction called dynamical magnetic turbulence and, furthermore, the asymptotic limit of a slab geometry. For the latter case, it is shown that particle diffusion perpendicular to the background magnetic field cannot exist, independent of the wave dispersion relation. This is explained by the reduced dimensionality of the slab turbulence in both the wave vector as well as configuration space. Furthermore, the FPCs providing for particle drifts, presumably due to the turbulence itself, are treated in detail.
Chapter 4 On Fokker-Planck Coefficients

4.1 Two Approaches for Modeling the Random Electromagnetic Fields

Two approaches for modeling the plasma turbulence in interplanetary and interstellar space are widespread in the literature. Besides the plasma wave viewpoint mentioned above, a second approach has become popular during the last two decades, namely the so-called dynamical magnetic turbulence model. This model can be considered as a special case (say strong restriction) of the plasma wave approach. Starting with the wave viewpoint as the more general approach, the two “schools of thought” will be introduced in turn.

4.1.1 The Plasma Wave Approach

In general, the calculation of FPCs is straightforward. One is tempted so say that the way of treating such coefficients is comparable with a recipe for cooking: first, one has to make a choice between longitudinal and transverse plasma waves. Since the first type of mode is usually strongly damped in interplanetary and interstellar space, the calculations of FPCs are limited to plasma waves having a transverse character. Second, the electric fields of these transverse plasma waves are then expressed by their magnetic counterparts, via Faraday’s law. At this point the plasma wave dispersion relation enters the calculations. Subsequently, all components of the electromagnetic correlation tensors \( \chi_{\alpha\beta} \) through \( \chi_{\alpha\gamma\delta} \) can be substituted by the corresponding elements of the magnetic correlation tensor \( P_{\alpha\beta} \). This has the advantage that all FPCs are then determined by the magnetic intensities of the turbulence, i.e. \( P_{\alpha\beta} \) (see appendix A).

Although it is straightforward to calculate the electromagnetic correlation tensors \( \chi_{\alpha\beta} \) through \( \chi_{\alpha\gamma\delta} \) for transverse plasma waves, the derivation will be illustrated here briefly. In order to start with the calculations, one usually assumes a superposition of \( N \) individual transverse plasma modes. The dispersion relations of these waves are given by the relation

\[
\omega_j = \omega_{R,j}(k) + i\Gamma_j(k)
\]  

(4.1)

where the subscript \( j \) refers to the superposition of the individual modes, it means \( j = 1, \ldots, N \). \( \omega_{R,j} \) and \( \Gamma_j \) denote frequencies and dissipation rates, due to damping, of the waves, respectively (see appendix C). Faraday’s law implies the origin of electric fields by time-varying magnetic fields, i.e.

\[
\nabla \times \delta E(x, t) = -\frac{1}{c} \frac{\partial \delta B(x, t)}{\partial t}
\]  

(4.2)

Making use of the Fourier representation

\[
\delta B(x, t) \bigg|_{\delta E(x, t)} = \sum_{j=1}^{N} \int d^3k e^{i(k \cdot x - \omega_j t)} \begin{cases} B^j(k) \\ E^j(k) \end{cases}
\]  

(4.3)

one obtains for Faraday’s law the expression

\[
k \times E^j(k) = \frac{\omega_j}{c} B^j(k)
\]  

(4.4)

The latter relation is expressible as

\[
E^j(k) = \frac{\omega_j}{ck^2} B^j(k) \times k
\]  

(4.5)

where the transversality condition \( k \cdot E^j(k) = 0 \) was used. As a first consequence of having expressed the electromagnetic fluctuations as plasma waves, the set of electromagnetic correlation
tensors (A.15a) through (A.15d) becomes

\[
\begin{align*}
P_{\alpha \beta}(k, s) &= < B_{\alpha}(k, t) B_{\beta}^*(k, t + s) > \\
Q_{\alpha \beta}(k, s) &= < E_{\alpha}(k, t) B_{\beta}^*(k, t + s) > \\
T_{\alpha \beta}(k, s) &= < B_{\alpha}(k, t) E_{\beta}^*(k, t + s) > \\
R_{\alpha \beta}(k, s) &= < E_{\alpha}(k, t) E_{\beta}^*(k, t + s) >
\end{align*}
\]

Here, the electromagnetic contributions of each wave mode \( j \) are defined by the following tensors:

\[
\begin{align*}
P_{\alpha \beta}^j(k) &= < B_{\alpha}^j(k) B_{\beta}^j(k) > \\
Q_{\alpha \beta}^j(k) &= < E_{\alpha}^j(k) B_{\beta}^j(k) > \\
T_{\alpha \beta}^j(k) &= < B_{\alpha}^j(k) E_{\beta}^j(k) > \\
R_{\alpha \beta}^j(k) &= < E_{\alpha}^j(k) E_{\beta}^j(k) >
\end{align*}
\]

As a second consequence, the time dependence of the correlation tensors (4.6) is manifested in a simple exponential expression. The \( s \)-integration in the FPCs (A.8) through (A.14) can then easily be performed, yielding

\[
\mathcal{R}_j = \mathcal{R}_j(k, \omega_j, n) = \int_0^\infty ds \exp \left[ -i(k\|v\| - \omega_j + n\Omega) s \right] = -\frac{\Gamma_j(k) - i(k\|v\| - \omega_R,j(k) + n\Omega_\alpha)}{\Gamma_j^2(k) + (k\|v\| - \omega_R,j(k) + n\Omega_\alpha)^2} (4.8)
\]

The real part of (4.8) is a Breit-Wigner type resonance function. For negligible plasma wave damping, i.e. \( \Gamma_j(k) \to 0 \), it can be expressed in terms of Dirac’s delta distribution \( \delta(x) \):

\[
\Re \mathcal{R}_j = \pi \delta(k\|v\| - \omega_R,j(k) + n\Omega_\alpha) (4.9)
\]

The argument of the delta distribution is also referred to as the frequency mismatch parameter. It represents a fundamental key for understanding particle diffusion in both the spatial configuration as well as momentum space, because it describes the nature of resonant wave-particle interactions. For harmonic numbers \( n \neq 0 \), the frequency mismatch parameter is a matching condition between the relativistic particle gyrofrequency and the Doppler-shifted wave frequency in the guiding center frame of the particle. Thus, these interactions are also called gyroresonant interactions. The most effective gyroresonance is \( |n| = 1 \), where \( n = +1 \) is referred to as the cyclotron resonance. The second resonance of importance for particle diffusion is \( n = 0 \). This resonance requires a compressional plasma wave field component aligned along the background magnetic field. An example for a compressional wave is given in section C.2 of appendix C. There it is shown that the transverse fast magnetosonic wave has a compressional magnetic field component parallel to the background magnetic field \( B_0 \). The effect of the \( n = 0 \)-interactions is to change the parallel particle momentum, whereas the perpendicular momentum is unaffected. Such interactions can result in huge particle energy gains and, therefore, in a non-thermal plasma wave damping. This damping, also referred to as transit-time damping, results from an effective exchange of energy via wave-particle interactions. If the compressional plasma wave component is represented by an electric field (which applies in the case of parallel propagating longitudinal waves) the energy exchange is due to the ordinary Landau damping. Therefore, transit-time damping is commonly considered as the magnetic analogue to the classical Landau damping (Stix, 1992).

Since the correlation tensors (A.15a) through (A.15d) are expressed in helical coordinates denoted by the subscripts \( L \), \( R \) and \( \parallel \), it is convenient to express the wave vector in helical coordinates too, so that

\[
k_{L,R} = \frac{1}{\sqrt{2}} (k_x \pm ik_y) \quad \text{and} \quad k_z = k_{\parallel} (4.10)
\]
The use of equation (4.5) then yields

\[ E_{L,R}^j = \mp \frac{i \omega_j}{ck^2} \left( B_{L,R}^j k || - B_{L,R}^j k || \right) \quad \text{and} \quad E_{||}^j = \frac{i \omega_j}{ck^2} \left( B_{L,R}^j k R - B_{L,R}^j k L \right) \]  

(4.11)

In order to express all electric contributions of \( Q_{j \alpha \beta}^j \), \( T_{j \alpha \beta}^j \) and \( R_{j \alpha \beta}^j \) by the corresponding magnetic components of \( P_{j \alpha \beta}^j \), one has to insert equations (4.10) and (4.11) into (4.7). This leads to a set of 27 components. For the sake of clarity, section A.3 gives a list of all these components. In view of this list one eventually could, of course, feel tempted to insert all these components directly into the coefficients (A.8) through (A.14). All FPCs would then be characterized solely by the helical magnetic correlation tensor \( P_{j \alpha \beta}^j \). However, since the helical representations of \( P_{j \alpha \beta}^j \) and \( k \) are introduced only to simplify the mathematical treatment of the FPCs, it is more useful and instructive to transform both the tensor \( P_{j \alpha \beta}^j \) and \( k \) back to Cartesian coordinates before inserting all tensor components into the FPC representations (A.8) to (A.14). For completeness, section A.4 provides a list of all transformed helical magnetic intensities. The elements of the corresponding Cartesian tensor \( P_{j \alpha \beta}^j \), where the subscripts \( l \) and \( m \) are related to the Cartesian coordinates, are very sensitive quantities in so far as they represent the evolution of the turbulence in wave vector space. In other words, \( P_{j \alpha \beta}^j \) provides for the geometry of the underlying turbulence and its temporal evolution. The components of \( P_{j \alpha \beta}^j \) and their forms and dependences on the wave vector characterize all FPCs and, therefore, the transport parameters (3.23a) to (3.23d). In this sense, \( P_{j \alpha \beta}^j \) is the most crucial quantity for the transport of charged particles in a turbulent plasma. Because of the lack of rigorous theories describing \( P_{j \alpha \beta}^j \) in all its details, it is also the most critical parameter entering all calculations. The geometrical concept of \( P_{j \alpha \beta}^j \) is considered in more detail in section 4.3. The wavenumber evolution of the turbulence power spectrum is the subject of chapter 5.

4.1.2 The Dynamical Magnetic Turbulence

Having first established the plasma wave approach, this section describes the second often used concept called dynamical magnetic turbulence. Here the fluctuations are purely magnetic, i.e. all electric components are neglected. To suppress all electric effects, it is instructive to restrict the plasma wave approach to the limit of a vanishing dispersion, i.e. \( \omega_{R,j} = \Gamma_j = 0 \). It immediately becomes clear that this approach leads to drastic simplifications of all FPCs, because \( P_{j \alpha \beta}^j \) is the only remaining correlation tensor (cf. the list in section A.3). The viewpoint of a superposition of individual plasma wave modes does not hold anymore. The subscript \( j \) will, therefore, be suppressed throughout considerations concerning the dynamical magnetic turbulence approach. To take into account the dynamical behavior of such a purely magnetic turbulence Bieber et al. [1994] defined two models. These are the damping and the random sweeping model, for which the temporal evolutions of the magnetic intensities \( P_{j \alpha \beta}^j \) are given by the relations

\[ T_D(k, s) = \exp\left[-\alpha s / s_0(k)\right] \quad ; \quad \text{damping model} \]  

(4.12a)

\[ T_R(k, s) = \exp\left[-\alpha s^2 / s_0^2(k)\right] \quad ; \quad \text{random sweeping model} \]  

(4.12b)

The parameter \( \alpha \in [0, 1] \) allows adjustment of the strength of the dynamical effects. These values are attained if the magnetostatic limit or a strongly dynamical magnetic turbulence are considered, respectively. The arbitrary function \( s_0(k) \) represents the half-life or decay time scale of magnetic fluctuations. In order to include the time functions (4.12a) and (4.12b) into the FPCs (A.8) through (A.14), one has to perform the \( s \)-integration. The real parts of the corresponding resonance
functions for the damping and the random sweeping model are then defined by

\[
\mathcal{R}R_D(k, n) = \mathcal{R} \int_0^\infty ds e^{-\ell(k||u|| + n\Omega_\alpha)s} T_D(k, s)
\]

\[
\mathcal{R}R_R(k, n) = \mathcal{R} \int_0^\infty ds e^{-\ell(k||u|| + n\Omega_\alpha)s} T_R(k, s)
\]

so that

\[
\mathcal{R}R_D(k, n) = \frac{\alpha/s_0(k)}{(k||u|| + n\Omega_\alpha)^2 + (\alpha/s_0(k))^2} ; \text{ damping model} \quad (4.14a)
\]

\[
\mathcal{R}R_R(k, n) = \frac{\sqrt{\pi} s_0(k)}{2\alpha} \exp\left(-\frac{(k||u|| + n\Omega_\alpha)^2 s_0^2(k)}{4\alpha^2}\right) ; \text{ random sweeping model} \quad (4.14b)
\]

Since \(s_0(k)\) characterizes the decay time scale of the turbulence, it represents a temporal decay process in two-points correlations. In analogy to the spectral theory of magnetic fluctuations, which will be described briefly in chapter 5, one can estimate a probable lower limit for \(s_0(k)\). Assuming that the turbulence exhibits an Alfvénic nature, \(s_0(k)\) is commonly approximated by the Alfvén time scale\(^1\) \(\tau_A = 1/(v_A k)\) (see, e.g., Matthaeus and Zhou, 1989; Bieber et al., 1994). Initially, the concept of the dynamical magnetic turbulence was developed for slab geometry (Bieber et al., 1994). However, due to the lack of rigorous theories for the fundamental three-dimensional turbulence physics, the restriction to a slab model is not applied here. For the same reason, it is questionable whether the simple assumption of the Alfvén time scaling of \(s_0(k)\) still holds for a more general treatment. It should be useful to consider the influence of more appropriate decay time scales in future studies.

### 4.2 Fokker-Planck Coefficients in Cartesian Coordinates

After having introduced the relevant quantities for the plasma wave approach and its limitation called dynamical magnetic turbulence, this section is about the FPCs in Cartesian coordinates. The further treatment of the coefficients (A.8) through (A.14) is as follows: first, concerning the Bessel functions of the first kind with order \(n\), it is convenient to make use of the identities (see formula (9.1.27) of Abramowitz and Stegun, 1972)

\[
J_{n-1}(W) + J_{n+1}(W) = \frac{2n}{W} J_n(W) \quad \text{and} \quad J_{n-1}(W) - J_{n+1}(W) = 2J'_n(W) \quad (4.15)
\]

The prime denotes the derivation with respect to the argument \(W\). The advantage of applying these identities to the FPCs (A.8) to (A.14) is the following: as it will be shown below, all coefficients can be expressed by a sum of three terms. Each contribution includes either \(J^2_n(W), J_n(W)J'_n(W)\) or \([J'_n(W)]^2\). Hence, the Bessel functions in all FPCs are of the same order, namely \(n\). Furthermore, each term is accompanied by a specific factor. These factors are different for each term and FPC, and they contain the components of the electromagnetic correlation tensors (A.15a) through (A.15d) and, therefore, the complete information about the underlying turbulence. Hence, these factors are referred to as the fluctuating electromagnetic fields (FEFs). According to the representation (4.11), the irregular electric contributions in these FEFs can be substituted by their magnetic counterparts. This leads to fluctuating magnetic fields (hereafter referred to as FMFs). Finally, the helical magnetic fields and wavenumbers in these FMFs will be expressed by their corresponding Cartesian representations. Then, one readily obtains all FPCs in Cartesian coordinates. Since the

\(^1\)Note that the Alfvén time scale \(\tau_A\) corresponds to the triple correlation decay time scale \(\tau_3\) in the Kraichnan and not the Kolmogorov phenomenology (see chapter 5).
underlying turbulence physics is shifted into the FMFs, the FPCs (A.8) through (A.14) collapse to relatively compact expressions. The rearranged forms reveal the mathematical structure of all FPCs in more detail than in earlier studies. Besides this, the new representations of the FPCs can be considered as an analogy to the conductivity tensor in a homogeneous hot magnetized plasma (see equation (C.57) in appendix C). It should be stressed that the new representations are expressed in terms of an arbitrary magnetic correlation tensor $P_{lm}$, and, therefore, an arbitrary turbulence geometry. This has not been done before in this generality. For ease of exposition, all mathematical details are described only at the example of the most important FPC, namely $D_{\mu\mu}$ given by equation (A.8).

The Fokker-Planck Coefficient $D_{\mu\mu}$

According to equation (3.23a), the coefficient $D_{\mu\mu}$ provides for the parallel spatial diffusion of charged particles in a turbulent plasma. In order to rearrange equation (A.8) the electromagnetic correlation tensors (4.6) are first substituted into the representation (A.8). One obtains, after making use of the Bessel function identities (4.15) and some algebra, the following form:

$$D_{\mu\mu} = \frac{Q_0^2}{2B_0^3} \sum_{j=\pm 1} \sum_{n=-\infty}^\infty \mathbb{R}\int k^3 R j \left[ J_n^2(W)H_1 + J_n(W)J_n'(W)H_2 + [J_n'(W)]^2 H_3 \right]$$

(4.16)

with $R j$ being the resonance function (4.8). Comparing equation (4.16) with its initial version (A.8), it immediately becomes clear that the new representation is relative compact. Further calculations depend on the auxiliary functions $H_1, H_2$ and $H_3$. They determine completely the complicated structure of the FPC (4.16). These auxiliary functions, representing the FEFs mentioned above, are then given by the following somewhat uncomfortable expressions:

$$H_1 = 2 \frac{c^2}{v^2} (1 - \mu^2) R_{||} \mu + \frac{c}{v} \frac{n}{W} \sqrt{2(1 - \mu^2)} \left[ e^{+i\psi} \left( T_{||}^j + Q_{||}^j + \mu \frac{c}{v} \left( R_{||}^j + R_{||}^j \right) \right) ight. \\
- \left. e^{-i\psi} \left( T_{||}^j + Q_{||}^j - \mu \frac{c}{v} \left( R_{||}^j + R_{||}^j \right) \right) \right] \\
+ \frac{n}{W} \frac{2}{W} \left[ P_{RR}^j + P_{LL}^j + \mu^2 \frac{c^2}{v^2} \left( R_{RR}^j + R_{LL}^j \right) + \mu \frac{c}{v} \left( Q_{RR}^j - T_{RR}^j + T_{LL}^j - Q_{LL}^j \right) \right. \\
- \left. e^{+2i\psi} \left( P_{RR}^j - \mu^2 \frac{c^2}{v^2} R_{RR}^j + \mu \frac{c}{v} \left( T_{RR}^j + Q_{RL}^j \right) \right) \right] \\
- \left. e^{-2i\psi} \left( P_{LR}^j - \mu^2 \frac{c^2}{v^2} R_{LR}^j - \mu \frac{c}{v} \left( T_{LR}^j + Q_{RL}^j \right) \right) \right]$$

(4.17)

$$H_2 = 2 \frac{n}{W} \left[ P_{LL}^j - P_{RR}^j + \mu^2 \frac{c^2}{v^2} \left( R_{LL}^j - R_{RR}^j \right) + \mu \frac{c}{v} \left( T_{LL}^j - Q_{LL}^j - Q_{RR}^j + T_{RR}^j \right) \right] \\
+ \frac{c}{W} \sqrt{2(1 - \mu^2)} \left[ e^{+i\psi} \left( Q_{||}^j - T_{||}^j + \mu \frac{c}{v} \left( R_{||}^j - R_{||}^j \right) \right) \right. \\
- \left. e^{-i\psi} \left( T_{||}^j - Q_{||}^j - \mu \frac{c}{v} \left( R_{||}^j - R_{||}^j \right) \right) \right]$$

(4.18)
Here, the following auxiliary functions were introduced:

\[ H_1 = T_{\mu\nu} \left( P_{RR}^j + P_{LL}^j + \mu^2 \frac{c^2}{v^2} \left( R_{RR}^j + R_{LL}^j \right) + \mu \frac{c}{v} \left( Q_{RR}^j - T_{RR}^j + T_{LL}^j - Q_{LL}^j \right) \right) \]

\[ + e^{+2\psi} \left( P_{RL}^j - \mu^2 \frac{c^2}{v^2} R_{RL}^j + \mu \frac{c}{v} \left( T_{RL}^j + Q_{RL}^j \right) \right) \]

\[ + e^{-2\psi} \left( P_{LR}^j - \mu^2 \frac{c^2}{v^2} R_{LR}^j - \mu \frac{c}{v} \left( T_{LR}^j + Q_{LR}^j \right) \right) \] (4.19)

A closer inspection of these auxiliary functions leads to the following intermediate results: first, \( H_1 \) consists of three terms where the first two contributions include the parallel fields of the turbulence. The third term of \( H_1 \) is determined only by the left- and right-handed components. Second, \( H_3 \) contains only left- and right-handed constituents, no parallel contributions occur. Third, \( H_2 \) is determined completely by differences of tensor components, whereas \( H_1 \) and \( H_3 \) include both the difference and the sum of tensor elements. It is shown in section 4.4 that \( H_2 \) represents the influence of the magnetic helicity \( \sigma^j \) on the FPCs and, therefore, on the particle transport. With the exception of the FPCs \( D_{XY} \) and \( D_{YX} \), this statement holds for all FPCs.

According to appendix A, all electric components of \( Q_{\alpha\beta}^j, T_{\alpha\beta}^j \) and \( R_{\alpha\beta}^j \) in equations (4.17) through (4.19) are expressible by their magnetic counterparts, i.e. by the corresponding components of \( P_{\alpha\beta}^j \). Making use of the list given in section A.3 and of equation (4.1), the functions \( H_1, H_2 \) and \( H_3 \) can be cast into the following forms:

\[ H_1 = T_{\mu\nu} \left( P_{RR}^j + P_{LL}^j - P_{RL}^j e^{+2\psi} - P_{LR}^j e^{-2\psi} \right) \] (4.20a)

\[ H_2 = 2S_{\mu\nu} A_{\mu\nu} \left( P_{LL}^j - P_{RR}^j \right) + \sqrt{2} \mu \frac{\omega_j}{v} k \frac{k}{k} S_{\mu\nu} \left[ \left( P_{||L}^j - P_{||R}^j \right) e^{+i\psi} + \left( P_{L}^j - P_{R}^j \right) e^{-i\psi} \right] \]

\[ + 2 \frac{\omega_j}{v} \left( 2 \mu k \frac{n}{W} + k \sqrt{1 - \mu^2} \left( P_{LL}^j - P_{RR}^j \right) (1 - \omega_{R,j}/|\omega_j|) \right) \]

\[ + \sqrt{2} \mu \frac{n}{W} \left( 1 - \omega_j^2 |\omega_j| \right) \left( P_{L}^j e^{+i\psi} - P_{L}^j e^{-i\psi} \right) + \left( 1 - \omega_j^2 |\omega_j| \right) \left( P_{R}^j e^{-i\psi} - P_{L}^j e^{+i\psi} \right) \]

\[ - 2\pi j(k) k \sqrt{1 - \mu^2} \frac{c}{v} \left( P_{LL}^j e^{-2\psi} - P_{LR}^j e^{+2\psi} \right) \] (4.20b)

\[ H_3 = \left( A_{\mu\nu}^2 + 2 \mu \frac{\omega_j}{v} k \frac{k}{k} \left( 1 - \omega_{R,j}/|\omega_j| \right) \right) \left[ P_{RR}^j + P_{LL}^j + P_{RL}^j e^{+2\psi} + P_{LR}^j e^{-2\psi} \right] \]

\[ + 2 \mu^2 \frac{\omega_j^2}{v^2 k^2} k^2 P_{||}^j + \sqrt{2} \mu \frac{\omega_j}{v} k \frac{k}{k} A_{\mu\nu} \left[ \left( P_{LL}^j + P_{RR}^j \right) e^{+i\psi} + \left( P_{L}^j + P_{R}^j \right) e^{-i\psi} \right] \]

\[ - \sqrt{2} \mu \frac{\omega_j}{v} k \frac{k}{k} \left( 1 - \omega_j^2 |\omega_j| \right) \left( P_{L}^j e^{+i\psi} + P_{L}^j e^{-i\psi} \right) + \left( 1 - \omega_j^2 |\omega_j| \right) \left( P_{R}^j e^{-i\psi} + P_{L}^j e^{+i\psi} \right) \] (4.20c)

Here, the following auxiliary functions were introduced:

\[ T_{\mu\nu} = S_{\mu\nu}^2 + 2 \frac{n}{W} \frac{\omega_j}{v} k \frac{k}{k} \left[ n \mu k \frac{n}{W} + k \sqrt{1 - \mu^2} \right] \left( 1 - \omega_{R,j}/|\omega_j| \right) \] (4.21)

\[ S_{\mu\nu} = n \frac{W}{A_{\mu\nu}} \frac{\omega_j}{v} k \frac{k}{k} \sqrt{1 - \mu^2} \] (4.22)
Chapter 4 On Fokker-Planck Coefficients

\[ A_{\mu\mu} = 1 - \frac{\mu |\omega_j| k_\parallel}{v k^2} \]  

(4.23)

Making use of \( k_x = k_\perp \cos \psi \) and \( k_y = k_\perp \sin \psi \) and expressing all helical magnetic intensities by their Cartesian counterparts, via section A.4, the auxiliary functions then read \( H_1 = 2F^J_{\mu\mu} \), \( H_2 = 2G^J_{\mu\mu} \) and \( H_3 = 2H^J_{\mu\mu} \). Here, the quantities \( F^J_{\mu\mu} \), \( G^J_{\mu\mu} \) and \( H^J_{\mu\mu} \) denote the FMFs mentioned above. They only depend on the Cartesian components of \( P_l^{n_m} \), the vector wave components and the plasma wave dispersion relation as follows:

\[ F^J_{\mu\mu} = T_{\mu\mu} \left[ k_y^2 P^J_{xx} + k_x^2 P^J_{yy} - k_x k_y \left( P^J_{xy} + P^J_{yx} \right) \right] / k_\perp^2 \]  

(4.24a)

\[ G^J_{\mu\mu} = S_{\mu\mu} \left[ A_{\mu\mu} \left( P^J_{yx} - P^J_{xy} \right) + \mu \frac{|\omega_j|}{v k^2} \left[ k_x \left( P^J_{y\parallel} - P^J_{y\perp} \right) + k_y \left( P^J_{x\parallel} - P^J_{x\perp} \right) \right] \right] + \frac{|\omega_j|}{v k^2} \left[ 2k_\parallel n - k_\perp \sqrt{1 - \mu^2} \right] \left( 1 - \omega_{R,j} / |\omega_j| \right) \left( P^J_{xy} - P^J_{yx} \right) + \mu \frac{n |\omega_j|}{v k^2} \left[ \left( k_y P^J_{x\parallel} - k_x P^J_{y\parallel} \right) \left( 1 - \omega_x^* / |\omega_j| \right) + \left( k_x P^J_{y\parallel} - k_y P^J_{x\parallel} \right) \left( 1 - \omega_y / |\omega_j| \right) \right] \]

\[ - \frac{1}{v k^2} \left( k_x^2 - k_y^2 \right) \left( P^J_{xy} + P^J_{yx} \right) - 2k_x k_y \left( P^J_{xy} - P^J_{yx} \right) \]  

(4.24b)

\[ H^J_{\mu\mu} = \frac{\mu^2 |\omega_j|^2 k_\perp^2 P^J_{\parallel\parallel}}{v^2 k^4} + \mu A_{\mu\mu} \frac{|\omega_j|}{v k^2} \left[ k_x \left( P^J_{y\parallel} + P^J_{y\perp} \right) + k_y \left( P^J_{x\parallel} + P^J_{x\perp} \right) \right] \]

\[ + \frac{1}{k_\perp} \left[ A^2_{\mu\mu} + 2 \mu k_\parallel |\omega_j| / v k^2 \left( 1 - \omega_{R,j} / |\omega_j| \right) \right] \left[ k_x^2 P^J_{xx} + k_y^2 P^J_{yy} + k_x k_y \left( P^J_{xy} + P^J_{yx} \right) \right] \]

\[ - \mu \frac{|\omega_j|}{v k^2} \left[ \left( k_x P^J_{y\parallel} + k_y P^J_{y\perp} \right) \left( 1 - \omega_y / |\omega_j| \right) + \left( k_x P^J_{x\parallel} + k_y P^J_{x\perp} \right) \left( 1 - \omega_x^* / |\omega_j| \right) \right] \]  

(4.24c)

In terms of these FMFs, the coefficient (4.16) can then be written as

\[ D_{\mu\mu} = \frac{\Omega_0^2}{B_0^2} (1 - \mu^2) \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \Re \int dk R_{\mu\mu} \left[ J^j_n(W) F^J_{\mu\mu} + i J^j_n(W) J^j_{n'}(W) G^J_{\mu\mu} + \left[ J^j_{n'}(W) \right]^2 H^J_{\mu\mu} \right] \]  

(4.25)

The Fokker-Planck coefficient \( D_{\mu\mu} \) is presented in this general form for the first time. It provides for the parallel particle diffusion in a turbulence consisting of damped plasma waves with arbitrary frequencies \( \omega_{j,R} \) and damping rates \( \Gamma_j \). Furthermore, the turbulence geometry is not specified so far. Equation (4.25) generalizes earlier calculations in so far as

(1) it has the advantage to include, via the turbulent magnetic fields (4.24a), (4.24b) and (4.24c), an arbitrary magnetic correlation tensor \( P^J_{lm} \). This tensor is determined by the underlying turbulence physics and involves the geometry of the turbulence and the evolution of the wave power spectrum in wavenumber space. Furthermore, the influence of more complex processes, e.g. non-linear wave-wave interactions (see chapter 5), enter the coefficient (4.25) by the components of \( P^J_{lm} \).

(2) it allows to include an arbitrary dispersion relation \( \omega_j = \omega_{j,R} + i \Gamma_j \). Generally, this enables one to investigate the effect of a changing plasma-\( \beta \) on the particle transport. Furthermore, more complex wavenumber dependences of \( \omega_{j,R} \) and \( \Gamma_j \) can be taken into account.
The second FPC \( D_{\mu p} \), given by equation (A.9), determines, in combination with \( D_{\mu \mu} \), the rate of adiabatic deceleration \( a_1 \), equation (3.23c). The final compact representation of (A.9) can be obtained similarly to the calculations performed for \( D_{\mu \mu} \). Making use of equations (4.6) and (4.15) as well as the sections A.3 and A.4, the representation (A.9) can be written as

\[
D_{\mu p} = \frac{q_s \Omega_n}{c B_0} (1 - \mu^2) \sum_{j = \pm 1} \sum_{n = -\infty}^{\infty} \Re \int d^3 k R_j \left[ \frac{\omega_j}{k^2} \right] \left[ j_n^2(W) F_{\mu p}^j + i J_n(W) J_n^j(W) G_{\mu p}^j + [J_n(W)]^2 H_{\mu p}^j \right]
\]

(4.26)

The FPC \( D_{\mu p} \) is presented in this general form for the first time. The corresponding FMFs are given by the following expressions:

\[
F_{\mu p}^j = S_{pp} \left[ S_{\mu \mu} - \frac{n}{W} \left( 1 - \frac{\omega_j^*}{|\omega_j|} \right) \right] \left[ k_y^2 P_{xy}^j + k_x^2 P_{yy}^j - k_x k_y \left( P_{xy}^j + P_{yx}^j \right) \right] / k^2
\]

(4.27)

\[
G_{\mu p}^j = \left[ \frac{k_x}{k^2} \left[ k_y^2 P_{yx}^j - k_y^2 P_{xy}^j + k_x k_y \left( P_{yy}^j - P_{xx}^j \right) \right] + k_y P_{x||}^j - k_x P_{y||}^j \right] \left( A_{\mu \mu} - \left( 1 - \frac{\omega_j^*}{|\omega_j|} \right) \right) S_{pp}
\]

\[
+ \mu \sqrt{\frac{\omega_j}{v} \frac{k_x}{k^2}} S_{pp} \left[ k_y P_{||x} - k_x P_{||y} \right]
\]

(4.28)

\[
H_{\mu p}^j = \left[ \frac{k_x}{k^2} \left[ k_y^2 P_{yx}^j + k_x^2 P_{yx}^j + k_x k_y \left( P_{yy}^j + P_{xy}^j \right) \right] - k_x P_{x||}^j - k_y P_{y||}^j \right] \left( A_{\mu \mu} - \left( 1 - \frac{\omega_j^*}{|\omega_j|} \right) \right)
\]

\[
- \mu \sqrt{\frac{\omega_j}{v} \frac{k_y}{k^2}} \left[ k_x^2 P_{x||}^j - k_y \left( k_x P_{||x} + k_y P_{||y} \right) \right]
\]

(4.29)

The function \( S_{pp} \) is defined by

\[
S_{pp} = \frac{n k_x}{W} \frac{\mu k_x}{\sqrt{1 - \mu^2}}
\]

(4.30)

while \( S_{\mu \mu} \) and \( A_{\mu \mu} \) were already introduced by equations (4.22) and (4.23), respectively. It is remarkable that the FPC (4.26) vanishes for the limit of a dynamical magnetic turbulence, i.e. if \( \omega_j = 0 \). As a consequence, the rate of adiabatic deceleration vanishes too, so that \( a_1 = 0 \).
The Fokker-Planck Coefficient $D_{pp}$

The third standard FPC, i.e. $D_{pp}$ given by equation (A.10), can be cast similarly. In addition to $D_{uu}$ and $D_{pp}$, it determines the momentum diffusion coefficient $a_2$, equation (3.23d). The rearranged form of (A.10) is as follows:

$$D_{pp} = \frac{q_0^2}{c^2} (1-\mu^2) \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \Re \int d^3 k R_j \frac{|\omega_j|^2}{k^4} \left[ J_n^2(W) F_{pp}^j + \mu J_n(W) J_n'(W) G_{pp}^j + [J_n'(W)]^2 H_{pp}^j \right]$$

(4.31)

The corresponding FMFs are defined by the expressions

$$F_{pp}^j = S_{pp}^2 \left[ k_x^2 P_{xx}^j + k_y^2 P_{yy}^j - k_x k_y \left( P_{xy}^j + P_{yx}^j \right) \right] / k_\perp^2$$

(4.32)

$$G_{pp}^j = S_{pp} \left[ k_{\parallel} \left( P_{xy}^j - P_{yx}^j \right) + k_x \left( P_{xx}^j - P_{yy}^j \right) + k_y \left( P_{xy}^j + P_{yx}^j \right) \right]$$

(4.33)

$$H_{pp}^j = k_\perp^2 P_{\parallel\parallel}^j - k_{\parallel} \left[ k_x \left( P_{xx}^j + P_{yy}^j \right) + k_y \left( P_{xy}^j + P_{yx}^j \right) \right]$$

(4.34)

where the function $S_{pp}$ was already defined in equation (4.30). As in the case of the previous coefficient $D_{pp}$, the FPC (4.31) vanishes for $\omega_j = 0$. Since $D_{pp}$ and $D_{pp}$ are equal to zero, the momentum diffusion transport parameter $a_2$ vanishes. Principally, this means that particle momentum diffusion and, therefore, stochastic acceleration of energetic particles cannot exist in the dynamical magnetic turbulence limit. A result, which is quite opposite to the often used and justified concept of stochastic acceleration of particles in space plasmas. This cast serious doubts on the dynamical magnetic turbulence approach.

The Fokker-Planck Coefficient $D_{XX}$

Having presented the standard FPCs in their general forms, this section turns the considerations to $D_{XX}$. On the basis of the transport parameter representation (3.23b), it contributes to the perpendicular particle diffusion along the $x$-axis. Substituting the tensors (4.6) and the identities (4.15) into equation (A.11) and, furthermore, making use of the sections A.3 and A.4, the general form of the coefficient $D_{XX}$ reads

$$D_{XX} = \frac{v^2}{B_0} \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \Re \int d^3 k R_j \left[ J_n^2(W) F_{XX}^j + \mu J_n(W) J_n'(W) G_{XX}^j + [J_n'(W)]^2 H_{XX}^j \right]$$

(4.35)

The corresponding FMFs are defined by the following expressions:

$$F_{XX}^j = a^2 \frac{k_x^2}{k_\perp^2} P_{\parallel\parallel}^j + ab \frac{k_x}{k_\perp} \left( P_{xx}^j + P_{yy}^j \right) + b^2 P_{xx}^j$$

$$+ \frac{2 |\omega_j|}{v k^2} \left( 1 - \frac{\omega_{R,j}}{|\omega_j|} \right) \left[ \sqrt{1 - \mu^2} \frac{n}{W} \frac{k_x^2}{k_\perp} P_{xx}^j + \mu k_{\parallel} P_{xx}^j \right]$$

$$- \frac{|\omega_j| k_x}{v k^2 k_\perp} \sqrt{1 - \mu^2} \left( 1 - \frac{\omega_j^2}{|\omega_j|^2} \right) P_{xx}^j$$

$$+ \left( 1 - \frac{\omega_j}{|\omega_j|} \right) P_{xx}^j$$
The corresponding FMFs are given by the following expressions:

\[ G^j_{XX} = b \sqrt{1 - \mu^2} \left( \frac{k_y}{k_\perp} \right) \left( P^j_{x||} - P^j_{x\parallel} \right) - 2i \sqrt{1 - \mu^2} \frac{\Gamma_j(k)}{v k^2} k_x k_y P^j_{y||} \]

\[ + \sqrt{1 - \mu^2} \frac{\omega_j}{k_\perp} \left[ \left( 1 - \frac{\omega_j^*}{|\omega_j|} \right) P^j_{x||} - \left( 1 - \frac{\omega_j}{|\omega_j|} \right) P^j_{x\parallel} \right] \quad (4.37) \]

\[ H^j_{XX} = \left( 1 - \mu^2 \right) \frac{k_y}{k_\perp} P^j_{y||} \quad (4.38) \]

where the following functions are introduced:

\[ a = \frac{n}{W} \sqrt{1 - \mu^2} - \frac{|\omega_j| k_\perp}{v k} \quad \text{and} \quad b = \frac{|\omega_j| k_\perp}{v k} - \mu \quad (4.39) \]

The FPC \( D_{XX} \) has not been presented in this general form before, neither for the plasma wave nor for the dynamical magnetic turbulence viewpoint.

**The Fokker-Planck Coefficient \( D_{YY} \)**

The coefficient \( D_{YY} \), equation \( (A.12) \), contributes to the perpendicular particle diffusion too, but along the \( y \)-axis. Again, using the representation \( (4.6) \), the identities \( (4.15) \) and sections \( A.3 \) and \( A.4 \), the following general form of \( (A.12) \) can be obtained:

\[ D_{YY} = \frac{v^2}{B_0^2} \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \Re \left[ \hat{d^{(k)R_j}} \left( J_n^2(W)F^j_{YY} + i J_n(W)J'_n(W)G^j_{YY} + [J'_n(W)]^2 H^j_{YY} \right) \right] \quad (4.40) \]

The corresponding FMFs are given by the following expressions:

\[ F^j_{YY} = a^2 \frac{k_y^2}{k_\perp^2} P^j_{y||} + ab \frac{k_y}{k_\perp} \left( P^j_{y||} + P^j_{y\parallel} \right) + b^2 P^j_{yy} \]

\[ + 2 \frac{|\omega_j|}{v k^2} \left( 1 - \frac{\omega_{R,j}}{|\omega_j|} \right) \left[ \sqrt{1 - \mu^2} \frac{n}{W} \frac{k_y}{k_\perp} P^j_{y||} + \mu k_\perp P^j_{y\parallel} \right] \]

\[ - \frac{|\omega_j|}{v k^2} \frac{k_y}{k_\perp} \sqrt{1 - \mu^2} \left[ \frac{n k_\parallel}{W} \left( 1 - \frac{\omega_j^*}{|\omega_j|} \right) P^j_{y||} + \left( 1 - \frac{\omega_j}{|\omega_j|} \right) P^j_{y\parallel} \right] \]

\[ + \frac{\mu k_\perp}{\sqrt{1 - \mu^2}} \left( 1 - \frac{\omega_j}{|\omega_j|} \right) P^j_{y\parallel} + \left( 1 - \frac{\omega_j^*}{|\omega_j|} \right) P^j_{y\parallel} \] \quad (4.41)

\[ G^j_{YY} = b \sqrt{1 - \mu^2} \left( \frac{k_x}{k_\perp} \right) \left( P^j_{y||} - P^j_{y\parallel} \right) + 2i \sqrt{1 - \mu^2} \frac{\Gamma_j(k)}{v k^2} k_x k_y P^j_{y||} \]

\[ + \sqrt{1 - \mu^2} \frac{|\omega_j| k_x k_\parallel}{k_\perp} \left[ \left( 1 - \frac{\omega_j}{|\omega_j|} \right) P^j_{y\parallel} - \left( 1 - \frac{\omega_j^*}{|\omega_j|} \right) P^j_{y\parallel} \right] \quad (4.42) \]

\[ H^j_{YY} = \left( 1 - \mu^2 \right) \frac{k_x}{k_\perp^2} P^j_{y||} \quad (4.43) \]
The functions \(a\) and \(b\) are given by equation (4.39). It is noteworthy that also \(D_{YY}\) has not been presented in this general form before, neither for the plasma wave nor for the dynamical magnetic turbulence approach.

**The Fokker-Planck Coefficient \(D_{\perp}\)**

The representations (4.35) and (4.40) can be summarized to yield a total coefficient \(D_{\perp}\). Considering the transport parameter (3.23b) in more detail, it becomes clear that the FPCs (4.35) and (4.40) already have the unit of a spatial diffusion coefficient, i.e. \(\text{cm}^2 \text{s}^{-1}\). Then, \(D_{XX}\) and \(D_{YY}\) can be expressed as

\[
D_{XX} \quad \text{and} \quad D_{YY} \quad \text{are given by equation (4.39). \(D_{\perp}\) has not been calculated before, neither for the plasma wave nor for the dynamical magnetic turbulence approach.}
\]

\[
D_{\perp} = D_{XX} + D_{YY} \quad (4.45)
\]

Taking the sum of equations (4.35) and (4.40) leads to

\[
D_{\perp} = \frac{v^2}{B_0^2} \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \Re \int d^3 k R_j \left[ J_n^2(W) F_{\perp} + \frac{\omega_n(W)}{P_{\|}} + \frac{\omega_n(W)}{P_{\|}} H_{\perp} \right] (4.46)
\]

The corresponding FMFs are then given by the sum of the FMFs of \(D_{XX}\) and \(D_{YY}\). One obtains

\[
F_{\perp} = a^2 P_{\|}^j + b^2 (P_{xx}^j + P_{yy}^j) + \frac{ab}{k_{\perp}} \left[ k_y (P_{jy}^j + P_{yj}^j) + k_x (P_{jx}^j + P_{yx}^j) \right] + \frac{2|\omega_j|}{v k^2} \left( 1 - \frac{\omega R}{\omega_j} \right) \left[ \sqrt{1 - \mu^2} \frac{n k_{xj}}{P_{\|}} + \frac{\mu k_{\perp}}{P_{xx}^j + P_{yy}^j} \right] - \frac{|\omega_j|}{v k^2} \left( 1 - \frac{\omega R}{\omega_j} \right) \left[ k_x P_{jj}^j + k_x P_{jy}^j + k_x P_{yj}^j + k_x P_{yy}^j \right] + \frac{\mu k_{\perp}}{\sqrt{1 - \mu^2}} \left[ \left( 1 - \frac{\omega_j}{|\omega_j|} \right) k_y P_{yy}^j + k_x P_{xj}^j + k_x P_{yj}^j \right] \right) (4.47)
\]

\[
G_{\perp} = b \sqrt{1 - \mu^2} \left[ k_x (P_{xj}^j - P_{yj}^j) + k_y (P_{jy}^j - P_{jx}^j) \right] / k_{\perp}
\]

\[
H_{\perp} = \left( 1 - \mu^2 \right) P_{\|}^j (4.49)
\]

where \(a\) and \(b\) are given by equation (4.39). \(D_{\perp}\) has not been calculated before, neither for the plasma wave nor for the dynamical magnetic turbulence approach.
4.3 The Magnetic Correlation Tensor

So far, the general FPC representations (4.25), (4.26), (4.31) and (4.46) enable one to consider an arbitrary turbulence geometry. This geometry is characterized by the Cartesian components of the magnetic fluctuation correlation tensor \( P^j_{lm} \). Whether for numerical or analytical calculations, a further treatment of the general FPCs requires assumptions which can be made in a twofold manner: first, a certain turbulence geometry can be assumed. This requires a certain representation of \( P^j_{lm} \). Second, the dispersion relation (4.1) of the wave modes of which the turbulence consists can be specified. For instance, a very popular and often used assumption is the case of undamped waves, i.e. \( \Gamma_j = 0 \). All FPCs then experience drastic simplifications in so far as, first, the resonance function (4.8) can be described by Dirac’s delta distribution, equation (4.9), and, second, all FMFs become relatively simple because of \( \omega_j = \omega_j^* = |\omega_j| \). However, even for the most simplified dispersion relation \( \omega_j = 0 \), as it is used in the context of the dynamical magnetic turbulence, the components of \( P^j_{lm} \) are required in any case. Without introducing assumptions about the dispersion relation the general FPC representations (4.25), (4.26), (4.31) and (4.46) experience drastic simplifications if additional assumptions are made with regard to the turbulence geometry. In other words, a specified magnetic fluctuation correlation tensor \( P^j_{lm} \) has to be introduced. Hence, it is more useful to choose, first, a specific magnetic fluctuation tensor \( P^j_{lm} \) and then, if necessary, to specify the dispersion relation. As stated above, the tensor \( P^j_{lm} \) is the most crucial quantity for all FPCs. Unfortunately, it is also the most critical one. Rigorous theories, being able to sufficiently explain a three-dimensional turbulence with all its dependences on the plasma parameters and its evolution in configuration as well as wavenumber space, do not exist so far. In view of the FMFs and their dependences on \( P^j_{lm} \), it is clear that different representations of \( P^j_{lm} \) will alter the underlying mathematical structure of all FMFs for the same FPC. Therefore, all diffusion processes in spatial and momentum space will depend sensitively on the definition of \( P^j_{lm} \).

However, in order to evaluate the general FPCs and their FMFs presented above, one can follow a standard procedure widespread in the literature and commonly used. For a magnetic turbulence tensor with no preferred direction, Batchelor [1953] pointed out that the nine correlation tensor components \( P^j_{lm} = \langle B^j_i(\mathbf{k})B^{*i}_m(\mathbf{k}) \rangle \) can be written in the form

\[
P^j_{lm}(\mathbf{k}) = A^j(\mathbf{k}) \left[ \delta_{lm} - \frac{k_lk_m}{k^2} + i \sigma^j(\mathbf{k}) \epsilon_{lmn} \frac{k_n}{k} \right] \tag{4.50}
\]

where the real quantity \( \sigma^j(\mathbf{k}) \) denotes the magnetic helicity and \( \delta_{lm} \) is Kronecker’s delta. \( \epsilon_{lmn} \) is the total antisymmetric tensor of rank 3. The function \( A^j(\mathbf{k}) \) includes an explicit wave power spectrum and, furthermore, provides for an appropriate normalization. The normalization requirement is that the magnetic energy density in wave component \( j \) is given by

\[
\left( \delta B^j \right)^2 = \int d^3k \left[ P^j_{xx}(\mathbf{k}) + P^j_{yy}(\mathbf{k}) + P^j_{zz}(\mathbf{k}) \right] = 2 \int d^3k A^j(\mathbf{k}) \tag{4.51}
\]

Although the tensor (4.50) does not agree with known polarization and damping properties of specific plasma wave modes at oblique propagation angles, as it has been shown for fast magneto-sonic waves (see, e.g., Tademaru, 1969; Lee and Völk, 1975), the representation (4.50) has been used in several studies performed in the past (see, e.g., Schlickeiser and Miller, 1998; Lerche and Schlickeiser, 2001a). Due to the lack of rigorous theories describing \( P^j_{lm} \) in all its details, equation (4.50) will be adopted for the following treatment of all FPCs. However, there has been some recent progress in determining the tensor \( P^j_{lm} \). Oughton et al. [1997] published a general correlation tensor for a homogeneous MHD turbulence which allows to consider standard cases as special limits. For instance, restricting the general correlation tensor to an isotropic turbulence, Oughton et al. [1997] demonstrated that the correlation tensor of second-rank can be, indeed, described by the result...
of Batchelor [1953], i.e. by equation (4.50). They also presented a correlation tensor for a two-dimensional symmetric turbulence which is perpendicular to the axis of symmetry. This is a further limit of the general correlation tensor. Nowadays, it is supposed that the solar wind turbulence can be described by a superposition of the two-dimensional with a slab turbulence (see Matthaeus et al., 1990; Bieber et al., 1996). The use of the more general correlation tensor derived by Oughton et al. [1997] in the context of FPCs will be the subject of future studies.

Since the tensor (4.50) involves not only the geometry of the underlying turbulence but also its evolution in wavenumber space, it is necessary to make assumptions concerning the fundamental wave power spectrum $A^j(k)$.

### 4.3.1 The Wave Power Spectrum for an Anisotropic Turbulence

Following Lerche and Schlickeiser [2001a], one can choose the wave power spectrum $A^j$ for an anisotropic turbulence to be of the form

$$A^j = g_0^j [k_{\parallel}^2 + \Lambda k_{\perp}^2]^{-(2+q)/2}$$  \hspace{1cm} (4.52)

Here, the anisotropy parameter $\Lambda$ enables one to distinguish between a strongly perpendicular turbulence ($\Lambda \ll 1$), a strongly parallel turbulence ($\Lambda \gg 1$) and an isotropic turbulence ($\Lambda = 1$). General theoretical considerations (Goldreich and Sridhar, 1995), linear Landau damping estimates for the decay of fast-mode waves in the turbulent interstellar medium (Lerche and Schlickeiser, 2001b) as well as observations of interstellar scintillations (Rickett, 1990; Spangler, 1991) support the assumption that the turbulence power in interstellar as well as interplanetary space can roughly be represented by a spectrum of the form (4.52). With the wavenumber representation $k_{\parallel} = k\eta$ and $k_{\perp} = k\sqrt{1-\eta^2}$, where $k$ is larger than a minimum wavenumber, $k_{\text{min}}$, and less than a maximum $k_{\text{max}}$, equation (4.52) reads

$$A^j = g_0^j k^{-(2+q)}[\eta^2 + \Lambda(1 - \eta^2)]^{-(2+q)/2} = g_0^j k^{-(2+q)} h(\eta)$$ \hspace{1cm} (4.53)

where $\eta$ is the cosine of the propagation angle of a plasma wave with respect to the background magnetic field. Substitution of the power spectrum (4.53) into the normalization (4.51) leads to

$$\left(\frac{\delta B^j}{B_0}\right)^2 = 8\pi g_0^j \int_{\eta_{\text{min}}}^{\eta_{\text{max}}} d\eta h(\eta) \int_{k_{\text{min}}}^{k_{\text{max}}} dk k^{-q}$$ \hspace{1cm} (4.54)

The integration with respect to $k$ can easily be carried out. It readily results in

$$\int_{k_{\text{min}}}^{k_{\text{max}}} dk k^{-q} = \frac{k_{\text{max}}^{1-q} - k_{\text{min}}^{1-q}}{q-1} \left[ 1 - \left( \frac{k_{\text{min}}}{k_{\text{max}}} \right)^{q-1} \right]$$ \hspace{1cm} (4.55)

In order to perform the integration with respect to $\eta$ it is convenient to introduce the new variable $t = \eta^2$. The integral then reads

$$\int_{0}^{1} d\eta h(\eta) = \frac{\Lambda^{-(2+q)/2}}{2} \int_{0}^{1} dt t^{-1/2} \left[ 1 - (1 - \Lambda^{-1})t \right]^{-(2+q)/2}$$ \hspace{1cm} (4.56)

Upon using the integral representation (15.3.1) of Abramowitz and Stegun [1972] the latter integral can be expressed in terms of the hypergeometric function $2F_1[a, b; c; z]$, yielding

$$\int_{0}^{1} d\eta h(\eta) = \Lambda^{-(2+q)/2} 2F_1[1 + q/2, 1/2; 3/2; 1 - \Lambda^{-1}]$$ \hspace{1cm} (4.57)
Including several corrections, the latter relation corresponds to equation (5) of Lerche and Schlickeiser [2001a]. The normalization $g^j_0$ is then given by

$$g^j_0 = \frac{(q - 1)}{8\pi} \left(\delta B^j\right)^2 \frac{k^{q-1}_{\min}}{1 - (k_{\min}/k_{\max})^{q-1}} \frac{\Lambda^{(2+q)/2}}{2F_1[1 + q/2, 1/2; 3/2; 1; 1 - \Lambda^{-1}]} \quad (4.58)$$

For a magnetic turbulence with no preferred direction, i.e. $\Lambda = 1$, the hypergeometric function becomes $2F_1[a, b; c; 0] = 1$. Although the turbulence power spectrum reveals an anisotropic nature, the assumed tensor $P^j_{\text{lin}}$ is valid only for an isotropic turbulence. Therefore, a more realistic approach should include not only an anisotropic wave power spectrum, but also a quite different representation for $P^j_{\text{lin}}$.

### 4.3.2 The Wave Power Spectrum for a Slab Model

The second turbulence geometry is the so-called slab model. In this approach it is assumed that the wave vectors of the fluctuations are all either parallel or antiparallel to the background magnetic field. The only non-vanishing elements of the magnetic correlation tensor ($4.50$) are then given by $P^j_{xx} = P^j_{yy} = A^j$ and $P^j_{yx} = P^j_{xy} = \sigma^j A^j$. In order to suppress the perpendicular wave vector component, the following power spectrum is usually chosen for a slab turbulence:

$$A^j = g^j(k^j) \delta(k^j) = g^j_0 k^j_\| \delta(k^j_\|) / k^j_\| \quad (4.59)$$

Here, $k^j_\|$ is larger than a parallel minimum wavenumber, $k_{\min}$, and less than a maximum $k_{\max}$. The normalization $g^j_0$ can be obtained by using equation (4.51). Substitution of (4.59) into (4.51) leads to

$$g^j_0 = \frac{(q - 1)}{4\pi} \left(\delta B^j\right)^2 \frac{k^{q-1}_{\min}}{1 - (k_{\min}/k_{\max})^{q-1}} \quad (4.60)$$

It should be noted that the tensor (4.50) represents the expected polarization states of transverse wave modes. The only non-vanishing real components are then $P^j_{xx}$ and $P^j_{yy}$.

The slab model offers the opportunity to get more insight into the meaning of the magnetic helicity $\sigma^j$. It determines the imaginary off-diagonal elements $P^j_{xy}$ and $P^j_{yx}$, respectively. To obtain an expression for $\sigma^j$ the relations $P^j_{xx} + P^j_{yy} = 2A^j$ and $P^j_{yx} - P^j_{xy} = -2\sigma^j A^j$ are used. Furthermore, it is convenient to recall the helical intensities $P^j_{LL}$ and $P^j_{RR}$ of section A.4. One obtains the expression

$$\sigma^j = \frac{P^j_{LL} - P^j_{RR}}{P^j_{LL} + P^j_{RR}} \quad (4.61)$$

The magnetic helicity represents the ratio of the intensities of left- to right-handed polarized waves of which a forward ($j = +1$) or backward ($j = -1$) propagating wave field consists. The magnetic helicity varies between $+1$ and $-1$. These values are attained if a wave field consists of only left- or right-handed polarized waves, respectively. For the case that a forward or backward propagating wave field represents a superposition of left- and right-handed polarized waves with the same intensity, the magnetic helicity vanishes. The resulting wave is then linearly polarized. The ratio of the intensities of forward to backward propagating waves is commonly referred to as the normalized cross helicity

$$h_c = \frac{g^j_0 - g^j_{\|}}{g^j_0 + g^j_{\|}} \quad (4.62)$$

The superscripts (+) and (−) denote quantities related to the forward ($j = +1$) and backward ($j = -1$) propagation direction, respectively. These superscripts will be used where it is appropriate, in particular in chapter 6. The normalized cross helicity varies between $-1$ and $+1$. These values are attained if only backward or forward moving waves are present, respectively.
4.4 Fokker-Planck Coefficients for an Isotropic Turbulence

As it was shown in section 4.2, each contribution of the general FPC representations (4.25), (4.26), (4.31) and (4.46) is accompanied by a specific FMF. All FMFs are solely determined by the components of the magnetic correlation tensor $P^j_{lm}$. Hence, a further treatment of the FPCs or, alternatively, FMFs requires a specific representation of $P^j_{lm}$. Such a representation was introduced in the previous section. Based on the tensor (4.50) all FMFs can be determined.

The Fluctuating Magnetic Fields of $D_{\mu \nu}$

In order to evaluate the FMFs of the FPC $D_{\mu \nu}$, the appropriate elements of (4.50) have to be inserted into the equations (4.24a), (4.24b) and (4.24c). Summarizing terms and performing some algebra yields

$$F^j_{\mu \nu} = A^j \left[ S^2_{\mu \nu} + 2n \frac{\omega_j}{W} k^2 + k_\perp \sqrt{1 - \mu^2} \left( 1 - \frac{\omega_{j,R}}{\omega_j} \right) \right]$$

(4.63)

$$G^j_{\mu \nu} = -2iA^j \sigma^j \left[ S_{\mu \nu} \left( \frac{k_j}{k} - \mu \frac{\omega_j}{v k} \right) + \frac{\omega_j}{v k^2} \left( \mu (k_\parallel^2 + k_\perp^2) + \frac{\omega_j}{v k^2} \right) \right]$$

(4.64)

$$H^j_{\mu \nu} = A^j \left[ \left( \frac{k_\parallel}{k} - \mu \frac{\omega_j}{v k} \right)^2 + 2\mu \frac{\omega_j}{v k^2} \left( 1 - \frac{\omega_{R,j}}{\omega_j} \right) \right]$$

(4.65)

where the function $T^j_{\mu \nu}$, equation (4.21), was used. At a first glance, it can be seen that the longish appearing FMFs (4.24a), (4.24b) and (4.24c) collapse to relatively simple expressions by using the tensor (4.50). The last term on the right-hand side of equation (4.24b) vanishes exactly. Then, the plasma wave damping function $\Gamma_j$ enters the calculations only by the resonance function $R_j$ and by $|\omega_j| = (\omega_{j,R}^2 + \Gamma_j^2)^{1/2}$. Note that $G^j_{\mu \nu}$ is solely characterized by the magnetic helicity $\sigma^j$. Furthermore, it is purely imaginary. By considering the second term in the brackets of the FPC (4.25) in more detail, it readily becomes clear that $D_{\mu \nu}$ is determined only by the real part of the resonance profile (4.8). At this point, it is easy to recognize that the FMFs (4.63) to (4.65) experience further simplifications for $\Gamma_j = 0$, because $|\omega_j| = \omega_{j,R}$. The limit of vanishing wave damping is considered in section 4.5.

The Fluctuating Magnetic Fields of $D_{\mu \nu}$

The FMFs of the coefficient $D_{\mu \nu}$ are treatable similarly. Substituting the appropriate components of (4.50) into equations (4.27), (4.28) and (4.29), the FMFs can be written as

$$F^j_{\mu \nu} = A^j S_{\mu \nu} \left[ -n \frac{\omega_j}{W} \left( 1 - \frac{\omega_j}{|\omega_j|} \right) \right]$$

(4.66)

$$G^j_{\mu \nu} = -\frac{i\sigma^j A^j}{k} \left[ (S_{\mu \nu} - n \frac{\omega_j}{W} \left( 1 - \frac{\omega_j}{|\omega_j|} \right)) k^2 - \mu \frac{\omega_j}{v k^2} S_{\mu \nu} k^2 + \left( A_{\mu \nu} - \left( 1 - \frac{\omega_j^*}{|\omega_j|} \right) \right) S_{\mu \nu} k^2 \right]$$

(4.67)

$$H^j_{\mu \nu} = A^j \left( \frac{\omega_j^*}{|\omega_j|^2} k - \mu \frac{\omega_j}{v} \right)$$

(4.68)

where the function $S_{\mu \nu}$ was defined in equation (4.30). The field $G^j_{\mu \nu}$ is characterized by the magnetic helicity $\sigma^j$ and is of purely imaginary nature. However, in this case the FMFs also include the complex plasma wave dispersion relation. Hence, if plasma wave damping is not neglected, the
coefficient $D_{pp}$, given by equation (4.26), has to be expressed in terms of both the real as well as the imaginary part of the resonance function (4.8). This fact has not been recognized before.

The Fluctuating Magnetic Fields of $D_{pp}$

The treatment of the FMFs (4.32), (4.33) and (4.34) is more simple and results, in combination with the tensor (4.50), in

\[
F_{pp}^j = A^j S_{pp}^2 \quad (4.69)
\]

\[
G_{pp}^j = -2iA^j \sigma^j k S_{pp} \quad (4.70)
\]

\[
H_{pp}^j = A^j k^2 \quad (4.71)
\]

The field $G_{pp}^j$ is the only imaginary contribution through which the magnetic helicity $\sigma^j$ enters the FPCs (4.31). It is remarkable that the representations (4.69), (4.70) and (4.71) are independent of the dispersion relation. The coefficient $D_{pp}$ is then solely determined by the real part of the resonance function (4.8).

The Fluctuating Magnetic Fields of $D_{\perp}$

Substituting the appropriate components of $P_{lm}$, i.e. equation (4.50), into the FMFs of the FPC $D_{\perp}$, equation (4.46), the representations (4.47), (4.48) and (4.49) can be cast into the following forms:

\[
\frac{F_{\perp}^j}{A^j} = \frac{(a k_{\perp} - b k_{||})^2}{k^2} + b^2 + 2 \frac{|\omega_j|}{v k^2} \left(1 - \frac{\omega_{R,j}}{|\omega_j|}\right) \left[\frac{n}{W} k_{\perp} \sqrt{1 - \mu^2} + 2\mu k_{||}\right] \quad (4.72)
\]

\[
\frac{G_{\perp}^j}{A^j} = -2i\sigma^j \sqrt{1 - \mu^2} k_{\perp} \left[ b - \frac{|\omega_j|}{v k^2} k_{||} \left(1 - \frac{\omega_{R,j}}{|\omega_j|}\right) \right] \quad (4.73)
\]

\[
\frac{H_{\perp}^j}{A^j} = (1 - \mu^2) \frac{k_{\perp}^2}{k^2} \quad (4.74)
\]

The functions $a$ and $b$ were already defined by the two relations in equation (4.39). The magnetic helicity $\sigma^j$ enters the coefficient $D_{\perp}$ only by $G_{\perp}^j$. The latter reveals a pure imaginary character, whereas $F_{\perp}^j$ and $H_{\perp}^j$ are real fields. Therefore, an evaluation of $D_{\perp}$ involves only the real part of the resonance function (4.8).

4.5 Reduced Representations for Special Cases

In order to get more physical as well as mathematical insight into the general representations (4.25), (4.26), (4.31) and (4.46) presented above, it is instructive to derive several representations of the FPCs for specific limits. To do so, the FMFs presented in section 4.4 are used.

4.5.1 The Limit of Vanishing Wave Damping

In the case of a vanishing plasma wave damping, the dispersion relation $\omega_j$ is a real quantity, i.e. $\omega_j = \omega_{R,j}$. On the basis of the FMFs (4.63), (4.64) and (4.65), one obtains for equation (4.25)
the form
\[
D_{\mu\nu} = \frac{\Omega_\alpha^2}{B_0^2}(1 - \mu^2) \sum_{j=\pm1} \sum_{n=-\infty}^{\infty} \Re \int d^3k R_j A^j(k) \quad (4.75)
\]
\[
\times \left( J_n^2(W)S_{\mu\mu}^2 + 2 \sigma^j(k)J_n(W)J_n'(W)S_{\mu\mu} \left( k_\parallel - \mu \frac{\omega_j}{v_k} \right) + \left[ J_n'(W)^2 \left( k_\parallel - \mu \frac{\omega_j}{v_k} \right) \right]^2 \right)
\]
where the Breit-Wigner type resonance function $R_j$ has to be expressed by Dirac’s delta distribution, equation (4.9). Considering the terms in brackets in more detail, it is obvious that the FPC $D_{\mu\nu}$ evolves according to the sum of binomial formulas if $\sigma^j = \pm 1$. In other words, if only undamped left- or right-handed polarized waves are present, the FPC (4.75) collapses to a binomial equation. This binomial structure is destroyed if the plasma waves are damped. Similarly, one can calculate $D_{\mu p}$ and $D_{p p}$ given by equations (4.26) and (4.31), respectively, for the limit of $\Gamma_j = 0$ and obtains the following forms:
\[
D_{\mu p} = \frac{\eta_\alpha\Omega_\alpha}{cB_0}(1 - \mu^2) \sum_{j=\pm1} \sum_{n=-\infty}^{\infty} \Re \int d^3k R_j \frac{\omega_j}{k^2} A^j(k) \left[ J_n^2(W)S_{\mu\mu}S_{p p} + \sigma^j(k)k^{-1}J_n(W)J_n'(W) \left[ S_{\mu\mu}k^2 + S_{p p} \left( k_\parallel - \mu \frac{\omega_j}{v_k} \right) \right] + \left[ J_n'(W)^2 \left( k_\parallel - \mu \frac{\omega_j}{v_k} \right) \right]^2 \right] \quad (4.76)
\]
\[
D_{p p} = \frac{\eta_\alpha^2}{c^2}(1 - \mu^2) \sum_{j=\pm1} \sum_{n=-\infty}^{\infty} \Re \int d^3k R_j \frac{\omega_j^2}{k^2} A^j(k) \times \left[ J_n^2(W)S_{p p}^2 + 2 \sigma^j(k)J_n(W)J_n'(W)kS_{p p} + \left[ J_n'(W)^2 \right]^2 k^2 \right] \quad (4.77)
\]
Before presenting the FPC $D_\perp$, equation (4.46), in its undamped version, it is convenient to consider the first term in $F^{\perp}_{\perp}$, equation (4.72), in more detail. By using equation (4.39), the numerator can be written as
\[
ak_\perp - bk_\parallel = \frac{1}{v} \left( k_\parallel v_\parallel - \omega_{R,j} + n\Omega_\alpha \right) \quad (4.78)
\]
The expression on the right-hand side is nothing else than the frequency mismatch parameter occurring in the resonance condition (4.9) for undamped waves. Therefore, one obtains $ak_\perp - bk_\parallel = 0$ for undamped wave modes. Equation (4.46) is then, in combination with the FMFs (4.47), (4.48) and (4.49), expressible as
\[
D_\perp = \frac{v^2}{B_0^2} \sum_{j=\pm1} \sum_{n=-\infty}^{\infty} \Re \int d^3k R_j A^j(k) \times \left[ J_n^2(W)b^2 + 2 \sigma^j(k)J_n(W)J_n'(W)b\sqrt{1 - \mu^2 k_\parallel^2/k^2} + \left[ J_n'(W)^2 \right]^2 (1 - \mu^2)k_\perp^2/k^2 \right] \quad (4.79)
\]
Again, the absence of wave damping leads to a binomial structure of (4.79) for $\sigma^j = \pm 1$.

4.5.2 The Limit of a Slab Geometry

Another quite interesting and important limit is the so-called slab model. The corresponding wave power spectrum $A^j$ was already introduced by equation (4.59) in section 4.3. In order to derive the FPCs for slab geometry, the following approximation process is applied to the FPC representations
(4.25), (4.26), (4.31) and (4.46): the argument $W = k_\perp v_\perp /\Omega_\alpha$ of all Bessel functions is assumed to be much less than unity. Then, $k_\perp$ is small compared to $k_\parallel$, but not equal to zero. This leads to a quasi-slab model. Then, the limit $k_\perp \to 0$ is considered. For ease of exposition and for illustrative reasons, detailed calculations are again presented only for the most important FPC, namely $D_{\mu\mu}$ given by (4.25). The treatment of the remaining FPCs is very similar for the quasi- and the pure-slab limits.

In order to start with the approximation scheme, one first considers in equation (4.25) the summation with respect to $n$. The sum can also be expressed as

$$\sum_{n=-\infty}^{\infty} I(n) = I(0) + \sum_{n=1}^{\infty} [I(-n) + I(n)] = \sum_{r=\pm 1}^{\infty} \sum_{n=0}^{\infty} I(rn) \quad (4.80)$$

where, for illustrative purposes, an arbitrary function $I(n)$ was introduced. Note that the sequence of summation should not be switched. On the basis of the sum representation (4.80), the FPC (4.25) then reads

$$D_{\mu\mu} = \frac{\Omega_\alpha^2}{B_0^2} (1 - \mu^2) \sum_{j=\pm 1}^{\infty} \sum_{r=\pm 1}^{\infty} \Re \int d^3k \mathcal{R}_j \times \left[ J_n^2(W) F_{\mu\mu}^j(rn) + \nu J_n(W) J_n'(W) G_{\mu\mu}^j(rn) + [J_n'(W)]^2 H_{\mu\mu}^j \right]$$

Note that the field $H_{\mu\mu}^j$ is independent of $n$. This will be also the case for the remaining FPCs. In order to evaluate (4.81) for a quasi-slab geometry, the asymptotic expression

$$J_n(W) \simeq \frac{1}{\Gamma(1 + n)} \left( \frac{W}{2} \right)^n$$

is used, representing the Bessel function for small arguments. Here, $\Gamma(x)$ denotes the Gamma function. Upon substituting the asymptotic representation (4.82) and its corresponding derivative into equation (4.81), one obtains

$$D_{\mu\mu} = \frac{\Omega_\alpha^2}{B_0^2} (1 - \mu^2) \sum_{j=\pm 1}^{\infty} \sum_{r=\pm 1}^{\infty} \Re \int d^3k \mathcal{R}_j \frac{W^{2n}}{2^{2n} \Gamma^2(1 + n)} \left[ F_{\mu\mu}^j(rn) + \nu G_{\mu\mu}^j(rn) \frac{n}{W} + H_{\mu\mu}^j \frac{n^2}{W^2} \right] \quad (4.83)$$

Equation (4.83) forms, in combination with the corresponding turbulent magnetic fields (4.63), (4.64) and (4.65), a suitable set to investigate a quasi-slab turbulence. This means that the turbulence consists of waves propagating quasi-parallel along the background magnetic field. In other words, the wave vector has a small but non-vanishing perpendicular component with $k_\perp \ll k_\parallel$. This will be left for future studies.

In order to evaluate (4.83) in the limit of a pure-slab geometry, i.e. $k_\perp = 0$, the FMFs (4.63), (4.64) and (4.65) have to be considered. They have different structures with regard to the ratio $n/W$: the field $F_{\mu\mu}^j$ is a second-order polynomial in $n/W$, whereas $G_{\mu\mu}^j$ represents a polynomial of first-order. Since $H_{\mu\mu}^j$ is independent of $n/W$, it represents a polynomial of zeroth-order. A detailed inspection of the terms in the brackets of equation (4.83) and, furthermore, of the FMFs (=polynomials in $n/W$ having different orders) leads to the following statement: the resulting FPC for slab geometry includes terms for $n = 1$ only, all other contributions are zero. In particular, transit-time damping ($n = 0$) cannot occur in a pure-slab turbulence. This can easily be explained by considering equation (4.83) and its dependence on $W$. The Fokker-Planck coefficient $D_{\mu\mu}$ is

\footnote{This approximation was already used in the context of the kinetic plasma wave approach, section C.3, where the conductivity tensor (C.37) was restricted to the case of purely parallel propagating waves.}
described as a polynomial function in \( W \) having a maximum order of \( 2n \). Therefore, all three terms in brackets can only contribute, for \( W = 0 \), to the result as follows: \( F^j_{\mu\mu} \) given by equation (4.21) is a second-order polynomial in \( W \). Therefore, the \( n = 1 \) term of the sum is the only non-vanishing contribution, since \( W^{2n-2} = 0 \) for all \( n \neq 1 \) if \( W = 0 \). Similarly, \( G^j_{\mu\mu} \) and \( H^j_{\mu\mu} \) can only contribute for \( n = 1 \), too, so that equation (4.83) is completely determined by the gyroresonant interaction of the first harmonic, i.e. \( n = 1 \). Making use of the power spectrum (4.59) and taking into account only the corresponding \( n/W \)-terms in the FMFs (4.63) and (4.64), one obtains for the Fokker-Planck coefficient \( D_{\mu\mu} \) in the limit of a pure-slab geometry the following form:

\[
D_{\mu\mu} = \pi \frac{\Omega^2}{B_0^2} \left( 1 - \mu^2 \right) \sum_{j=\pm 1} \sum_{r=\pm 1} \Re \int dk_k g^j(k_k) \mathcal{R}_j \left( 1 + r \sigma^j_k \right)
\]

\[
\times \left[ \left( 1 - \frac{\omega_j}{v_k k_k} \right)^2 + 2 \mu \left| \frac{\omega_j}{k_k} \right| \left( 1 - \frac{\omega_{R,j}}{\omega_j} \right) \right]
\]

(4.84)

Applying the same approximation processes to the two remaining standard FPCs \( D_{\mu p} \) and \( D_{p p} \), given by equations (4.26) and (4.31), respectively, the following expressions can be derived for the limit of a pure-slab geometry:

\[
D_{\mu p} = \pi \frac{q_0 \Omega_o}{c B_0} \left( 1 - \mu^2 \right) \sum_{j=\pm 1} \sum_{r=\pm 1} \Re \int dk_k g^j(k_k) \mathcal{R}_j \frac{\omega_j}{k_k} \left( 1 + r \sigma^j_k \right) \left( \frac{\omega_j^*}{\omega_j} - \mu \frac{\omega_j}{v_k k_k} \right)
\]

(4.85)

\[
D_{p p} = \pi \frac{q_0^2}{c^2} \left( 1 - \mu^2 \right) \sum_{j=\pm 1} \sum_{r=\pm 1} \Re \int dk_k g^j(k_k) \mathcal{R}_j \frac{\omega_j^2}{k_k} \left( 1 + r \sigma^j_k \right)
\]

(4.86)

Equations (4.84), (4.85) and (4.86) represent the fundamental diffusion processes of energetic particles for a turbulence consisting of strictly parallel or antiparallel propagating transverse wave modes. Note that \( \omega_j \) also includes, besides the real frequency \( \omega_{R,j} \), the plasma wave damping function \( \Gamma_j(k_k) \). Both \( \omega_{R,j} \) as well as \( \Gamma_j(k_k) \) are arbitrary functions in \( k_k \). Equation (4.86) agrees with earlier results presented by Schlickeiser and Achatz [1993] and Schlickeiser [1994]. The FPCs (4.84) and (4.85), however, reveal new features with regard to their structures in \( \omega_j \). The coefficient (4.84) has, if compared with equation (13.2.4) of Schlickeiser [2002], an additional term. This term is determined by \( 1 - \omega_{R,j}/|\omega_j| \). It results from the fact that the influence of wave damping is not neglected when the electromagnetic correlation tensors are expressed by their corresponding magnetic counterparts. This was not taken into account before. For the case \( \Gamma_j = 0 \), the last term of equation (4.84) vanishes and earlier results are retained. Furthermore, considering the limit \( \omega_j = 0 \), equation (4.84) simplifies to the expression (33) of Schlickeiser and Achatz [1993]. Formally, this corresponds to the dynamical magnetic turbulence approach. The new representation of \( D_{\mu p} \) differs from earlier expressions (e.g. equation (13.2.4) of Schlickeiser, 2002) in so far as it includes the complex conjugated form of \( \omega_j \). This is important if damped waves are considered. Then, the imaginary part of the resonance function (4.8) is required. Of course, this complicates the evaluation of (4.85), even for slab geometry, substantially. The occurrence of \( \omega_j^* \) results from the fact that damping is not neglected when expressing, via Faraday’s law, all electric fields by magnetic fields. In view of this, the FPCs (4.84) and (4.85) are here presented, for the first time, in a more general form than before.

In order to demonstrate the potential and flexibility of equations (4.84), (4.85) and (4.86), the following considerations are restricted to the case of undamped dispersionless Alfvén waves, i.e. \( \omega_{R,j} = jv_A k_k \). With \( g^j(k_k) = g_0^j k_k^{-q} \) (see equation (4.59) of section 4.3), the resonance function
and substituting the asymptotic expression (4.2.9) to consider the Fokker-Planck coefficient (13.2.9) of 

The further treatment of (4.9) and substituting the asymptotic expression (4.82) into (4.46) yields

\[ D_{\perp} = \frac{v^2}{B_0^2} \sum_{j=\pm 1} \sum_{r=\pm 1} \sum_{n=0}^{\infty} \Re \int dk |g(k)| R_j \int_0^\infty \frac{k^2}{k^2 + k_\perp^2} \delta(k_\perp) = 0 \]  

The further treatment of (4.90) depends on \( F_{1,1}^j \), \( G_{1,1}^j \) and \( H_{1,1}^j \) given by equations (4.72), (4.73) and (4.74), respectively. For the limit \( W \to 0 \), the FMF \( F_{1,1}^j \) can only contribute to (4.90) if it contains a term with \( n^2/W^2 \). The function \( b \) is independent of \( n/W \). Thus, the first term in equation (4.72) is the only contributing term. It involves, via the function \( a \), a ratio of the form \( n^2/W^2 \). In order to contribute to \( D_{\perp} \), the field \( G_{1,1}^j \), equation (4.73), has to be a first-order polynomial in \( n/W \). However, \( G_{1,1}^j \) is independent of \( n/W \) so that the second term in the brackets of equation (4.90) vanishes for \( W \to 0 \). The field \( H_{1,1}^j \) is a zeroth-order polynomial in \( n/W \) and, therefore, also contributes to (4.90). By using equation (4.49), the contributing term of \( F_{1,1}^j \) and the power spectrum (4.59), equation (4.90) can be evaluated for \( n = 1 \) (all other contributions vanish). One obtains, after some algebra, the following expression:

\[ D_{\perp} = \pi v^2 \sum_{j=\pm 1} \sum_{r=\pm 1} \Re \int dk |g(k)| R_j \int_0^\infty \frac{k^2}{k^2 + k_\perp^2} \delta(k_\perp) = 0 \]  

Then, the transport parameter (3.23b) implies

\[ \kappa_{\perp} = 0 \]  

The perpendicular particle diffusion coefficient vanishes for a slab turbulence. Note that no specific dispersion relation was assumed and that \( \kappa_{\perp} \) vanishes even for damped waves. Furthermore, the magnetic helicity \( \sigma^j \) does not occur in equation (4.91) and, therefore, has no influence. In other words, energetic particles cannot diffuse perpendicular to the background magnetic field if the turbulence consists only of either parallel or antiparallel propagating waves. In section 6.1, however, it will be shown that \( \kappa_{\perp} \neq 0 \) for an isotropic turbulence. This means that the particle motion normal to the background magnetic field is always suppressed if a three-dimensional turbulence is reduced in its wave vector dimensions. This can be explained nicely on the basis of the magnetic correlation tensor \( D_{ln}^j \) and its proper meaning. Initially, this tensor was introduced to include the spatial components of the plasma wave magnetic field \( \delta B \) into all FPCs listed in appendix A. For a turbulence fully developed in three-dimensional configuration space, each vector component of \( \delta B \)
enters the FPCs by the equation of motion (3.3). The spatial components of $\delta B$ are subjected to a Fourier-Laplace transformation (see equation (4.3) in section 4.1.1), so that they represent functions in wave vector components. Although considered in wave vector space, the transformed fluctuating magnetic field components still possess the initial spatial directions they had in spatial configuration space before the transformation. The transformed spatial magnetic field components are then summarized in the expression $P_{lm}^j = \langle B_j^l(k) B_m^j(k) \rangle$, where $l$ and $m$ denote Cartesian coordinates. But this means that $P_{lm}^j$ represents not only the evolution of a turbulence in wavenumber space, but also the spatial distribution of this turbulence. In order to illustrate this, it is instructive to consider a turbulence having slab geometry. The wave vector and the background magnetic field are then aligned along the same spatial axis, in this case the $z$-axis. It immediately becomes clear that the wave propagation direction coincides with the spatial $z$-axis. However, for slab geometry, $P_{jx}^j$ and $P_{yy}^j$ are the only non-vanishing components and no contribution along the $z$-axis occurs. Although $P_{jx}^j$ and $P_{yy}^j$ depend, via the power spectrum $A^j$, on the parallel wavenumber $k_\parallel$ they only form a two-dimensional plane perpendicular to this $z$-axis. In other words, the turbulence is reduced in its spatial dimensions or, since no contributions occur with respect to the $z$-axis, is ignored in $z$-direction. For an isotropic turbulence, however, the magnetic correlation tensor $P_{lm}^j$ is fully developed and has contributions along the $z$-axis. In view of these explanations, the following conclusion may be drawn: a turbulence with reduced dimensionality in wave vector space corresponds to a turbulence with ignorable spatial coordinates. Equation (4.92) is a result of the reduced dimensionality of the slab turbulence, in which charged particles cannot move in perpendicular direction and remains tied to a magnetic line of force. Nevertheless, slab geometry allows particle diffusion along the background magnetic field so that $\kappa_\parallel \neq 0$. This is the topic of section 6.3.

The spatial dimensionality of electromagnetic fields and its importance for particle transport was considered earlier by Jokipii et al. [1993] or, more recently, Jones et al. [1998]. They showed that charged particles in an arbitrary electromagnetic field with at least one ignorable spatial coordinate remain tied to a given magnetic field line. Giacalone and Jokipii [1994] used synthesized magnetic field turbulence and verified that particles cannot cross magnetic field lines if the spectrum does not depend on all three spatial coordinates. For a fully three-dimensional turbulence, they showed that cross field diffusion of charged particles is evident. The considerations presented above support their conclusions. Here, however, the plasma wave approach is used to show that perpendicular particle diffusion requires a turbulence geometry with more than one dimension and that, therefore, the slab model is not suitable for discussing perpendicular particle diffusion.

4.5.3 The Limit of a Dynamical Magnetic Turbulence

This section represents FPCs for the second approach, i.e. the dynamical magnetic turbulence. Following the considerations presented in section 4.1.2, one is willing to say that this concept represents a strong restriction of the plasma wave approach, because all electric fields are neglected. This limit can easily be achieved by making the simplest choice for the dispersion relation, namely $\omega_j = 0$. This implies a vanishing damping function, and this limit was discussed above. Therefore, it is sufficient to consider the representations (4.75) and (4.79) for the limit $\omega_j, R = 0$ and to replace the plasma wave resonance function $R_j$ by (4.14a) or (4.14b) for the damping or the random sweeping model, respectively. Furthermore, the (super-)subscript $j$ will be dropped.

It was already shown in section 4.2 that $D_{\mu\mu}$ and $D_{pp}$ vanish for a dynamical magnetic turbulence. Since $S_{\mu\mu} = n/\omega_j$ for $\omega_j = 0$, one obtains for $D_{\mu\mu}$, equation (4.75), the following expression:

$$
D_{\mu\mu} = \frac{\Omega_0^2}{B_0^2} (1 - \mu^2) \sum_{r=\pm 1} \sum_{n=0}^{\infty} \Re \int d^3k R_{D,R} A(k)
$$
4.6 The Fokker-Planck Coefficients for Particle Drift

where the subscripts $D$ and $R$ refer to the damping and the random sweeping model, respectively.

The real parts of the corresponding resonance functions are then given by (4.14a) and (4.14b), respectively. Note that $D_{μμ}$ is not restricted to the slab model initially used. Therefore, it generalizes earlier presentations and is presented in this form for the first time. The treatment of the second, non-vanishing FPC $D_⊥$ is similar. With $ω_j = 0$ in the magnetic turbulent fields (4.47), (4.48) and (4.49), equation (4.79) can readily be cast into the form

$$D_⊥ = \frac{v^2}{B_0^2} \sum_{r=±1} \sum_{n=0}^{∞} \Re \int dk_R R_{D,R} A(k) k^{-2} \left[ J_n^2(W) \left( \frac{rn_{⊥} k_{∥}}{W} \sqrt{1 - μ^2 + μk_{∥}} \right)^2 + μ^2 k_{∥}^2 \right]$$

$$- 2σ J_n(W) J_n’(W) μ \sqrt{1 - μ^2} k_{∥} + [J_n’(W)]^2 (1 - μ^2) k_{∥}^2$$

where the resonance function behaves according to (4.14a) or (4.14b) for the damping or the random sweeping model, respectively. For the first time, a FPC for perpendicular particle diffusion is derived in this general form, based on the plasma wave approach.

In order to restrict equation (4.93) to the originally introduced slab model and, furthermore, to prove this general version, the power spectrum (4.59) for a slab turbulence is used. Evaluation of the Bessel functions for a vanishing argument yields

$$D_{μμ} = \frac{πΩ_α^2}{B_0^2} (1 - μ^2) \sum_{r=±1} \Re \int dk_{∥} g(k_{∥}) R_{D,R} \left( 1 + r σ \frac{k_{∥}}{|k_{∥}|} \right)$$

Equation (4.95) agrees, apart from a dimensionless constant, with results presented earlier by Schlickeiser and Achatz [1993]. Of course, equation (4.95) can also simply be derived from the FPC (4.84) derived on the basis of the plasma wave viewpoint approach and the slab limit. Following Schlickeiser and Achatz [1993], it is instructive to restrict equation (4.95) to the magnetostatic limit, i.e. $α = 0$. This choice leads, for the resonance function in the damping model, to the expression

$$R_D = π δ(k_{∥} v_{∥} + rΩ_α) = \frac{π}{v_{∥} |μ|} δ \left( k_{∥} + \frac{rΩ_α}{μ |μ|} \right)$$

Substitution of the latter relation into equation (4.95) yields

$$D_{μμ} = \frac{π^2 Ω_α^2}{v_{∥} |μ| B_0^2} (1 - μ^2) g \left( k_{∥} = \frac{|Ω_α|}{μ |μ|} \right) \left[ 1 - σ \left( \frac{|Ω_α|}{μ |μ|} \right) \frac{μΩ_α}{μ |μ|} \right]$$

As it was pointed out by Schlickeiser and Achatz [1993], equation (4.97) shows that particles with $μΩ_α > 0$ interact resonantly with a right-hand polarized turbulence, while particles with $μΩ_α < 0$ interact resonantly with left-hand polarized waves (see Hasselmann and Wibberenz, 1968). Furthermore, eq. (4.97) exhibits no resonance gap for particles having the pitch-angle $μ = 0$. In the general case of a non-vanishing $α$, the resonance function (4.14a) leads to a broadening, since it is a Lorentzian-type resonance function.

4.6 The Fokker-Planck Coefficients for Particle Drift

So far, the standard FPCs $D_{μμ}$, $D_{μp}$ and $D_{pp}$ as well as the perpendicular coefficient $D_⊥$ were only considered and presented in their general forms. However, the transport parameter representation
(3.23b) also enables one to calculate not only the perpendicular diffusion coefficient $\kappa_\perp$, but also the drift coefficients $\kappa_{XY}$ and $\kappa_{YX}$. They are determined by the FPCs $D_{XY}$ and $D_{YX}$ and enter the diffusion tensor (3.25) via the first drift tensor $K_T$. As explained in section 3.3.2, the drift coefficients $\kappa_{XY}$ and $\kappa_{YX}$ are, presumably, related to particle drifts due to the turbulence. It has to be stressed that the drifts in $K_T$ have nothing in common with particle drift effects due to the large-scale magnetic field embedded in the background plasma, i.e. $K_D$. The aim of this section is to investigate $D_{XY}$ and $D_{YX}$ and their mathematical structure in more detail and, furthermore, to provide a mathematical basis for section 6.2. There, the calculations of the drift coefficients $\kappa_{XY}$ and $\kappa_{YX}$ will be presented.

The scheme for deriving general forms of $D_{XY}$ and $D_{YX}$ is the same as for the FPCs presented in section 4.2. This means, the representation (4.6) has first to be inserted into the FPCs (A.13) and (A.14) given in appendix A. Then, the Bessel function identities (4.15) are applied to (A.13) and (A.14). Having done this, the electromagnetic tensors $Q^j_{\alpha\beta}$, $T^j_{\alpha\beta}$ and $R^j_{\alpha\beta}$ are expressed by the components of the helical magnetic correlation tensor $P^j_{\alpha\beta}$ (see section A.3). Subsequently, all helical magnetic intensities are expressed by Cartesian expressions. The corresponding list is given in section A.4. One obtains, after arduous algebra, the following general representations:

$$D_{XY} = \frac{\nu^2}{B^2_0} \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \Re \int k R_j \left[ J^j_n(W) F^j_{XY} - J^j_n(W) J^j_n(W) G^j_{XY} - [J^j_n(W)]^2 H^j_{XY} \right]$$

$$D_{YX} = \frac{\nu^2}{B^2_0} \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \Re \int k R_j \left[ J^j_n(W) F^j_{YX} + J^j_n(W) J^j_n(W) G^j_{YX} - [J^j_n(W)]^2 H^j_{YX} \right]$$

where the resonance function $R_j$ is still given by equation (4.8). The corresponding FMFs of $D_{XY}$ and $D_{YX}$ are found to be

$$F^j_{XY} = a^2 \frac{k_x k_y}{k^2_\perp} P^j_{\parallel \parallel} + b^2 P^j_{\perp \parallel} + \frac{ab}{k_\perp} \left[ k_x P^j_{\parallel \perp} + k_y P^j_{\perp \perp} \right]$$

$$F^j_{YX} = a^2 \frac{k_x k_y}{k^2_\perp} P^j_{\parallel \perp} + b^2 P^j_{\perp \perp} + \frac{ab}{k_\perp} \left[ k_x P^j_{\perp \parallel} + k_y P^j_{\perp \perp} \right]$$

$$G^j_{XY} = \sqrt{1 - \mu^2} \left[ a P^j_{\parallel \parallel} + \frac{b}{k_\perp} \left[ k_x P^j_{\perp \parallel} + k_y P^j_{\perp \perp} \right] \right]$$

$$G^j_{YX} = \sqrt{1 - \mu^2} \left[ a P^j_{\parallel \perp} + \frac{b}{k_\perp} \left[ k_x P^j_{\perp \parallel} + k_y P^j_{\perp \perp} \right] \right]$$

$$H^j_{XY} = H^j_{YX} = \left( 1 - \mu^2 \right) \frac{k_x k_y}{k^2_\perp} P^j_{\parallel \perp}$$

The functions $a$ and $b$ were already defined in equation (4.39). By substituting the elements of the magnetic correlation tensor (4.50) into (4.100) through (4.104), the FMFs can be manipulated to obtain

$$F^j_{XY} = \frac{A^j}{k^2 k^2_\perp} \left[ k_x k_y \left( (ak_\perp - bk_\parallel)^2 - b^2 k^2 \right) - i\sigma^j k^2_\perp k(ak_\perp - bk_\parallel) \right]$$

$$F^j_{YX} = \frac{A^j}{k^2 k^2_\perp} \left[ k_x k_y \left( (ak_\perp - bk_\parallel)^2 - b^2 k^2 \right) + i\sigma^j k^2_\perp k(ak_\perp - bk_\parallel) \right]$$
\[
\frac{G_{XY}}{A^j} = \frac{A^j}{k_\perp k^2} \sqrt{1 - \mu^2} \left[ k_\perp^2 (a k_\perp - b k_\parallel) - i 2 \sigma^j b k k_x k_y \right] 
\]  
(4.107)

\[
\frac{G_{YX}}{A^j} = \frac{A^j}{k_\perp k^2} \sqrt{1 - \mu^2} \left[ k_\perp^2 (a k_\perp - b k_\parallel) + i 2 \sigma^j b k k_x k_y \right] 
\]  
(4.108)

\[
H_{XY}^j = H_{YX}^j = A^j (1 - \mu^2) \frac{k_x k_y}{k^2} 
\]  
(4.109)

At a second glance, the reason for presenting the calculations of \(D_{XY}\) and \(D_{YX}\) in a separated section becomes clear now. A closer inspection of the FMFs (4.105) through (4.109) and a comparison with the FMFs of, for instance, \(D_{\mu\mu}\) (see equations (4.63), (4.65) and (4.63) as well as and the comments following it) results in the following findings: first, the magnetic helicity \(\sigma^j\) enters the FMFs of \(D_{XY}\) and \(D_{YX}\) not only by \(G_{XY}^j\) and \(G_{YX}^j\), but also by \(F_{XY}^j\) and \(F_{YX}^j\). Second, \(\sigma^j\) is not a simple factor in \(G_{XY}^j\) and \(G_{YX}^j\) anymore, it only determines one contribution of the fields. Third, as a consequence of this, \(G_{XY}^j\) and \(G_{YX}^j\) are not purely imaginary anymore, they also reveal a real part. Since \(\sigma^j\) enters \(F_{XY}^j\) and \(F_{YX}^j\), these fields are not purely real anymore, they also have imaginary parts. These three modifications did not occur in earlier considerations, neither in the case of the standard FPCs \(D_{\mu\mu}\), \(D_{\mu p}\) and \(D_{pp}\), nor in the case of \(D_{\perp}\). Since \(F_{XY}^j\), \(F_{YX}^j\), \(G_{XY}^j\) and \(G_{YX}^j\) are complex fields now, these modifications imply that the resonance function \(\Re J_j\), equation (4.8), has to be split into its real and imaginary part, and both contributions have to be taken into account. This was not necessary before, because the other FPCs only involved the real part of (4.8), i.e. resonant interactions. It will be shown in turn that the resonant interaction terms of \(D_{XY}\) and \(D_{YX}\) vanish exactly. The only non-vanishing term is then represented by the imaginary part of the resonance function (4.8).

To enable a further treatment of equations (4.98) and (4.99), it is convenient to consider first the imaginary part of the resonance function (4.8). By making use of the relation (4.78), the imaginary part of (4.8) can also be written in the form

\[
\Im \Re J_j = - \frac{(a k_\perp - b k_\parallel)/v}{\Gamma^j/v^2 + (a k_\perp - b k_\parallel)^2} 
\]  
(4.110)

Now, the fields \(F_{XY}^j\), \(G_{XY}^j\) and \(H_{XY}^j\) are substituted into equation (4.98). Separating the real and imaginary parts of \(F_{XY}^j\) and \(G_{XY}^j\) and, furthermore, taking into account that one has to take the real part of these contributions, the following form of equation (4.98) can be obtained:

\[
D_{XY} = \frac{v^2 B_0^2}{A^j} \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \int d^3 k k_x k_y \left[ \Re J_n^j (W) \frac{A^j}{k^2 k_\perp^2} \right] J_n^2 (W) \left( (a k_\perp - b k_\parallel)^2 - b^2 k^2 \right) 
\]  
(4.111)

\[
- 2 \sigma^j J_n (W) J'_n (W) \sqrt{1 - \mu^2} b k k_\perp - \left[ J'_n (W) \right]^2 (1 - \mu^2) k_\perp^2 
\]

\[
- \frac{v B_0^2}{A^j} \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \int d^3 k \frac{(a k_\perp - b k_\parallel)^2}{\Gamma_j^2/v^2 + (a k_\perp - b k_\parallel)^2} \frac{A^j}{k^2} \left[ \sigma^j b k J_n^2 (W) + J_n (W) J'_n (W) \sqrt{1 - \mu^2} k_\perp \right] 
\]

where it is made use of \(\Re (i \Re J_j) = -\Im \Re J_j\). Furthermore, the imaginary part \(\Im \Re J_j\) is already expressed by equation (4.110). The first contribution is characterized by the real part of the resonance function (4.8), i.e. resonant wave-particle interactions. The second term, however, is solely determined by the non-resonant part of (4.8), i.e. expression (4.110). Considering equation (4.111) for the limit
of vanishing wave damping, the first contribution is determined by the expression (4.9), i.e. by Dirac’s delta distribution. This was also the case for the other FPCs presented above. In the second term, however, the expression \( ak_\perp - bk_\parallel \) is automatically dropped for the limit \( \Gamma_j \to 0 \). Furthermore, an additional dependence on the particle speed \( v \) vanishes (in the denominator of the first fraction appearing in the integrand). Performing the same calculations for \( D_{YX} \), equations (4.98) and (4.99) are expressible as follows:

\[
\begin{align*}
D_{XY} & = -R k_x k_y + \begin{cases} -N \\ +N \end{cases} \\
D_{YX} & = \begin{cases} -N \\ +N \end{cases}
\end{align*}
\]

Here, the integral operator \( R \) is given by

\[
R = \pi \frac{v^2}{B_0^2} \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \int d^3 k \delta (k_\parallel v) - \omega_{R,j} + n \Omega_\alpha) \frac{A_j}{k^2 k_\perp^2} \\
\times \left[ J_n^2(W) b^2 k^2 + 2 \sigma_j J_n(W) J_n'(W) \sqrt{1 - \mu^2 b k k_\perp^2} + [J_n'(W)]^2 \right]
\]

The second contribution, \( N \), can be written as

\[
N = \frac{v}{B_0^2} \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \int d^3 k A_j k^{-2} \left[ \sigma_j b k J_n^2(W) + J_n(W) J_n'(W) \sqrt{1 - \mu^2 k_\perp^2} \right]
\]

In view of the wavenumber representation \( k_x = k_\perp \sin \phi \) and \( k_y = k_\perp \cos \phi \), it becomes obvious that the resonant contribution \( R \) vanishes exactly. If no additional assumptions are made concerning an additional \( \phi \)-dependence of the wave power spectrum \( A_j \), the integration with respect to \( \phi \) is equal to zero. Therefore, the following important result can be obtained:

\[
D_{XY} = -D_{YX} = -N
\]

At a first glance, \( D_{XY} \) and \( D_{YX} \) are antisymmetric and completely characterized by the non-resonant contribution \( N \). The transport parameter representation (3.23b) then implies

\[
\kappa_T = \kappa_{XY} = -\kappa_{YX} = -\frac{1}{2} \int_{-1}^{1} d\mu N
\]

The drift tensor \( K_T \), introduced in section 3.3.2 by the diffusion tensor (3.25), then indeed reveals an antisymmetric nature for \( \kappa_{XY} = -\kappa_{YX} = \kappa_T \), with \( \kappa_T \) being a so far undetermined drift transport parameter. This has already been stated in section 3.3.2, and the result (4.116) is the proof for it. Equation (4.116) offers a tractable tool for investigating particle drifts, due to the turbulence, in an isotropic plasma wave turbulence. It should be noted that this has never been done before. As described in section 3.3.2, earlier results are related to curvature and gradient drifts of charged particles in the large-scale averaged magnetic field. So, they strictly have to be distinguished from the drift concept considered here. It should also be noted that equation (4.116) allows to include plasma wave damping in a simple manner. The first term of equation (4.111) vanishes in any case if \( \Gamma_j, A_j \) and \( \omega_j \) are independent of the azimuthal angle. Then, the non-resonant contribution would be given by the second term of equation (4.111). In general, this would allow to investigate the influence of a changing plasma-\( \beta \) on particle drifts. Furthermore, the additional dependence on the particle speed \( v \) would be taken into account. A topic which will be left for future studies. For the sake of simplicity, only undamped waves are assumed for a further treatment of the non-resonant contribution \( N \). The evaluation of \( N \) and the calculation of \( \kappa_T \) are the topic of section 6.2.
4.7 Summary and Conclusions

In this chapter, a thorough insight into the mathematical structure and the proper meaning of all FPCs derived in appendix A is offered. For this, the Bessel function representation of all FPCs is rearranged and the helical description of small-scale electromagnetic fields is dropped and expressed by Cartesian coordinates. This procedure has the advantage that each coefficient is expressible as a sum of three terms. It was shown that each contribution includes either $J_2^2(W)$, $J_n(W)J'_n(W)$ or $[J'_n(W)]^2$ and, furthermore, is accompanied by a specific factor. These factors are different for each sum term and FPC. All electric fields in these factors are then expressed by their magnetic counterparts. Hence, these fields are called fluctuating magnetic fields (FMFs). They contain not only the plasma wave dispersion relation but also the components of the Cartesian magnetic correlation tensor $P_{jm}$ and, therefore, the complete information about the underlying turbulence and its evolution in the three-dimensional wavenumber space. Since the underlying turbulence physics is shifted into the FMFs, the coefficients collapse to more compact expressions. The new FPC representations reveal the mathematical structure of all FPCs much clearer than earlier presentations. Moreover, they are valid for an arbitrary turbulence geometry and an arbitrary plasma wave dispersion relation. Therefore, the new FPCs allow to include a general plasma wave damping function into future calculations. Comparable considerations of FPCs on such a general level are not known.

In order to obtain more tractable coefficients, a specific magnetic correlation tensor for an isotropic turbulence is used. The corresponding FMFs are calculated and it is shown that the magnetic helicity enters the standard as well as the perpendicular FPCs in a characteristic manner. The isotropic FPCs are presented for three different restrictive limits. First, a vanishing plasma wave damping is considered. Apart from $D_{XY}$ and $D_{YX}$, it is shown that the FPCs are expressible as binomial formulas for a turbulence consisting only of either left- or right-handed polarized waves. For the limit of a slab geometry it is shown that earlier results of Schlickeiser [1989a] and Schlickeiser and Achatz [1993] are generalized and, for a vanishing wave damping, retained. Furthermore, the FPC for perpendicular particle diffusion is explicitly calculated on the basis of the plasma wave approach. The quintessence is that particle motion normal to the background magnetic field cannot exist for a slab turbulence. The particles remain tied to the background magnetic field. This is explained by the reduced dimensionality of the slab geometry in both the wavenumber as well as the spatial configuration space. The calculations support the JRG theorem derived by Jokipii et al. [1993] and vice versa. The limit of a dynamical magnetic turbulence is achieved by assuming a vanishing plasma wave dispersion relation. For the dynamical magnetic turbulence, the FPCs for isotropic geometry are presented for the first time. Earlier results of Schlickeiser and Achatz [1993] are obtained for the slab limit of the dynamical magnetic turbulence approach. Finally, the FPCs $D_{XY}$ and $D_{YX}$ are investigated in detail. It is shown that the resonant contribution vanishes. Both FPCs are then determined only by the imaginary part of the fundamental resonance function. As a consequence of this, $D_{XY}$ and $D_{YX}$ are solely characterized by a non-resonant term and reveal an antisymmetric feature. This implies an antisymmetric structure of the drift tensor $\kappa_T$ and, thus, supports the suggestion that $\kappa_T$ represents particle drifts in a plasma wave turbulence.
Chapter 4 On Fokker-Planck Coefficients
Chapter 5
On Solar Wind Magnetic Fluctuation Spectra

In the previous chapter, the FPCs and their dependences on the plasma wave dispersion relation and the magnetic correlation tensor were considered in detail. The elements of the magnetic correlation tensor involve the wave power spectrum. This spectrum represents the evolution of the turbulence intensity, i.e. fluctuating magnetic energy, in wavenumber space. The physics of this wavenumber evolution and, furthermore, of the spatial variation of the turbulence may be considered as a separated research field within heliospheric physics. The investigation of the power spectrum behavior in both the wavenumber as well as spatial space is beyond the purview of this thesis. Hence, the focus of this chapter is the ubiquitous solar wind feature called magnetic fluctuations and their evolution only in wavenumber space.

Solar wind magnetic fluctuation power spectra at frequencies below 1 Hz are commonly observed to have an approximate power law dependence with spectral index \(-5/3\). These observations may commonly be described by Kolmogorov diffusion in wavenumber space which defines what is called the inertial range. At higher frequencies, spectra often are observed to have steeper power laws. This intermediate wavelength regime is sometimes called the dissipation range, because it has been assumed that the steepening is a consequence of collisionless damping of Alfvén and magneto sonic/whistler waves. However, based on numerical calculations, it is suggested that the damping of both wave modes cannot explain the steepening at higher wavenumbers and is, therefore, not the only physical basis for explaining this steepening in the dissipation range. In view of this discrepancy, an alternative scenario is presented. It is argued that the steepening at intermediate wavenumbers is more a result of the dispersion of magneto sonic/whistler waves than of collisionless damping. For this, it is assumed that the diffusion rate in wavenumber space is increased due to this dispersion. Numerical calculations for small values of plasma beta yield fluctuation spectra with steep power laws at intermediate wavenumbers and sharp cut-offs, due to electron cyclotron damping, at still shorter wavelengths. The dissipation range is, therefore, replaced by the dispersion range for a low plasma beta. With increasing \(\beta_p\), proton cyclotron damping of the magneto sonic/whistler mode becomes more appreciable. The dispersion range disappears and is then replaced by the proton cyclotron dissipation range.

5.1 Basic Concepts

From the Helios and Mariner as well as the Voyager and Pioneer spacecraft propagating through interplanetary space and reaching the inner and outer heliosphere at approximately 0.3 AU and 70 AU, respectively, detailed measurements and associated data of large- and small-scale properties of the solar wind and its embedded turbulence are available over a broad range of heliocentric distances. Plasma experiments on board of these spacecraft, and others such as WIND, allow to determine the fully, three-dimensional plasma distribution function of electrons, protons, and some ion species with great precision.
Figure 5.1: A sketch of a solar wind magnetic power spectrum illustrating the energy range at small scales below $k_s$, the inertial range at intermediate wavenumbers where turbulent cascade processes determine the form of the spectrum, and the dissipation range at higher frequencies, i.e. wavenumbers above $k_d$. $W(k)$ denotes the energy per wavenumber.

Additionally, search-coil magnetometers and other plasma wave instruments provide a detailed characterization of electromagnetic fluctuations being present in the outward expanding solar wind plasma. Analysis of interplanetary magnetic and velocity field fluctuation data obtained by such detectors revealed that the spectra of fluctuations can roughly be divided into three different frequency or, alternatively, wavenumber regimes. These ranges are illustrated schematically in figure 5.1. At the largest scales, i.e. the wavenumbers below $k_s$, observed power spectra can often be fitted by power laws of the form $k^{-1}$. This large scale regime, which corresponds roughly to frequencies lower than around $10^{-3}$ Hz, is usually called the energy range. It is suggested that large-scale processes, such as shear flows near the Sun, release turbulence energy. This energy is injected in $k$ space at small wavenumbers and forms, subsequently, the $k^{-1}$ spectrum. It was suggested by Matthaeus and Goldstein [1986] that such spectra can be interpreted in terms of a superposition of signals stemming from different sources. This large-scale range contains the whole power of the fluctuations and, therefore, represents a pool of energy tapped by processes setting in at smaller scales, i.e. around the illustrated wavenumber $k_s$. It is commonly assumed that nonlinear interaction processes, such as turbulent cascades due to wave-wave interactions, become important at wavenumbers larger than about $k_s$ and, therefore, determine substantially the further evolution of the turbulence power in $k$ space. Since the turbulent cascade taps the energy containing range, the $k^{-1}$ spectrum gradually erodes to the often observed $k^{-5/3}$ spectrum. Based on Helios data, Bavassano et al. [1982] pointed out that the transition around $k_s$ is shifted to smaller wavenumbers and, furthermore, that the spectral index of the low-frequency energy range varies with increasing heliocentric distance, respectively; a finding which is attributed to the dynamical evolution of the expanding solar wind plasma. The regime in wavenumber space produced by the gradual erosion is called the inertial range of fluctuation power spectra. It is sketched in figure 5.1. In this inertial range, the nonlinear interactions allow the fluctuation energy to cascade to successively higher frequencies, i.e. to larger values of the wavenumber $k$. At smaller scales (larger wavenumbers), the high-frequency limit of the inertial range coincides with the Doppler-shifted proton gyrofrequency $\Omega_p$. The latter is usually observed at frequencies from around 0.1 Hz going up to 1 Hz. This is illustrated in figure 5.1 by the wavenumber $k_d$. Observations at frequencies higher than this point show that magnetic power spectra can roughly be described by power laws with a spectral index covering a range lower than about 4.5 and greater than 2.5. This means the power spectrum

\[ W(k) \sim k^{-1} \]

\[ W(k) \sim k^{-5/3} \]

\[ W(k) \sim k^{-3} \]

\[ W(k) \sim k^{-3} \]

\[ W(k) \sim k^{-3} \]

Assuming that a single correspondence exists between the spacecraft rest frame frequency $\nu$ and the wavenumber $k$, the spectral fluctuation energy $W$ is considered in wavenumber $k$ space instead of the frequency.
breaks at the inertial high-frequency limit and becomes steeper in order to have, for instance, a slope around 3 (see figure 5.1).

The nature of this steepening is, still nowadays, a subject for intensive studies. Since the spectral break wavenumber $k_d$ seems to be associated with the proton cyclotron frequency $\Omega_p$, it has been suggested that the onset of magnetic dissipation at frequencies around this spectral breakpoint leads to the distinct steepening of solar wind power spectra observed above the Doppler-shifted proton gyrofrequency. Thus, this high-frequency regime is also referred to as the dissipation range of fluctuation spectra, indicating that the break in power laws around $\Omega_p$ is a consequence of the onset of dissipation around $k_d$. Therefore, $k_d$ is also referred to as the dissipation wavenumber. Under the fundamental assumption that observed magnetic fluctuations represent an ensemble of linear plasma wave modes, the process of dissipation is commonly attributed to plasma wave damping due to wave-particle interactions. In order to describe this dissipation, several authors used collisionless damping of linear Alfvén and/or magnetosonic wave modes (see, e.g., Marsch, 1991; Leamon et al., 1998; Gary, 1999; Marsch, 1999).

However, another school of thought considers the effect of rugged invariants on the rate of nonresonant turbulent cascades at the proton cyclotron scales (see Gosh et al. [1996] and references therein). Such a global invariant, namely the so-called hybrid helicity, was first reported by Turner [1986] in the context of ideal homogeneous incompressible MHD including the nondissipative Hall term. This was extended to compressible Hall MHD by Gosh et al. [1996]. In this picture, no dissipation due to wave-particle interactions is required to explain the steepening around the proton cyclotron frequency.

5.2 The Turbulence Model

In order to investigate the evolution of the solar wind turbulence in heliocentric distance and time, a variety of transport models has been presented during the last four decades. Based on measurements performed on the Mariner 5 spacecraft, Belcher and Davis [1971] pointed out that the interplanetary turbulence might consist of Alfvén waves with large amplitudes. To understand these measurements from a theoretical point of view, the first transport models were developed subsequently by Hollweg [1973a, 1973b, 1974] and many others (e.g., Barnes and Hollweg, 1974; Whang, 1973) on the basis of the WKB theory. The fundamental concept of this approach is that the observed fluctuations are described by plasma waves, i.e. Alfvén wave modes, traveling as passive remnants to the outer heliosphere. However, the WKB theory is generally restricted to a weak inhomogeneous medium carrying small-amplitude fluctuations. Furthermore, it is valid only for the case of noninteracting waves and, therefore, excludes inward propagating waves. Nevertheless, WKB transport models reflect the spatial dependence of the observed turbulence energy density quite reasonably over a broad range of heliocentric distances, but, however, they completely fail to explain other important features of the solar wind plasma such as observed proton temperature profiles in heliocentric distance (see, e.g., Coleman, 1968; Matthaeus et al., 1999). While WKB theory predicts a strictly monotonically decreasing proton temperature with increasing heliocentric distance, observations indicate a strong nonadiabatic behavior of temperature profiles (see Matthaeus et al. [1999], figure 3). This discrepancy results from the fact that WKB theory explicitly excludes wave-wave interactions supporting turbulence cascades and, therefore, does not allow particle heating throughout all heliocentric distances. As a further example: WKB theory cannot explain the above mentioned shift of the transition wavenumber $k_s$ to smaller wavenumbers with increasing heliocentric distance. In order to broaden the potential of the WKB approach of the solar wind plasma, various extensions were subsequently presented to include turbulent cascade by phenomenological heating terms (e.g., Heinemann and Olbert, 1980; Tu et al., 1984, 1989; Tu, 1988).
In order to investigate the transport of fluctuations with and in the solar wind plasma in more detail, a general approach has been presented by several scientists (Marsch and Tu, 1989; Tu and Marsch, 1990; Zhou and Matthaeus, 1989, 1990a, 1990b). Based on compressible MHD equations, they developed a non-WKB multiple-scale spectral transport theory which reveals some similarities to the above mentioned traditional WKB method. Using the Elsässer variables $\mathbf{z}^\pm = \mathbf{V} \pm \mathbf{v}_A$, where the + and − signs refer to outward and inward propagating waves and $\mathbf{V}$ and $\mathbf{v}_A$ denote the bulk and Alfvén velocity, respectively, Zhou and Matthaeus [1990a] derived the following transport equations:

$$\frac{\partial P^\pm}{\partial t} + L^\pm P^\pm + M^\pm F^\pm = N^\pm_L$$

Here, $P^\pm$ is the reduced fluctuation power spectrum (the power in $\mathbf{z}^\pm$). Neglecting any effects of the cross helicity, $P^\pm$ represents the fluctuating energy (per unit mass) according to $P^\pm \propto (\mathbf{z}^+ \cdot \mathbf{z}^+ + \mathbf{z}^- \cdot \mathbf{z}^-) = W_v + W_B$, where $W_v$ and $W_B$ denote the kinetic and the magnetic energy, respectively. The differential operator

$$L^\pm = (\mathbf{V} \mp \mathbf{v}_A) \cdot \nabla \mp \nabla \cdot \left( \frac{\mathbf{V}}{2} \pm \mathbf{v}_A \right)$$

is closely related to the standard differential operator used in WKB theory (e.g. Hollweg, 1973a, 1973b, 1974) and describes spatial transport effects such as convection and expansion on the fluctuation power spectrum. Essentially, $F^\pm$ denotes the difference of the kinetic and the magnetic energy. The quantity $M^\pm$ represents the coupling, i.e. reflection, scattering or mixing, between the inward and outward propagating fluctuations. It involves gradients of both the large-scale magnetic field and the small-scale turbulence (see Zhou and Matthaeus, 1990a). The term $N^\pm_L$ on the right-hand side of equation (5.1) describes complex phenomena such as turbulent cascades due to nonlinear interactions. In general, it represents an energy change and, therefore, a diffusion of turbulence energy in wavenumber space. This term and its influence on interplanetary power spectra is the focus of this chapter.

### 5.2.1 Diffusion of Turbulence Energy in Wavenumber Space

The approach to describe the evolution of turbulence by diffusion of fluctuation energy in wavenumber space was pioneered by Leith [1967] in hydrodynamics. In order to provide a simple framework to consider the evolution of turbulence in space physics, Zhou and Matthaeus [1990c] subsequently introduced this approach to the context of interplanetary magnetohydrodynamics. Using a diffusion approximation as well as phenomenological and scaling arguments, they derived a transport equation for the fluctuation spectral energy density in magnetohydrodynamics.

Neglecting any spatial dependences, i.e. $L^\pm P^\pm = 0$, and, furthermore, assuming equal kinetic and magnetic energy densities, i.e. $F^\pm = 0$, equation (5.1) readily yields

$$\frac{\partial P^+}{\partial t} = N^+_L \quad \text{and} \quad \frac{\partial P^-}{\partial t} = N^-_L$$

Since $P^+ + P^- \propto \hat{W}$, these two equations can be summarized to yield a transport equation for the three-dimensional spectral energy density $\hat{W}(\mathbf{k}, t)$. Assuming that the time evolution of $\hat{W}$ is, in addition to the nonlinear processes, determined by dissipative processes and a source, the corresponding transport equation reads

$$\frac{\partial \hat{W}}{\partial t} = N_L + Q + \left. \frac{\partial \hat{W}}{\partial t} \right|_{\text{dis}}$$

Here, the first term on the right-hand side is the contribution resulting from the cascading transport of turbulent energy. The second term represents a source function $Q$ with which fluctuation energy
is injected at low wavenumbers into the energy range. The last term is related to sinks of the modal energy spectrum. These sinks are due to dissipative processes. In general, the cascade term $N_L$ can be described as a sum of several contributions resulting from different wave-wave couplings covering the whole wavenumber space (see, e.g., Kraichnan, 1965; Batchelor, 1953). Zhou and Matthaeus [1990c] introduced a simple approach by restricting their calculations to a net spectral energy transfer rate, which is a local quantity in wavenumber space and results from the sum of all nonlinear wave-wave couplings. This means, a continuous flux of fluctuation energy density is present in wavenumber space. Denoting this turbulent flux in three-dimensional wavenumber space with $\mathbf{F}(\mathbf{k})$, the time rate of the fluctuation energy change in wavenumber space can be expressed via the conservation equation

$$N_L = -\nabla_k \cdot \mathbf{F}(\mathbf{k})$$ \hspace{1cm} (5.5)

where $N_L$ is now given by a divergence of the turbulent energy flux in wavenumber space. Zhou and Matthaeus [1990c] applied a diffusion approximation to the latter equation by adopting a Fick’s law relation between the fluctuating energy flux $\mathbf{F}$ and the modal spectral energy density $\hat{W}$. Making use of the expression

$$\mathbf{F} = -D \nabla_k \hat{W}$$ \hspace{1cm} (5.6)

the term $N_L$ can be manipulated to obtain the form

$$N_L = \nabla_k \left( D \nabla_k \hat{W} \right)$$ \hspace{1cm} (5.7)

The nonlinear term is now expressed by a diffusive term. In general, it involves the wavenumber diffusion tensor $D(\mathbf{k})$, which may depends on any relevant turbulence parameters. Substituting equation (5.7) into (5.4), one readily arrives at

$$\frac{\partial \hat{W}}{\partial t} = \nabla_k \cdot \left[ D \cdot \nabla_k \hat{W} \right] + \Gamma \hat{W} + Q$$ \hspace{1cm} (5.8)

Here, it was made use of the assumption that dissipative processes acting at small scales are characterized by a plasma wave damping function $\Gamma(k) < 0$, so that the rate for dissipation is expressed as follows:

$$\left. \frac{\partial \hat{W}}{\partial t} \right|_{\text{dis}} = \Gamma(k) \hat{W}(k)$$ \hspace{1cm} (5.9)

Assuming an isotropic turbulence in wavenumber space and introducing the one-dimensional and omnidirectional power spectrum $W(k,t) = 4\pi k^2 \hat{W}(k,t)$, equation (5.8) can be written as

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial k} \left[ k^2 D_{kk} \frac{\partial}{\partial k} \left( k^{-2} \hat{W} \right) \right] + \Gamma W + S$$ \hspace{1cm} (5.10)

Here, the source function $S(k,t) = 4\pi k^2 Q(k,t)$ allows to inject turbulence energy into wavenumber space at a certain position. In analogy to the spatial diffusion tensor $K$, equation (3.32), described in the context of the diffusion-convection equation in chapter 3, the component $D_{kk}$ represents the diffusion of fluctuation energy in radial direction of the wavenumber space. $D_{kk}$ characterizes wave-wave interactions permitting the interplanetary magnetic field fluctuation energy to cascade to higher frequencies and/or wavenumbers where resonant wave-particle interactions become more important. These nonlinear couplings, which determine the inertial range at intermediate wavenumbers, do not change the total amount of energy, they only rearrange the energy from larger scales to higher wavenumbers. In view of this, the source function $S$ and the dissipation function $\Gamma$ are the only energy-changing contributions in equation (5.10). In general, this means that the inertial range is well separated from the $k$ space regions associated with sources and dissipation. In particular for steady state conditions, the inertial range is characterized completely by the energy-conserving nonlinear wave-wave interactions, i.e. $D_{kk}$. The presence of inertial range energy spectra is a characteristic feature of turbulent fields exhibiting a high Reynolds number.
5.2.2 The Kolmogorov and Kraichnan Phenomenologies

The scalar wavenumber diffusion coefficient $D_{kk}$, which is so far still undetermined, depends significantly upon the phenomenology of wave-wave interactions describing the underlying cascade mechanisms. Two approaches and their applications to astrophysical and heliospheric problems are well-known in turbulence theory, namely the so-called Kolmogorov approach and the Kraichnan theory (see Zhou and Matthaeus, 1990c). Using the formulation of Kraichnan [1965] one obtains for the inertial energy flux, i.e. the energy transfer rate, the expression

$$\epsilon = A^2 \tau_3(k) k^4 W^2(k)$$

(5.11)

where $A$ and $\tau_3$ denote a constant and the decay time scale of triple correlations, respectively. The latter determines cascade processes and induces the turbulent spectral transfer in wavenumber space. In other words, $\tau_3$ is the time over which a fluctuation, i.e. a plasma wave with wavelength $\lambda$, interacts with other fluctuations having wavelengths around $\lambda$. In general, $\tau_3$ may depend significantly on any important turbulence parameters.

In order to make further progress, it is instructive to consider the nonlinear cascade term in equation (5.10) in more detail.

$$\frac{\partial}{\partial k} \left[ k^2 D_{kk} \frac{\partial}{\partial k} \left( k^{-2} W \right) \right] = \frac{\partial \epsilon}{\partial k}$$

(5.12)

On the basis of equation (5.12) and dimensional analysis, the structure of the nonlinear diffusion coefficient in wavenumber space can readily be determined to be as

$$D_{kk} = \frac{\epsilon k}{W}$$

(5.13)

Upon substituting the Kraichnan relation (5.11) into the last equation, one derives for the functional form of the nonlinear wavenumber diffusion coefficient the expression

$$D_{kk} = A^2 \tau_3(k) k^5 W(k)$$

(5.14)

Since $D_{kk}$ represents the transfer speed of a spectral element through $k$ space, it is directly proportional to the triple correlation time scale $\tau_3$, a longer lifetime $\tau_3$ increases the transfer speed and, therefore, the coefficient $D_{kk}$. The total decay rate of triple correlations, i.e. the reciprocal of $\tau_3$, can generally be presented as the sum of decay rates resulting from different effects, i.e.

$$\frac{1}{\tau_3} = \sum_i \frac{1}{\tau_i}$$

(5.15)

where the sum index $i$ refers to these various processes. At this point, several approaches have been introduced to estimate the decay rate $1/\tau_3$. Initially, Kolmogorov [1941] considered for homogeneous stationary isotropic turbulence the nonlinear eddy turn-over time scale $\tau_{nl}$, a longer lifetime $\tau_3$ increases the transfer speed and, therefore, the coefficient $D_{kk}$. The total decay rate of triple correlations, i.e. the reciprocal of $\tau_3$, can generally be presented as the sum of decay rates resulting from different effects, i.e.
that the lifetime of triple correlations scales as the Alfvén time scale \( \tau_A \). The phenomenologies by Kolmogorov [1941] and Kraichnan [1965] are most popular and were summarized nicely by Zhou and Matthaeus [1990c]. They also presented an extended approach to consider both phenomenologies as special limits. This is not considered here.

In order to determine \( D_{kk} \) for the Kolmogorov and the Kraichnan phenomenologies, it is instructive to note that the velocity fluctuation \( \delta v \) is related to the magnetic field of the plasma wave, i.e. \( \delta B \), by the expression

\[
(\delta v)^2 = v_A^2 \frac{(\delta B)^2}{B_0^2} = v_A^2 k \left[ \frac{W(k)}{2U_B} \right]
\]

(5.16)

Here, \( (\delta B)^2 \) is expressed by the spectral energy density \( W(k) \). \( U_B = B_0^2/8\pi \) denotes the energy density of the background magnetic field \( B_0 \). With \( \lambda = 1/k \), the nonlinear time scale \( \tau_{nl} = \lambda/\delta v \) may be written as

\[
\tau_{nl} = \frac{1}{v_A^2 k^{3/2}} \sqrt{\frac{2U_B}{W(k)}}
\]

(5.17)

Considering first the Kolmogorov phenomenology, plasma waves of a comparable wavelength interact in one turn-over time \( \tau_{nl} \). This means the triple lifetime \( \tau_3 \) has to be chosen to be the wavenumber dependent eddy turnover time scale \( \tau_{nl} \). Under this condition, equation (5.17) can be substituted into (5.14) to yield, for the Kolmogorov phenomenology, the wavenumber diffusion coefficient

\[
D_{kk} = A^2 v_A k^{7/2} \left[ \frac{W(k)}{2U_B} \right]^{1/2}
\]

(5.18)

Within the framework of the Kraichnan approach, \( \tau_3 \) scales as the Alfvén time scale \( \tau_A = 1/v_A k \). The resulting diffusion coefficient reads

\[
D_{kk} = A^2 v_A k^4 \left[ \frac{W(k)}{2U_B} \right]
\]

(5.19)

With these diffusion coefficients, equation (5.10) is in either case nonlinear. Upon substituting the coefficients (5.18) and (5.19) into equation (5.10) and assuming steady state conditions with no dissipation, i.e. \( \Gamma(k) = 0 \), one indeed obtains the usual power law behavior \( W(k) \propto k^{-q} \) in the inertial range where \( q = 5/3 \) and \( q = 3/2 \) for the Kolmogorov and the Kraichnan approach, respectively.

### 5.3 Plasma Wave Damping Functions and Dispersion Properties

For a further treatment of equation (5.10), the collisionless damping function \( \Gamma(k) \) has to be specified. Since the solar wind plasma is observed to be Maxwellian-like to a good approximation, a linear Vlasov code of S.P. Gary is used. This code allows to consider fully electromagnetic fluctuations in a collisionless, homogeneous, magnetized plasma. It is assumed that the plasma consists of electrons and protons, where both constituents are described by Maxwell velocity distributions with the same temperature, i.e. \( T_p = T_e \). Here, two modes are of interest, namely the left-hand circular polarized Alfvén wave and the right-handed magnetosonic/whistler wave mode propagating parallel with respect to the background magnetic field \( B_0 \), respectively.

For the Alfvén mode, figure 5.2 shows numerical model results (dots) obtained by the linear Vlasov code for two different values of \( \beta_p \). It illustrates \( \Gamma/\Omega_p \) as a function of the normalized wavenumber \( kc/\omega_p \) for \( \beta_p = 0.1 \) and 1.0. \( \Omega_p \) and \( \omega_p = \sqrt{4\pi n_p q_p^2/m_p} \) are the cyclotron and plasma frequency of the protons, respectively. \( n_p = n_e \) is the proton number density, \( q_p \) and \( m_p \) are the charge and the rest mass of a proton, respectively.
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Figure 5.2: Computational model results showing the damping rate of the Alfvén mode for \( \beta_p = 0.1 \) and 1.0. The dots denote the results obtained by using the code of S.P. Gary. The solid lines represent the fitting function (5.20) for \( \beta_p = 0.1 \) and 1.0, respectively.

The primary damping process for the Alfvén wave mode is attributed to the proton cyclotron resonance. As illustrated in figure 1 of Gary [1999], Alfvén wave mode damping is strongest for the case of parallel propagation. Figure 5.2 illustrates that damping of Alfvén waves at \( k \times B_0 = 0 \) is very weak at small wavenumbers, i.e. \( kc/\omega_p \ll 1 \). However, damping due to the proton cyclotron resonance becomes stronger within a relatively small increment of increasing wavenumber. Furthermore, the onset of the damping is shifted to longer wavelengths (smaller wavenumbers) with increasing plasma beta. In order to obtain an analytical expression for the damping rate, a form as suggested by Gary [1993] (see his equation (6.1.8)) is assumed:

\[
\frac{\Gamma(k)}{\Omega_p} = -m_1 \left( \frac{k^2c^2}{\omega_p^2} \right)^{m_2} \exp \left( -\frac{m_3\omega_p^2}{k^2c^2} \right) \tag{5.20}
\]

Considering wavenumbers corresponding to the range \( \Gamma/\Omega_p \in [-1, 0] \), the fitting parameters \( m_1, m_2 \) and \( m_3 \) can be estimated for the range \( \beta_p \in [0.01, 1.0] \) by the following relations:

\[
m_1 = 0.60\beta_p^{0.36}, \quad m_2 = 0.77\beta_p^{0.03}, \quad m_3 = 0.32/\beta_p^{0.65} \tag{5.21}
\]

The solid lines shown in figure 5.2 represent the function (5.20) for the two values of \( \beta_p \).

In contrast to the Alfvén wave the magnetosonic mode has its strongest damping at oblique propagation. The primary interaction is the Landau resonance \( \omega_r = k_\parallel v_\parallel \) (Barnes, 1966). Damping of the magnetosonic mode becomes stronger with increasing wave propagation angle. At a certain oblique propagation direction it reaches its maximum rate and decreases within further increments of the propagation angle. For quasi-perpendicular conditions (see figure 1 of Gary [1999]), it becomes less appreciable. Panel (a) of figure 5.3 illustrates linear Vlasov solutions (dots) for the damping rate \( \Gamma/\Omega_p \) of the magnetosonic mode at \( k \times B_0 = 0 \) for \( \beta_p = 1.5, 2.5 \) and 3.5 (going from the top to the bottom), respectively. The damping rates of the magnetosonic mode are given for the wavenumber range where significant proton cyclotron damping appears. For low \( \beta_p \), i.e. \( \beta_p < 2.5 \), magnetosonic mode damping is relatively weak but becomes more important with increasing \( \beta_p \). An appropriate fitting function for the damping rate in the proton cyclotron damping regime is

\[
\frac{\Gamma(k)}{\Omega_p} = -\mu_1 \exp \left( -\mu_2 k^2 c^2 / \omega_p^2 \right) \exp \left( -\mu_3 \omega_p^2 / k^2 c^2 \right) \tag{5.22}
\]
5.3 Plasma Wave Damping Functions and Dispersion Properties

(a) Proton cyclotron damping rates

(b) Magnetosonic/whistler mode: frequency

Figure 5.3: Computational model results showing in panel (a) and (b) the damping rate and the real frequency $\omega_r$ of the magnetosonic/whistler mode at $k \times B_0 = 0$ for three different values of the proton plasma $\beta_p$, respectively. The values are 1.5, 2.5 and 3.5 (going in panel (a) from the top to the bottom). For these values, the solid lines given in panel (a) represent the damping rate (5.22). The extrapolation of the dispersionless relation $\omega_r = v_A k$ to shorter wavelengths is indicated in panel (b).

For the domain $\beta_p \in [0.5, 10]$, the fitting parameters $\mu_1$, $\mu_2$ and $\mu_3$ are given by the relations

$$\mu_1 = 0.33\beta_p^{0.54} \exp \left( -3.97/\beta_p^2 \right), \quad \mu_2 = 0.80/\beta_p^{1.07}, \quad \mu_3 = 1.73/\beta_p^{0.91} \quad (5.23)$$

The solid curves given in panel (a) represent the damping rate (5.22) for the corresponding $\beta_p$ values. Panel (b) illustrates model results for the magnetosonic mode frequency $\omega_r/\Omega_p$ at $k \times B_0 = 0$ for the same $\beta_p$ values used in panel (a). The frequency of the magnetosonic/whistler mode is relatively independent of $\beta_p$ in the proton cyclotron damping regime. This agrees with figure 6.3 of Gary [1993]. Performing an approximate empirical fit on the wavenumber domain of panel (b) yields the following analytical expression:

$$\frac{\omega_r}{\Omega_p} = \left( \frac{kc}{\omega_p} \right) + \frac{3}{4} \left( \frac{kc}{\omega_p} \right)^2 \quad (5.24)$$

Since damping of the magnetosonic mode is weak for small values of $\beta_p$, the wave propagation continues to much higher wavenumbers. This is illustrated in panel (b) of figure 5.4. It shows numerical results (dots) for the frequency of the magnetosonic/whistler mode for $\beta_p = \beta_e = 0.1$, 0.25 and 1.0, respectively. The results are limited to damping rates such that $|\Gamma| < \omega_r$. For low $\beta_p$ values, damping is due to the electron cyclotron resonance dominating at much higher wavenumbers. This is shown in panel (a) of figure 5.4 where code results (dots) are presented. Considering, for instance, the case $\beta_p = \beta_e = 0.25$, electron cyclotron damping begins at approximately $kc/\omega_p \sim 26$ and increases monotonically with wavenumber $k$. With increasing plasma $\beta_p$, the onset of the damping is significantly shifted to lower wavenumbers, where proton cyclotron damping becomes
more important for larger values of $\beta_p$. The shape of the damping rates looks very similar to the damping rates of the Alfvén mode presented in figure 5.2. The appropriate fitting function is therefore assumed to be of the following form:

$$\frac{\Gamma(k)}{\Omega_p} = -m_1 \left( \frac{k^2 c^2}{\omega_p^2} \right)^{m_2} \exp \left[ -\frac{m_3 \omega_p^2}{k^2 c^2} \right]$$

(5.25)

This equation has the same structure as the analytical expression (5.2), but $m_1$, $m_2$ and $m_3$ are quite different. For the domain $\beta_p \in [0.1, 10]$, the corresponding fitting parameters of equation (5.25) are given by

$$m_1 = 0.46 \beta_p^{0.26}, \quad m_2 = 1.0, \quad m_3 = 893 / \beta_p^{0.57}$$

(5.26)

The damping rate (5.25) with (5.26) is illustrated in panel (a) for the two corresponding values of $\beta_p$ (solid lines).

![Graph showing electron cyclotron damping rates and magnetosonic/whistler mode frequency](image)

(a) Electron cyclotron damping rates
(b) Magnetosonic/whistler mode: frequency

**Figure 5.4:** Linear Vlasov theory results (dots) displaying in panel (a) the electron cyclotron damping rate of the magnetosonic/whistler mode at $k \times B_0 = 0$ for $\beta_p = 0.25$ and $\beta_p = 1.0$. The solid lines represent damping rates obtained by using the fitting function (5.25) with (5.26) for both values of $\beta_p$. Panel (b): The frequency of the magnetosonic/whistler mode at $k \times B_0 = 0$ for $\beta_p = 0.1, 0.25$ and 1.0. The results are limited to damping rates shown in panel (a) such that $|\Gamma| < \omega_r$.

### 5.4 Numerical Calculations for Kolmogorov Diffusion

Having established the damping rates of both the Alfvén and the magnetosonic/whistler mode, this section introduces numerical solutions of the wave transport equation (5.10) describing the transfer of fluctuation energy in wavenumber space for the Kolmogorov wavenumber diffusion coefficient (5.18). In order to solve equation (5.10) numerically, a code and method developed by Miller and Roberts [1995] is used. This code is changed in so far as it now includes the damping rates derived
5.4 Numerical Calculations for Kolmogorov Diffusion

in the previous section. The algorithm is based on the introduction of logarithmic derivatives in the wave transport equation (5.10). The resulting equation is then finite-differenced according to the Crank-Nicholson technique (for details, see Miller and Roberts, 1995). For all numerical calculations presented below, the turbulence energy is injected at $k c / \omega_p = 2 \cdot 10^{-3}$ at a rate of $10^{-15}$ erg cm$^{-3}$ s$^{-1}$. The background magnetic field $B_0$ and the Alfvén speed $v_A$ are assumed to be $10^{-4}$ Gauss and 30 km s$^{-1}$, respectively.

5.4.1 Magnetic Power Spectra for the Alfvén Mode

In this section, numerical results from solving equation (5.10) are presented. Here, the Kolmogorov diffusion coefficient (5.18) and the Alfvén wave damping rate (5.20) are included. For $\beta_p = 0.5$, figure 5.5 illustrates a typical numerical result representing the temporal evolution of the spectral energy density $W(k)$ as a function of the normalized wavenumber $k c / \omega_p$. Steady-state conditions are attained relatively quickly. The late-time power spectrum follows a $k^{-5/3}$ power law in the inertial range, but it shows an exponential cut-off, or steep roll-over, towards higher wavenumbers. The exponential cut-offs may be understood from the fact that the energy transfer rate in wavenumber space becomes slower than the strongly increasing Alfvén damping rate. In other words, collisionless Alfvén wave damping due to proton cyclotron resonances overwhelms the fluctuating energy transfer rate at wavenumbers above the inertial range.

Figure 5.5: Computational power spectrum results illustrating the temporal evolution of the spectral energy density $W(k)$ as a function of wavenumber for $\beta_p = 0.5$. To solve equation (5.10) numerically, the Kolmogorov diffusion coefficient (5.18) and the Alfvén wave damping rate (5.20) are used.

Figure 5.6 shows late-time solutions of equation (5.10) for $\beta_p = 0.1$ and 1.0, respectively. Furthermore, it illustrates the evolution of $W(k)$ in wavenumber space for undamped waves, i.e. $\Gamma(k) = 0$. For this case, no break occurs and the fluctuation energy cascades successively to higher $k$. As expected, the slope of the late-time fluctuation power spectrum evolves in $k$ space according to $k^{-5/3}$. This is the Kolmogorov spectrum above the injection wavenumber $k c / \omega_p = 2 \cdot 10^{-3}$. By comparing the curves for $\beta_p = 0.1$ and 1.0, it can be seen that the onset of the cut-off is shifted to smaller wavenumbers for an increased $\beta_p$. The shifts of the cut-off can be explained on the basis of figure 5.2. There, Alfvén damping becomes stronger with increasing plasma $\beta_p$ for a certain wavenumber $k c / \omega_p$. This means that the onset of significant damping is shifted to smaller wavenumbers. As a consequence of this, the cut-off of $W(k)$ is shifted to longer wavelengths.
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Figure 5.6: Power spectrum code results: late-time values of the magnetic fluctuation spectra at two given values of $\beta_p$ as functions of wavenumber. Furthermore, a power spectrum for undamped wave modes, i.e. $\Gamma(k) = 0$, is presented. For the numerical solution of equation (5.10), the Kolmogorov diffusion coefficient (5.18) and the Alfvén wave damping rate (5.20) are used.

Apparently, the nonlinear diffusion transport equation (5.10), combined with the Alfvén wave damping rate (5.20) and the Kolmogorov diffusion coefficient (5.18), cannot explain the gradual evolution of the Kolmogorov spectrum $k^{-5/3}$ to steeper power laws often observed in the dissipation range. The numerical calculations show steep cut-offs of $W(k)$. In other words, collisionless damping of Alfvén waves seems to be unsuitable to explain the occurrence of the dissipation range. The key physical reason for this discrepancy is the mismatch between the turbulent energy cascade that is governed by the Kolmogorov diffusion coefficient and the plasma wave damping function.

When damping of Alfvén waves becomes important in the dissipation regime, it is instructive to look for power law like solutions to equation (5.10). Substituting the Kolmogorov coefficient (5.18) into (5.10) and assuming that the spectral energy density $W(k)$ obeys $W(k) = W_0 k^{-q}$, one readily obtains for the damping rate $\Gamma(k)$ under steady state conditions the relation

$$\Gamma(k) \propto (2 + q)(5 - 3q) \left( \frac{W_0}{2U_B} \right)^{1/2} k^{(3q - 5)/2}$$

(5.27)

Considering a damping rate of the form $\Gamma(k) = -\Gamma_0 k^\alpha$, one obtains for the amplitude $\Gamma_0$ and the spectral index $q$ the following relations:

$$\Gamma_0 \propto (2 + q)(3q - 5) \left( \frac{W_0}{2U_B} \right)^{1/2} \text{ and } q = 3 - 2\alpha$$

(5.28)

The first relation shows that $\Gamma_0 = 0$ for the Kolmogorov inertial range, i.e. $q = 5/3$. However, for larger $q$ values being typical for the dissipation range, the damping rate has to have the right amplitude and wavenumber dependence. For instance, $q = 3$ requires $\alpha = 0$, i.e. a constant damping rate. For $q > 3$, the exponent $\alpha$ has to be negative. Neither for a constant nor a decreasing damping rate with increasing wavenumber is this consistent with the Alfvén wave damping rate (5.20) for which $m_\beta > 0$. In addition to that, (5.20) reveals an exponential expression which effects the evolution of $W(k)$. This means: Alfvénic damping increases too strongly with $k$ so that it completely damps out the spectral energy input from the cascade. As a consequence of this, no power laws can occur for Alfvénic damping in the dissipation range. This supports the numerical calculations presented above.
5.4 Numerical Calculations for Kolmogorov Diffusion

5.4.2 Magnetic Power Spectra for the Magnetosonic/Whistler Mode

In the previous section, the Alfvén wave damping rate (5.20) is used and its influence on the temporal and wavenumber evolution of the spectral energy density $W(k)$ was considered. Here, the magnetosonic/whistler damping rates for the proton- and electron-cyclotron regime, i.e. equations (5.22) and (5.25), respectively, are used for the numerical solution of equation (5.10). The turbulent cascade process is given by the Kolmogorov diffusion coefficient (5.18).

Figure 5.7 illustrates three numerical late-time solutions of equation (5.10) for the proton cyclotron damping rate (5.22) of the magnetosonic/whistler mode. The values of $\beta_p$ are given by 0.5, 2.5 and 4.5, respectively. For $\beta_p = 0.5$, no break in the late-time power spectrum occurs. The damping of the magnetosonic/whistler mode in the proton-cyclotron range, i.e. $kc/\omega_p \in [0, 5]$, is too weak (see panel (a) of figure 5.3). The fluctuation energy cascades successively to higher $k$ and the spectrum is determined by Kolmogorov diffusion throughout the wavenumber space above $kc/\omega_p = 2 \cdot 10^{-3}$. For the case $\beta_p = 2.5$, only a low level of damping is present within a small range of $k$. The spectrum reveals a drop but quickly recovers and continues on with a Kolmogorov power law. In this wavenumber regime, damping becomes negligible again (see panel (a) of figure 5.3). This was not the case for the Alfvén wave damping rate, the spectra have shown steep cut-offs, even for lower values of $\beta_p$ (cf. figure 5.6). This is a consequence of the fact that damping of the magnetosonic/whistler mode is less efficient than Alfvén damping in the proton-cyclotron regime. The steady-state spectrum for $\beta_p = 4.5$ cascades Kolmogorov-like to higher $k$ until the damping cuts it off. This is very similar to the Alfvén case. For $\beta_p = 4.5$, damping of the magnetosonic/whistler mode is too strong so that it overwhelms the cascade rate.

Figure 5.7: Power spectrum code results: late-time values of the magnetic fluctuation spectra for the magnetosonic/whistler mode for three different values of $\beta_p$. The diffusion coefficient is given by (5.18), the proton cyclotron damping is given by equation (5.22) with its corresponding fitting parameters (5.23).

Figure 5.8 shows steady state results from solving equation (5.10) numerically for three different values of $\beta_p$. Going from the right to the left, the values are given by 0.5, 2.5 and 4.5, respectively. In addition to the proton-cyclotron damping rate (5.22), the electron-cyclotron damping rate (5.25) of the magnetosonic/whistler mode is included. For $\beta_p = 0.5$, the late-time power spectrum has no break since damping in the proton-cyclotron range is too weak. The fluctuation energy cascades to higher $k$ and the spectrum is determined by Kolmogorov diffusion throughout the wavenumber space until electron-cyclotron damping sets in at higher wavenumbers at about 20. Similar to the Alfvén damping in the proton-cyclotron regime for small values of $\beta_p$, the electron-cyclotron damping
overwhelms the turbulent cascade rate, leading to a strong roll-over at about $k_c/\omega_p = 2 \cdot 10^1$. An increase of $\beta_p$ to 2.5 results in the following finding: the spectrum reveals a drop but recovers and continues on with a $k^{-5/3}$ power law within a small range of $k$. Proton cyclotron damping, equation (5.22), is on a low level (see panel (a) of figure 5.3) and becomes negligible until electron-cyclotron damping sets in. The latter damping becomes important within a relatively small increment of increasing $k$. As a result of this, the spectrum cuts off due to the damping rate (5.25). This is not the case if proton-cyclotron damping is included only. For $\beta_p = 4.5$, the steady-state spectrum cascades with $k^{-5/3}$ to higher $k$ until damping, due to proton-cyclotron resonances, cuts it off. The damping in the proton-cyclotron regime is too strong. It completely damps out the fluctuation energy input from the inertial range.

Figure 5.8: Power spectrum code results: late-time values of the magnetic fluctuation spectra for the magnetosonic/whistler mode at three given values of $\beta_p$. The cascade process is described by the Kolmogorov diffusion coefficient (5.18), the proton cyclotron damping is given by equation (5.22) with fitting parameters (5.23), and the electron cyclotron damping rate is given by (5.25) with (5.26).

5.5 Intermediate Results

The numerical calculations presented above show for both the Alfvén and the magnetosonic/whistler mode always one characteristic feature in the dissipation range, namely strong cut-offs instead of steepened power laws. All calculated Alfvénic power spectra exhibit cut-offs in the dissipation range at intermediate wavenumbers, even for low values of $\beta_p$. Magnetosonic/whistler power spectra show a relatively small drop in the dissipation range, depending on $\beta_p$, and a strong roll-over at higher wavenumbers. In view of this, the following conclusion may be drawn: collisionless damping of Alfvén and magnetosonic/whistler wave modes cannot explain the steepened power laws observed in the so-called dissipation range.

5.6 A Modified Diffusion Coefficient

In view of this intermediate result, one can take the position that Kolmogorov wavenumber diffusion may not represent fluctuation energy transfer at intermediate and higher wavenumbers. This viewpoint is not quite unreasonable since the underlying cascade process may be altered by dissipation as well as other properties such as dispersion. Unfortunately, a rigorous theory describing the influence of damping and dispersion on the turbulent cascade mechanism is not developed so far.
In order to make further progress, it is assumed that the Kolmogorov diffusion coefficient (5.18) is unsuitable to describe energy transfer at intermediate and higher wavenumbers where collisionless wave damping and, furthermore, dispersion may become important. Since Kolmogorov diffusion is then representative only for wavenumbers within the inertial range, a modified representation has to be derived for shorter wavelengths. To do so, one can turn the problem around and ask the question: given a diffusion equation for the transport of turbulence energy in wavenumber and a damping rate, what are the properties of the diffusion coefficient which are necessary to yield power law spectra observed in the different ranges?

Assuming, for simplicity, steady-state conditions and neglecting the source function, one can rearrange equation (5.10) into a simplified inhomogeneous first-order differential equation for the diffusion coefficient:

\[
W'(k) - \frac{2W(k)}{k} D'_kk(k) + \left[ W''(k) - \frac{2W'(k)}{k} + \frac{2W(k)}{k^2} \right] D_{kk}(k) = -\Gamma(k)W(k) \tag{5.29}
\]

This equation can easily be manipulated to obtain the form

\[
D'_kk(k) + \left[ \frac{k^2W''(k) - 2kW'(k) + 2W(k)}{k^2W'(k) - 2kW(k)} \right] D_{kk}(k) = \frac{\Gamma(k)W(k)k}{2W(k) - kW'(k)} \tag{5.30}
\]

Introducing the quantity

\[
\rho(k) \equiv \exp \left[ \int dk k^2W''(k) - 2kW'(k) + 2W(k) \right] \frac{2W(k)}{k^2W'(k) - 2kW(k)} \tag{5.31}
\]

equation (5.30) can be cast into the form

\[
\frac{d}{dk} \rho(k)D_{kk}(k) = \frac{\Gamma(k)W(k)\rho(k)k}{2W(k) - kW'(k)} \tag{5.32}
\]

The general solution of equation (5.32) is then given by the following expression:

\[
D_{kk} = \frac{1}{\rho(k)} \int dz z\Gamma(z)\rho(z)\frac{W(z)}{2W(z) - zW'(z)} + \frac{C_I}{\rho(k)} \tag{5.33}
\]

with \(C_I\) being an integration constant. Assuming a power law behavior of the spectral energy density, i.e. \(W(k) = W_0k^{-q}\), the general solution yields, in combination with \(\rho(k) = k^{-(q+1)}\), the following relation:

\[
D_{kk} = \left[ \frac{1}{2 + q} \int dz z^{-q}\Gamma(z) + C_I \right] k^{1+q} \tag{5.34}
\]

In order to determine the unknown integration constant \(C_I\), the inertial range is first considered. Neglecting damping, equation (5.34) yields with \(q = 5/3\) the expression \(D_{kk} = C_Ik^{8/3}\). Comparing this with the Kolmogorov diffusion coefficient (5.18), one readily obtains the relation

\[
C_I = A^2v_A \sqrt{\frac{W_0}{2U_B}} \tag{5.35}
\]

At intermediate wavenumbers the spectral index \(q\) is assumed to be 3, as suggested by solar wind observations. For simplicity, an influence of the damping rate on the cascade mechanism is neglected, so that equation (5.34) yields \(D_{kk,\omega} = C_Ik^4\). Here the subscript \(\omega\) points out that the diffusion coefficient is valid only for the intermediate wavenumber range where dispersion of the
magnetosonic/whistler mode becomes important. In order to determine $C_I$ for this wavenumber regime, it is assumed that the Kolmogorov diffusion coefficient (5.18) matches $D_{kk,\omega}$ at a certain wavenumber $k_\omega$, i.e. $D_{kk}(k_\omega) = D_{kk,\omega}(k_\omega)$. Using this condition, one obtains the following modified diffusion coefficient:

$$
D_{kk} = \begin{cases} 
A^2v_Ak^{7/2} \sqrt{\frac{W(k)}{2U_B}} & \text{for } k \leq k_\omega \\
A^2v_Ak_\omega^{7/2} \left(\frac{k}{k_\omega}\right)^{4+q/2} \sqrt{\frac{W(k)}{2U_B}} & \text{for } k > k_\omega
\end{cases}
$$

(5.36)

For the range $k \leq k_\omega$, the coefficient (5.36) is given by the well-known Kolmogorov coefficient (5.18), whereas a modified representation is required to yield power law solutions for $k > k_\omega$. The index $q$ appearing in the lower relation denotes the spectral index valid for the inertial range spectrum given by the upper relation. In order to determine the wavenumber $k_\omega$, the following function is defined:

$$
G(k) = \frac{\left(\partial \omega_r / \partial k\right) - \omega_r / k}{\omega_r / k}
$$

(5.37)

It is representative for a significant departure of $\omega_r$ from the dispersionless relation $\omega_r / k = \text{constant}$. Upon substituting the dispersion relation (5.24) into the function $G(k)$, one readily arrives at

$$
G(k) = \frac{3}{4} \left(\frac{kc}{\omega_p}\right) \sqrt{1 + \frac{3}{4} \left(\frac{kc}{\omega_p}\right)}
$$

(5.38)

For the further treatment, a normalized wavenumber $kc/\omega_p$ has to be determined for which the dispersion relation (5.24) exhibits a clear departure near $kc/\omega_p \sim 1$. This corresponds to $G(k) \approx 0.43$. Defining the dispersion wavenumber as corresponding to $G(k_\omega) = 0.4$ and, furthermore, using the linear Vlasov code by S.P. Gary, the dispersion wavenumber for the magnetosonic/whistler mode may found to be

$$
\frac{k_c}{\omega_p} \approx 0.9
$$

(5.39)

A second criterion is the dissipation wavenumber $k_d$. It represents the wavenumber at which the onset of significant dissipation takes place. Gary [1999] defined $k_d$ as the smallest wavenumber corresponding to $\Gamma / \Omega_p = -0.10$. In order to generalize this definition, the dissipation wavenumber is here given by $(k_{dc}/\omega_p)^2 = \mu_3$ or $m_3$, corresponding to the onset of proton- or electron-cyclotron damping in the rates (5.22) or (5.25), respectively. Then, the dissipation wavenumber for the magnetosonic/whistler mode at $k \times B_0 = 0$ is given for the proton-cyclotron regime by

$$
\frac{k_{dc}}{\omega_p} = \frac{1.32}{\beta_p^{0.46}}
$$

(5.40)

over $2.5 \leq \beta_p \leq 10$. At $\beta_p < 2.5$, there is no dissipation wavenumber for the magnetosonic mode because damping is weak so that $\Gamma / \Omega_p < -0.10$. Using the third relation of equation (5.26), the electron-cyclotron dissipation wavenumber behaves over the range $0.1 \leq \beta_p \leq 10.0$ as follows:

$$
\frac{k_{dc}}{\omega_p} = \frac{29.9}{\beta_p^{0.29}}
$$

(5.41)

Figure 5.9 shows the dispersion wavenumber $k_\omega$ (solid line), the dissipation wavenumbers for the proton-cyclotron regime (dots) and the electron-cyclotron dissipation wavenumber (triangles) for
the magnetosonic/whistler mode as functions of the plasma $\beta_p$. At $\beta_p > 2.5$ the dissipation wavenumber for the proton-cyclotron regime is lower than the dispersion wavenumber, indicating that this damping process becomes more important than dispersion at such high $\beta_p$. In contrast to this, there is more than one order of magnitude difference between the dispersion and the electron-cyclotron dissipation wavenumber. Since damping due to proton-cyclotron resonances is negligible for low values of $\beta_p$, the dispersion of the magnetosonic/whistler mode should significantly effect the fluctuation energy transfer rate at intermediate wavenumbers.

![Figure 5.9](image_url)

**Figure 5.9:** Dispersion and dissipation wavenumbers for the magnetosonic/whistler mode. The solid line represents the dispersion wavenumber $k_\omega$, the dots and triangles indicate the dissipation wavenumbers $k_d$ for the proton and electron cyclotron regime, respectively.

### 5.7 Numerical Calculations for Modified Wavenumber Diffusion

In analogy to the numerical calculations presented in section 5.4, the modified diffusion coefficient (5.36) is here used instead of the Kolmogorov diffusion coefficient (5.18). To solve the transport equation (5.10) numerically for the magnetosonic/whistler mode, the proton and electron cyclotron damping rates (5.22) and (5.25) are included into the code. As in section 5.4, the turbulence energy is injected through the source function $S(k)$ at $kc/\omega_p = 2 \cdot 10^{-3}$ at a rate of $10^{-15}$ erg cm$^{-3}$ s$^{-1}$. The background magnetic field $B_0$ and the Alfvén speed $v_A$ are assumed to be $10^{-4}$ Gauss and 30 km s$^{-1}$, respectively.

Figure 5.10 illustrates numerical steady-state solutions of equation (5.10) for three different values of $\beta_p$. As indicated in the figure, the values are 0.5, 2.5 and 4.0. The usual $k^{-5/3}$ steady state power spectrum in the inertial range is obtained in each case. The turbulent energy transfer, due to cascades, is described by Kolmogorov diffusion until a breakpoint is reached where dispersion commences. Near the dispersion wavenumber, $k_\omega c/\omega_p \simeq 1$, the character of the power spectra changes, but the properties of the resulting spectra depend sensitively on $\beta_p$. At $\beta_p = 0.5$, $W(k)$ breaks at $k_\omega$ to a steeper power law with $q = 3$. The formation of the $k^{-3}$ spectrum at intermediate wavenumbers results from a faster energy transfer rate due to dispersion. In this regime, the damping of the magnetosonic/whistler mode is very weak for low values of $\beta_p$ and is, therefore, negligible. At higher wavenumbers, where collisionless electron-cyclotron damping dominates, the $k^{-3}$ power law spectrum is followed by an even more rapidly diminishing cut-off spectrum beyond the electron cyclotron dissipation wavenumber $k_d c/\omega_p \simeq 30$. The cascade mechanism is too weak to sustain power law spectra. Because damping is negligible at intermediate wavenumbers for a low $\beta_p$ but dispersion becomes important at wavenumbers beyond $k_\omega$, it is suggested to drop the often used label “dissipation range” and to replace it by the “dispersion range”. The true dissipation range is at
still higher wavenumbers above \( k c/\omega_p \simeq 30 \) where electron-cyclotron damping becomes important. With increasing \( \beta_p \), the damping in the proton-cyclotron regime becomes more appreciable over a limited range of \( k \) in the dispersion regime. This can be seen by inspecting the curve for the case \( \beta_p = 2.5 \). Due to the increased damping rate, the spectrum reveals a drop but quickly recovers and continues on with the dispersion spectrum \( k^{-3} \) where damping becomes negligible again. At still higher wavenumbers, where electron-cyclotron damping dominates, the spectrum cuts off. Finally, at \( \beta_p = 4.5 \), proton cyclotron damping becomes strong enough to completely absorb the fluctuation energy cascading down from longer wavelengths, no dispersion range is evident. At \( k_d c/\omega_p \simeq 0.4 \) the dissipation range with its sharply cut-off power spectrum begins. In summary, increasing \( \beta_p \) leads to a shrinking of the dispersion range, resulting from the shifted onset of the electron cyclotron dissipation range to smaller wavenumbers. Finally, at higher \( \beta_p \) values the dispersion range disappears and is replaced by the proton-cyclotron dissipation range with its strong cut-offs.

\[ Figure \ 5.10: \] Power spectrum code results: steady-state power spectra from solving equation (5.10) numerically by using the modified diffusion coefficient (5.36) and the magnetosonic/whistler mode damping rates (5.22) and (5.25) for three different values of \( \beta_p \).

5.8 Summary and Conclusions

A model is constructed which may represent the contributions of magnetosonic/whistler mode dispersion to solar wind magnetic power spectra. For this, weakly turbulent electromagnetic fluctuations in a collisionless, homogeneous, magnetized plasma are assumed. The plasma consists of protons and electrons having Maxwellian velocity distributions with the same temperature. By using a linear Vlasov theory code, the damping rates of both the left-hand circular polarized Alfvén wave and the right-handed magnetosonic/whistler mode at \( k \times B_0 = 0 \) are deduced and represented by fitting functions. The governing wave transport equation describing the transfer of fluctuation energy in wavenumber space is solved numerically for two different approaches:

- First, a Kolmogorov energy transfer is assumed which describes the wave cascading not only in the inertial range but also in the dissipation regime of power spectra. Using damping rates of both the Alfvén and the magnetosonic/whistler mode, it is shown that damping of both wave modes cannot explain the steepening at higher wavenumbers. In contrast to observed steepened power laws, the calculations always show exponential cut-offs (or a steep roll-over). For the Alfvén mode, this is explained by the strongly increasing damping rate.
The fluctuation energy transfer rate in wavenumber space via diffusion becomes slower than Alfvénic damping at intermediate wavelengths. For the magnetosonic/whistler mode, proton cyclotron damping is weak for low values of $\beta_p$ and the fluctuation energy cascades Kolmogorov-like to shorter wavelengths where electron cyclotron damping dominates. With increasing $\beta_p$, proton cyclotron damping of the magnetosonic/whistler mode becomes more appreciable. The resulting power spectrum is then very similar to the Alfvén case. Since the damping rate is smaller than that of the Alfvén wave for a certain value of $\beta_p$, the cut-offs are larger, but the behavior of strong roll-over in power spectra remains. In view of these numerical results, the conclusion is drawn that collisionless damping of Alfvén and magnetosonic/whistler wave modes is not the only physical key for explaining the steepening in the power spectrum dissipation range.

- The second approach is based on the assumption that Kolmogorov wavenumber diffusion may not represent the energy transfer of the magnetosonic/whistler mode at intermediate and higher wavenumbers. Assuming that wave dispersion affects the underlying cascade process, a modified wavenumber diffusion coefficient is derived. For this, dispersion properties of the magnetosonic/whistler mode are taken into account by using the linear Vlasov code. Furthermore, appropriate proton and electron cyclotron damping rates are included. Using this scenario in the numerical code, computed power spectra lead to the following findings: at long wavelengths, with no dispersion and dissipation, the inertial range appears. At intermediate wavelengths, the character of the power spectra changes, but the properties of the resulting spectra are sensitive functions of $\beta_p$. For negligible proton cyclotron damping, dispersion becomes important and results in steeper power laws. It is therefore suggested to call this regime the dispersion range of solar wind magnetic fluctuation spectra. At shorter wavelengths, collisionless electron cyclotron damping arises, leading to strong cut-offs at higher wavenumbers. With increasing $\beta_p$, proton cyclotron damping becomes more important at intermediate wavenumbers. Then, the dispersion range disappears for relatively large values of $\beta_p$ and is replaced by the proton cyclotron dissipation range with its strongly cut-off spectra.
Chapter 6

On Transport Parameters

After having provided in chapters 4 and 5 an insight into the FPCs and the wavenumber evolution of the turbulence power spectrum, respectively, calculations of several transport parameters are presented in this chapter. The focus of the first section is the calculation of the perpendicular particle diffusion coefficient for an isotropic turbulence by using the plasma wave approach. This has never been done before. As a first important result, the new calculations indicate that \( \kappa_\perp \) does not vanish for an isotropic turbulence geometry. This supports the considerations performed in section 4.5.2. There, it is shown that the transport parameter for perpendicular particle diffusion vanishes for slab geometry, due to the reduced dimensionality in both wave vector and spatial configuration space. For the isotropic turbulence, it is assumed that it consists of undamped fast magnetosonic waves contributing, for simplicity, only via the \( n = 0 \) resonance. The power spectrum is assumed to be Kolmogorov-like above a minimum wavenumber \( k_{\text{min}} \) or, alternatively, below a maximum turbulence correlation length. Although it is required that this correlation length is larger than the particle’s Larmor radius, the new calculations allow to consider approximately, via a correction term, the influence of the turbulence correlation length on \( \kappa_\perp \) for different particle energies. For very large values of this correlation length, it is shown that the new perpendicular diffusion coefficient reduces to a result derived earlier by using field line random walk. The latter process is governed by the turbulence power at zero wavenumber \( \omega \), or, equivalently, an infinite maximum correlation length. However, in particular for higher particle energies, it is argued that the influence of a finite correlation length and, furthermore, of a power spectrum with broken power laws may be important for the understanding of particle motion normal to the background magnetic field.

The second section of this chapter is about the drift parameter \( \kappa_T \). The mathematical basis for its treatment is already presented in section 4.6. The calculations are performed in two ways: first, an isotropic turbulence is considered. For this, it is assumed that the turbulence consists of undamped fast magnetosonic waves. It is shown that \( \kappa_T \) vanishes for this case, due to the dimensionality of the turbulence. The second approach is based on a slab turbulence produced by undamped Alfvén waves. The drift parameter is then solely determined by the normalized cross and, in particular, magnetic helicity of the underlying plasma wave turbulence. As it is expected for a drift parameter, the sign of \( \kappa_T \) depends characteristically on the polarization states and intensities of forward and backward propagating Alfvén waves. For a vanishing magnetic helicity, i.e. no net polarization of the waves, no drift effects occur. The occurrence of the new drift term is explained by the \( \mathbf{E} \times \mathbf{B} \) drift.

Finally, in the last section of this chapter, the transport parameters for parallel diffusion, adiabatic cooling and momentum diffusion are derived and considered in detail for a “classical” slab Alfvénic turbulence. For different intensity ratios of parallel to antiparallel propagating waves, each transport parameter is solved numerically and then compared with analytical expressions. The results generalize and confirm earlier calculations available so far.
6.1 The Coefficient for Perpendicular Spatial Diffusion

In this section the coefficient for perpendicular particle diffusion is considered. For simplification, it is assumed that the turbulence consists of undamped linearly polarized plasma waves. The appropriate FPC $D_\perp$ is then given by the representation (4.79) together with the magnetic helicity $\sigma^2 = 0$. Upon using the wavenumber representation $k_\parallel = k \eta$ and $k_\perp = k \sqrt{1 - \eta^2}$, where $\eta$ is the cosine of the wave propagation angle with respect to the background magnetic field, equation (4.79) can be written as

$$D_\perp = 2\pi v^2 B_0^2 \sum_{j=\pm 1} \int_{k_{min}}^{+\infty} d\eta h(\eta) \int dkk^{-q} R(k, \omega_j, n) \left[ J_n^2(W) \delta^2 + [J_n'(W)]^2(1 - \mu^2)(1 - \eta^2) \right]$$

Here, the wave power spectrum (4.53) is already used. Furthermore, it is assumed that $k_{max} \to \infty$. The argument of the Bessel function is now given by $W = k R_L \sqrt{(1 - \mu^2)(1 - \eta^2)}$ with $R_L = v/\Omega_\alpha$ being the Larmor radius of the particle. The functions $b$ and $h(\eta)$ were already introduced with equations (4.39) and (4.53), respectively. Because undamped waves are assumed, the real part of the resonance function $R(k, \omega_j, n)$ is given by equation (4.9). In order to enable a simple treatment of equation (6.1), the considerations presented in this section are restricted to the transit-time damping contribution of fast magnetosonic waves, i.e. $n = 0$. Making use of (4.39) and substituting the dispersion relation $\omega_j = jv_A k$ (see also equation (C.37) in appendix C) and (4.9) into equation (6.1), the integration with respect to $\eta$ can be performed. Equation (6.1) is then expressible as $D_\perp = D_{\perp,1} + D_{\perp,2}$ with

$$D_{\perp,1} = \frac{2\pi^2 v}{B_0^2} \sum_{j=\pm 1} g_0^j |\mu| H[|\mu| - \epsilon] \left( 1 - \frac{\epsilon^2}{\mu^2} \right)^2 h(\eta = \epsilon/\mu) \int_{k_{min}}^{\infty} dkk^{-q} J_0^2(W)$$

$$D_{\perp,2} = \frac{2\pi^2 v}{B_0^2} \sum_{j=\pm 1} g_0^j H[|\mu| - \epsilon] \left( 1 - \frac{\epsilon^2}{\mu^2} \right) \int_{k_{min}}^{\infty} dkk^{-q} J_1^2(W)$$

Here, the function $H[x] = 1(= 0)$ for $x \geq 0(< 0)$ denotes Heaviside’s step function. The ratio of the Alfvén to the particle speed is given by the abbreviation $\epsilon = v_A/v$. The two integrations in equations (6.2) and (6.3) can be solved by using formula (6.574.2) of Gradshteyn and Ryzhik [1965]. Assuming that $R_L k_{min} \ll 1$ and integrating by parts, the integral in (6.2) is then approximately given by

$$\int_{k_{min}}^{\infty} dkk^{-q} J_0^2(W) \simeq \left( \frac{k_{min}^{-q}}{q} \right)$$

for $1 < q < 2$. The integral in equation (6.3) can be solved analogously. One arrives at

$$\int_{k_{min}}^{\infty} dkk^{-q} J_1^2(W) \simeq K(q) R_L^q (1 - \mu^2)(1 - \epsilon^2/\mu^2)^{q/2}$$

where the dimensionless quantity $K(q)$ depends only on the spectral index $q$. In terms of the Gamma function $\Gamma(x)$ it can be written as

$$K(q) = \frac{\Gamma(q) \Gamma(1 - q/2)}{2^{q-3} q^2 (2 + q) \Gamma^3(q/2)}$$
By using the definition of the function $h(\eta)$ introduced by equation (4.53), the terms (6.2) and (6.3) can be cast into the following forms:

$$D_{\perp, 1} = \frac{2\pi^2 v}{qB_0^2} k_{\text{min}}^{-q} \sum_{j=\pm 1} g_j^0 |\mu| H(|\mu| - \epsilon) \left(1 - \frac{\epsilon^2}{\mu^2}\right)^2 \left[\frac{\epsilon^2}{\mu^2} + \Lambda \left(1 - \frac{\epsilon^2}{\mu^2}\right)\right]^{-(2+q)/2}$$  \hspace{1cm} (6.7)

$$D_{\perp, 2} = \frac{2\pi^2 v}{B_0^2} R_L^q K(q) \sum_{j=\pm 1} g_j^0 H(|\mu| - \epsilon) \left(\frac{(1 - \mu^2)(1 - \epsilon^2/\mu^2)}{\epsilon^2/\mu^2 + \Lambda (1 - \epsilon^2/\mu^2)}\right)^{(2+q)/2}$$  \hspace{1cm} (6.8)

Both equations (6.7) and (6.8) represent symmetric functions in $\mu$. The representation (3.23b) of the perpendicular diffusion coefficient is then expressible as

$$\kappa_{\perp} = \frac{1}{2} \int_{-1}^{1} d\mu D_{\perp} = \frac{1}{2} \int_{-1}^{1} d\mu \left(D_{\perp, 1} + D_{\perp, 2}\right) = \int_{0}^{1} d\mu \left(D_{\perp, 1} + D_{\perp, 2}\right) = \kappa_{\perp, 1} + \kappa_{\perp, 2}$$ \hspace{1cm} (6.9)

For further simplifications it is assumed that $\Lambda = 1$, i.e. the turbulence is isotropic. Substituting expression (6.8) into equation (6.9) results, for the first contribution $\kappa_{\perp, 1}$, in

$$\kappa_{\perp, 1} = \frac{2\pi^2 v}{qB_0^2} k_{\text{min}}^{-q} \sum_{j=\pm 1} g_j^0 \int_{\epsilon}^{1} d\mu \mu \left(1 - \frac{\epsilon^2}{\mu^2}\right)^2$$ \hspace{1cm} (6.10)

The integral with respect to $\mu$ is readily solvable. Making use of the normalization (4.58) and taking into account that $\epsilon \ll 1$, the latter equation leads to

$$\kappa_{\perp, 1} = \frac{\pi(q - 1)v}{4q k_{\text{min}}} \sum_{j=\pm 1} \left(\frac{\delta B_j}{B_0}\right)^2$$ \hspace{1cm} (6.11)

The calculation of the second contribution, $\kappa_{\perp, 2}$, is more complicated. Substitution of equation (6.8) into (6.9) yields

$$\kappa_{\perp, 2} = \frac{2\pi^2 v}{B_0^2} R_L^q K(q) \sum_{j=\pm 1} g_j^0 I(\alpha, \epsilon)$$ \hspace{1cm} (6.12)

Here, the abbreviation $\alpha = (2 + q)/2$ is introduced. The quantity $I(\alpha, \epsilon)$ defines the integral

$$I(\alpha, \epsilon) = \int_{\epsilon}^{1} d\mu \mu^{-1} \left[(1 - \mu^2)(1 - \epsilon^2/\mu^2)\right]^\alpha$$ \hspace{1cm} (6.13)

Making use of $y = \sqrt{(1 - \mu^2)(1 - \epsilon^2/\mu^2)}$, the integral can be manipulated to obtain

$$I(\alpha, \epsilon) = \frac{1}{2} \int_{0}^{1} dy y^{2\alpha} \left[\frac{1}{\varphi_2} \frac{d\varphi_2}{dy} - \frac{1}{\varphi_1} \frac{d\varphi_1}{dy}\right]$$ \hspace{1cm} (6.14)

where the auxiliary quantities $\varphi_1$ and $\varphi_2$ are given by $\varphi_{1, 2} = [1 + \epsilon^2 - y^2 \pm \sqrt{(1 + \epsilon^2 - y^2) - 4\epsilon^2}]/2$.

The derivations of $\varphi_1$ and $\varphi_2$ can readily be calculated, so that

$$I(\alpha, \epsilon) = 2 \int_{0}^{1} dy \frac{y^{2\alpha+1}}{\sqrt{(1 + \epsilon^2 - y^2)(1 - \epsilon^2 - y^2)}} = \frac{(1 - \epsilon)^{2\alpha+1}}{(1 + \epsilon)} \int_{0}^{\frac{1}{2}} \frac{t^\alpha}{\sqrt{(1 - t)(1 - at)}}$$ \hspace{1cm} (6.15)
with $a = (1 - \epsilon)^2/(1 + \epsilon)^2$. According to formula (15.3.1) of Abramowitz and Stegun [1972], the latter relation can be written in terms of the hypergeometric function $2F_1$. One arrives at

$$I(\alpha, \epsilon) = \frac{2\sqrt{\pi}a}{1 + 2\alpha} \frac{\Gamma(\alpha)}{\Gamma(\alpha + 1/2)} \frac{(1 - \epsilon)^{2\alpha + 1}}{(1 + \epsilon)} 2F_1[1/2, 1 + \alpha; 3/2 + \alpha; a] \tag{6.16}$$

On the basis of the quadratic transformation formula (15.4.14) and the representation (8.2.7) of Abramowitz and Stegun [1972], respectively, the hypergeometric function is expressible by the associated Legendre function of the second kind of zeroth-order and degree $\alpha$, i.e. $Q_\alpha$. Equation (6.13) then reads

$$I(\alpha, \epsilon) = (1 - \epsilon)^{\alpha} Q_\alpha \left(\frac{1 + \epsilon^2}{1 - \epsilon^2}\right) \tag{6.17}$$

Inserting the latter relation into (6.12) and, furthermore, using the normalization (4.58), equation (6.12) may be cast into the form

$$\kappa_{\perp,2} = \frac{\pi(q - 1)v}{4k_{\min}} \sum_{j=\pm 1} \left(\frac{\delta B_j}{R_0}\right)^2 \left(R_L/k_{\min}\right)^q K(q)(1 - \epsilon^2)^{(2+q)/2} Q_\alpha \left(\frac{1 + \epsilon^2}{1 - \epsilon^2}\right) \tag{6.18}$$

The argument of the Legendre function is near unity for $\epsilon \ll 1$ so that it reaches a singular point. Then, $Q_\alpha$ diverges logarithmically. The asymptotic behavior of $Q_\alpha$ can be represented, according to table (4.8.2) of Magnus et al. [1966], by the expression

$$\lim_{\epsilon \to 0^+} Q_\alpha \left(\frac{1 + \epsilon^2}{1 - \epsilon^2}\right) \simeq -\gamma_E - \psi(\alpha + 1) + \ln \epsilon^{-1} \approx \ln \epsilon^{-1} \tag{6.19}$$

where $\gamma_E = 0.5772$ and $\psi$ are Euler’s constant and the Digamma function, respectively. Inserting (6.19) into (6.18) and making use of (6.11), the coefficient for perpendicular diffusion finally reads

$$\kappa_{\perp} = \kappa_{\perp,1} + \kappa_{\perp,2} = \frac{(q - 1)v}{4q} l_v \left(\frac{\delta B}{B_0}\right)^2 + \frac{(q - 1)K(q)}{4} \left(\frac{\delta B}{B_0}\right)^2 \frac{l_v}{c} \left(\frac{2\pi R_L}{l_c}\right)^q \ln \left(\frac{v}{v_A}\right) \tag{6.20}$$

with $(\delta B)^2 = (\delta B^+)^2 + (\delta B^-)^2$ being the total intensity of the turbulence. The scale length $l_c = 2\pi/k_{\min}$ denotes the maximum correlation length of the turbulence. At a first glance, it can be seen that $\kappa_{\perp,1}$ is directly proportional to the particle’s speed $v$, the scale length $l_c$ and $(\delta B)^2$. It therefore reveals the same dependences on $v, l_c$ and $(\delta B)^2$ as equation (3.30). As it has been shown in section 3.3.2, the latter equation is the weak scattering limit result of the Forman et al. [1974] model. Furthermore, it is remarkable that $\kappa_{\perp,1}$ agrees, apart from the additional dependence on $q$, with formula (B5) of Giacalone and Jokipii [1999] published more recently. The diffusion coefficient (6.20) is, however, not only determined by $\kappa_{\perp,1}$ but also by the second contribution, $\kappa_{\perp,2}$. In order to understand the properties of $\kappa_{\perp,2}$ and its importance for $\kappa_{\perp}$, figure 6.1 presents two computations of equation (6.20). Both panels show the perpendicular coefficient $\kappa_{\perp}$ in $10^{18}$ cm$^2$ s$^{-1}$ as a function

\[\text{Note that they have calculated $\kappa_{\perp}$ for a purely magnetic slab turbulence. Furthermore, they have not included any plasma wave dispersion relations in their calculations, nor Jokipii [1966] and Giacalone and Jokipii [1999]. Furthermore, it has been shown by equation (4.91), section 4.5.2, that perpendicular diffusion requires more than one dimension, implying $\kappa_{\perp} = 0$ for a slab geometry. In other words, a slab model is not suitable in the context of perpendicular particle transport, since the particles are always tied to the same magnetic field line. Note that Jokipii et al. [1993] came to the same conclusion. Here, however, $\kappa_{\perp}$ is calculated from a more general viewpoint, meaning that the turbulence consists of fast magnetosonic waves. In this sense, the dispersion relation of fast magnetosonic waves is explicitly involved into the calculations for the first time. The only assumptions made above are the isotropic and undamped state of the turbulence and the waves, respectively. In view of equation (4.91) and the contradicting geometry assumptions, the slab turbulence result of Forman et al. [1974] is, in this sense, strictly excluded.}\]
of kinetic energy in MeV nucleon\(^{-1}\). The calculations are restricted to protons only. For the background magnetic field \(B_0\) and the spectral index \(q\), typical values such as \(4 \times 10^{-5}\) Gauss and \(5/3\) are used. The Alfvén speed \(v_A\) is set to 50 km s\(^{-1}\).

Figure 6.1: Computation of equation (6.20) showing \(\kappa\) in \(10^{18}\) cm\(^2\) s\(^{-1}\) as a function of a proton’s kinetic energy in MeV nucleon\(^{-1}\). The left panel illustrates the contribution \(\kappa_{\perp,1}\) of equation (6.20) for the values \(\delta B/B_0 = 0.05\) and \(l_c = 1.5 \cdot 10^9\) km (solid line), \(\delta B/B_0 = 0.1\) and \(l_c = 1.5 \cdot 10^7\) km (dotted line) as well as \(\delta B/B_0 = 0.5\) and \(l_c = 1.5 \cdot 10^5\) km (dashed line). The spectral index \(q\) is set to \(5/3\) and the heliospheric background magnetic field is assumed to be \(4 \times 10^{-5}\) Gauss. Right panel: Same values and line styles as used for the computations shown in the left panel, but the calculations are performed for the total perpendicular diffusion coefficient \(\kappa = \kappa_{\perp,1} + \kappa_{\perp,2}\) given by (6.20).

The left panel shows \(\kappa\) for the case that \(\kappa_{\perp,2}\) is neglected in equation (6.20). \(\kappa = \kappa_{\perp,1}\) is shown for three different parameter sets of \(\delta B/B_0\) and \(l_c\). The solid line corresponds to \(\delta B/B_0 = 0.05\) and \(l_c = 1.5 \cdot 10^9\) km. The set \(\delta B/B_0 = 0.1\) and \(l_c = 1.5 \cdot 10^7\) km is represented by the dotted curve. The dashed line illustrates the calculation for \(\delta B/B_0 = 0.5\) and \(l_c = 1.5 \cdot 10^5\) km. A closer inspection of the left panel results in the following findings: \(\kappa_{\perp,1}\) increases monotonically with increasing kinetic energy and changes its slope to become a constant at energies beyond the proton’s rest energy (\(\sim 936\) MeV). The amplitude of \(\kappa_{\perp,1}\) varies in the order of two magnitudes, depending on the relative huge range of \(\delta B/B_0\) and \(l_c\) values given above. Note that both the amplitudes and shapes of \(\kappa_{\perp,1}\) are in excellent agreement with the quasi-linear result presented by Giacalone and Jokipii [1999] (cf. the solid line in their figure 7). The right panel of figure 6.1 illustrates the influence of \(\kappa_{\perp,2}\) on the total perpendicular diffusion coefficient, i.e. \(\kappa = \kappa_{\perp,1} + \kappa_{\perp,2}\) given by equation (6.20). The assumed values of \(\delta B/B_0\) and \(l_c\) as well as the line styles are the same as those used for the computations presented in the left panel. At a first glance, it seems that \(\kappa_{\perp,2}\) has a tremendous influence on the perpendicular diffusion coefficient, but depending on the chosen turbulence correlation length \(l_c\). For the largest value, i.e. \(l_c = 1.5 \cdot 10^9\) km, the amplitude and shape of the corresponding solid curve is preserved if \(\kappa_{\perp,2}\) is taken into account. This can
easily be understood by considering the term \( R_L k_{\min} = 2\pi R_L l_c^{-1} \) in \( \kappa_{\perp} \) of equation (6.20) and, furthermore, by recalling the condition \( R_L k_{\min} \ll 1 \) or, alternatively, \( R_L \ll l_c \).

Figure 6.2 illustrates the quantity \( 2\pi R_L l_c^{-1} \) as a function of kinetic energy for the correlation lengths used above. For the kinetic energy range shown in figures 6.1 and 6.2, the particle Larmor radius \( R_L \) becomes maximally around 5 \( \cdot \) 10^5 km. From figure 6.2 it becomes clear that the condition \( R_L \ll l_c \) is not fulfilled if \( l_c \) becomes too short. In other words, the resonant wavenumbers at long wavelengths with which protons of higher energies interact are not available anymore, the fluctuation power spectrum is cut-off below \( k_{\min} \). Because of this, the dashed curve in the right panel of figure 6.1 is invalid for all kinetic energies considered in this section. The calculation for the case \( l_c = 1.5 \cdot 10^7 \) km (dotted line) seems to be valid for energies below several times 10 MeV. For higher energies the approximation \( R_L k_{\min} \ll 1 \) does not hold either (cf. the dotted line in figure 6.2 and compare it with its counterpart in the right panel of figure 6.1). In contrast to this, the calculation for \( l_c = 1.5 \cdot 10^9 \) km is valid throughout the whole energy range.

The validity of all three cases in energy space is also indicated in figure 6.3. It illustrates the relative importance of \( \kappa_{\perp,2} \) with respect to the total diffusion coefficient \( \kappa_{\perp} = \kappa_{\perp,1} + \kappa_{\perp,2} \). The dashed curve \( (l_c = 1.5 \cdot 10^5 \) km) confirms that \( \kappa_{\perp,2} \) overwhelms \( \kappa_{\perp,1} \) throughout all energies if the value of \( l_c \) is too short. This implies that equation (6.20) becomes invalid for such short \( l_c \), because \( R_L \geq l_c \). The influence of \( \kappa_{\perp,2} \) on \( \kappa_{\perp,1} \) becomes less appreciable with increasing \( l_c \)-values.

The condition \( R_L k_{\min} \ll 1 \) seems to be satisfied to a sufficient approximation for \( l_c = 1.5 \cdot 10^7 \) km (dotted curve) at lower energies below several times 10 MeV. A further increase in \( l_c \) extends the validity of \( R_L k_{\min} \ll 1 \) to higher energies. This is indicated by the solid curve corresponding to \( l_c = 1.5 \cdot 10^9 \) km. In this case, \( \kappa_{\perp,1} \) overwhelms \( \kappa_{\perp,2} \) for such long scale lengths completely. At higher kinetic energies, the second term \( \kappa_{\perp,2} \) is totally suppressed only if \( l_c \) is sufficient long, so that the approximation \( R_L k_{\min} \ll 1 \) still holds. Presumed that the turbulence correlation length is long enough, the contribution \( \kappa_{\perp,2} \) may be neglected in equation (6.20). For this long correlation length limit the perpendicular diffusion coefficient is then given by \( \kappa_{\perp,1} \), so that

\[
\kappa_{\perp} = \frac{(q - 1)}{4q} l_c v \left( \frac{\delta B}{B_0} \right)^2
\] (6.21)
This expression includes the earlier results of Forman et al. [1974] and Giacalone and Jokipii [1999] and generalizes them in a twofold manner: first, equation (6.21) contains a dependence on the spectral index $q$. Second, and this is more important, it generalizes those earlier results in so far as it is obtained and, furthermore, only valid for higher energies if the correlation length is long enough.

The essential agreement of equation (6.21) with the representations derived by Forman et al. [1974] and Giacalone and Jokipii [1999] is therefore not surprising, because their theories were developed on the assumption that the perpendicular diffusion coefficient is given in terms of a magnetic power spectrum at zero wavenumber. Actually, this corresponds to an infinite correlation length $l_c$. Here, however, the perpendicular diffusion coefficient is presented for a large but still finite $l_c$. As a consequence of this the contribution $\kappa_{\perp,2}$ occurs in (6.20). This term is determined sensitively by the ratio of the particle’s Larmor radius $R_L$ to $l_c$. The explanations presented above indicate that the long correlation length limit (6.21) is valid only if $R_L$ is much less than the maximum correlation length. However, this may be not always the case. The Larmor radius $R_L$ increases with particle’s kinetic energy and, therefore, becomes more appreciable relative to $l_c$. Then, correction terms are required to take a finite correlation length into account. These terms supplement the long correlation length limit (6.21). In the case that the approximation $R_L k_{\text{min}} \ll 1$ marginally holds, the component $\kappa_{\perp,2}$ may be representative for such a correction term. If the particle kinetic energy increases further, the Larmor radius exceeds the correlation length. The $k$-integral in equation (6.1) then has to be solved in its general form. This will be the subject of forthcoming studies.

![Figure 6.3: Computations of the ratio $\kappa_{\perp,2}/(\kappa_{\perp,1} + \kappa_{\perp,2})$ indicating the relative importance of the contribution $\kappa_{\perp,2}$ in equation (6.20). The ratio is given as a function of kinetic energy in MeV nucleon$^{-1}$. The correlation lengths and line styles are the same as those used in figure 6.1.](image)

Before finishing this section it might be worthy to consider the diffusion coefficient (6.20) for specific parameters for which the approximation $R_L \ll l_c$ may be regarded as marginally valid in the energy range between 100 to 300 MeV. Therefore, the following considerations have to be considered more as an quantitative outlook than a qualitative treatment.

It was mentioned above that $R_L$ increases up to several times $10^5$ km for the energy range considered here. For typical correlation lengths around $3.2 \cdot 10^5$ km (see Chalov and Fahr [1996] and, furthermore, chapter 7 where this value is used in the context of stochastic acceleration of PUIs) it was shown that the representation (6.20) fails for energies ranging from around 100 keV to 10 GeV.
Chapter 6: On Transport Parameters

However, an increase in $l_c$ extends the validity range of $R_L k_{\text{min}} \ll 1$ and, therefore, of equation (6.20) to higher energies in energy space. The case $l_c = 1.5 \cdot 10^7$ km (dotted curves in figures 6.1, 6.2 and 6.3) pointed out that equation (6.20) may be considered as marginally valid for energies at several times 10 MeV. If one is willing to accept $R_L/l_c \approx 0.15$ as an upper limit for which the approximation is marginally valid, equation (6.20) may then be just suitable even for 80-100 MeV. It should be noted that $\kappa_{\perp,2}$ consists not only of the term $R_L k_{\text{min}}$ but also includes other dependences on $v$ or, alternatively, kinetic energy. This means $\kappa_{\perp,2}$ contributes, even if only weakly, to $\kappa_{\perp}$ at energies where the approximation is satisfied. This can nicely be verified by comparing the dotted curve in both panels of figure 6.1 at 40 MeV. A closer inspection of the corresponding $\kappa_{\perp}$ value in the right panel shows a small deviation from the dotted curve illustrated in the left panel. $\kappa_{\perp}$ is slightly increased in the right panel. By inspecting the dotted curve in figure 6.2, the corresponding value of $2\pi R_L/l_c$ can be estimated to result in $\sim 0.09$, meaning that the Larmor radius is less than $l_c$. Figure 6.3 shows that $\kappa_{\perp,2}$ contributes appreciably to $\kappa_{\perp}$. In view of these explanations concerning the approximation, one may increase $l_c$ slightly to $2.3 \cdot 10^7$ km. The approximation might then be considered as marginally valid at energies around $100-300$ MeV.

Figure 6.4 presents computations of equation (6.20) for $l_c = 2.3 \cdot 10^7$ km. For the background magnetic field $B_0$, the ratio $\delta B/B_0$ and the spectral index $q$, the values $4 \cdot 10^{-5}$ Gauss, 0.1 and $5/3$ are used, respectively. The Alfvén speed $v_A$ is assumed to be 50 km s$^{-1}$. The figure is split into two panels. The upper panel shows $\kappa_{\perp}$ as a function of kinetic energy. The lower panel (separated from the upper panel by the horizontal solid line) illustrates, similarly to figure 6.3, the relative importance of $\kappa_{\perp,2}$ by the ratio $\kappa_{\perp,2}/\kappa_{\perp}$ (solid line in the lower panel). Furthermore, it also gives the quantity $2\pi R_L/l_c$ visualized by the dashed curve.

Figure 6.4: Equation (6.20) compared with simulation results for an isotropic turbulence (solid dots) derived by Giacalone and Jokipii [1999]. The figure is split into two panels separated by the solid horizontal line. Upper panel: the solid curve illustrates (6.20) for the parameters $l_c = 2.3 \cdot 10^7$ km, $\delta B/B_0 = 0.1$, $v_A = 50$ km s$^{-1}$ and $q = 5/3$. The dashed curve represents the long correlation length limit given by equation (6.21). Lower panel: the solid and dashed lines show the corresponding ratios $\kappa_{\perp,2}/\kappa_{\perp}$ and $2\pi R_L/l_c$, respectively.

The upper panel shows a comparison of equation (6.20) with numerical results presented recently by Giacalone and Jokipii [1999]. They used a simulation that integrates the trajectories of an ensemble of test particles. By using this simulation, they deduced the parallel and perpendicular diffusion coefficient on the basis of particle’s motions for a composite, i.e. a superposition of a slab and two-dimensional component, as well as an isotropic turbulence spectrum (see their paper for further details). The solid dots shown in the upper panel correspond to their numerical results for the isotropic turbulence (cf. Giacalone and Jokipii [1999], the solid dots in their figure 7).
6.2 The Coefficient for Particle Drift

The dashed curve gives the long correlation length limit (6.21). The solid line illustrates equation (6.20). The long correlation length limit is neither at very low nor high kinetic energies consistent with the numerical results presented by Giacalone and Jokipii [1999]. In particular at higher energies, it appears that equation (6.21) fails completely, since it becomes a constant. A behavior which is quite contrary to the simulation results. The solid line, however, seems to reproduce the numerical simulation not only at relative low energies between 3 and 100 MeV, but also at energies around 100 – 300 MeV (indicated by the vertical dotted lines), where the numerical results show an increasing coefficient $\kappa_\perp$. Although the approximation $R_L k_{\text{min}} \ll 1$ might be marginally valid at such energies (because $R_L k_{\text{min}} \sim 0.09$ at 100 MeV), figure 6.4 and the explanations presented above point out that the inclusion of a finite $k_{\text{min}}$ is important for perpendicular diffusion at intermediate (energy range within the dotted curves) and higher energies. In the case that the turbulence power spectrum becomes flatter at wavenumbers below $k_{\text{min}}$, it is supposed that $\kappa_\perp$ becomes more appreciable at higher energies than expected on the basis of the long correlation length limit. Furthermore, a maximum $k_{\text{max}}$ or, alternatively, minimum scale length $l_{c,\text{min}}$ may also be important for perpendicular diffusion. For instance, considering wavenumbers larger than a $k_{\text{max}}$ and taking into account that the power spectrum becomes steeper in this short-wavelength regime, the fluctuation energy pool becomes less appreciable. Since turbulent energy at such large wavenumbers provide for the scattering of low-energetic particles, perpendicular diffusion becomes less efficient. In other words, $\kappa_\perp$ presumably decreases faster if a finite $k_{\text{max}}$ would be included into the calculations. Therefore, it is supposed that the inclusion of $k_{\text{max}}$ plays an important role for perpendicular diffusion of less energetic particles. The deviation of the numerical result of Giacalone and Jokipii [1999] at 1 MeV (first solid dot on the left-hand side) from the quasi-linear approach points out that $\kappa_\perp$ decreases faster than expected.

6.2 The Coefficient for Particle Drift

The aim of this section is to present calculations of the drift transport parameter $\kappa_T$. The mathematical basis was already developed in section 4.6 where the FPCs $D_{XY}$ and $D_{YX}$ were considered in thorough detail. Following the explanations given in section 3.3.2, the coefficient $\kappa_T$ determines the off-diagonal elements $\kappa_{XY}$ and $\kappa_{YX}$ of the drift tensor $K_T$. The latter tensor is manifested in the spatial diffusion tensor $K$, equation (3.25). Based on the Fokker-Planck approach, it was already shown in section 4.6 that the corresponding FPCs satisfy the relation $D_{XY} = -D_{YX} = -N$ (see equation (4.115) and the comments following it) with $N$ being a non-resonant term given by equation (4.114). It was argued that this condition even holds for undamped waves, but only if anisotropies in a plane perpendicular to the background magnetic field are not included. In view of this simplification, the off-diagonal elements of $K_T$ obey the condition $\kappa_{XY} = -\kappa_{YX} = \kappa_T$. Then, $K_T$ indeed reveals an antisymmetric structure. Since $N$ is not evaluated so far, the drift coefficient $\kappa_T$ is still unknown. In the following, two derivations are presented.

6.2.1 The Drift Coefficient for an Isotropic Turbulence

According to equation (4.116) of section 4.6, $\kappa_T$ is given as a pitch angle integral over the non-resonant contribution $N$, equation (4.114). The latter term can be rearranged as follows: since the wave power spectrum $A^l$, the magnetic helicity $\sigma^l$ and the function $b$ (see equation (4.39) for its definition) are independent of $n$, the sum sign can be shifted into the brackets of (4.114). Note that this step cannot be carried out for undamped waves, because the resonance function (4.110) then has an additional $n$-contribution. Furthermore, a specific wave power spectrum has to be assumed. For this, equation (4.53) is used with $\Lambda = 1$. Making use of the wavenumber representation $k_\parallel = k\eta$
and \( k_\perp = k\sqrt{1 - \eta^2} \), the non-resonant contribution (4.114) can then be cast into the form

\[
N = 2\pi \frac{v}{B_0^2} \sum_{j=\pm 1} g_0^j \int \frac{1}{d\eta} \int_{-1}^{\infty} dk \, k^{-(1+\eta)} \left[ \sigma^j b \sum_{n=-\infty}^{\infty} J_n^4(W) + \sqrt{(1 - \mu^2)(1 - \eta^2)} \sum_{n=-\infty}^{\infty} J_n(W)J_n'(W) \right]
\]

(6.22)

The sum in the first contribution can also be expressed by

\[
\sum_{n=-\infty}^{\infty} J_n^2(W) = J_0(W) + 2 \sum_{n=1}^{\infty} J_n^2(W) = 1
\]

(6.23)

Here it was made use of the relation \( J_{-n}(W) = (-1)^n J_n(W) \) and of formula (9.1.76) of Abramowitz and Stegun [1972]. The treatment of the second term is more complicated. By using the Bessel function identity on the right-hand side of equation (4.15), the sum can be manipulated as follows:

\[
\sum_{n=-\infty}^{\infty} J_n(W)J_n'(W) = \frac{1}{2} \sum_{n=-\infty}^{\infty} J_n(W) [J_{n-1}(W) - J_{n+1}(W)]
\]

\[
= -J_0(W)J_1(W) + \sum_{n=1}^{\infty} J_n(W) [J_{n-1}(W) - J_{n+1}(W)]
\]

\[
= -J_0(W)J_1(W) + J_1(W)J_0(W) - J_1(W)J_2(W) + \cdots
\]

\[
= -\lim_{n\to\infty} J_n(W)J_{n+1}(W)
\]

(6.24)

With the latter relation and the addition theorem (6.23), equation (6.22) reads

\[
N = 2\pi \frac{v}{B_0^2} \sum_{j=\pm 1} g_0^j \int \frac{1}{d\eta} \int_{-1}^{\infty} dk \, k^{-(1+\eta)} \left[ \sigma^j b - \sqrt{(1 - \mu^2)(1 - \eta^2)} \lim_{n\to\infty} J_n(W)J_{n+1}(W) \right]
\]

(6.25)

In order to evaluate the wavenumber integration of the second term, the argument \( W \) is substituted by \( z = k_\perp v_\perp / \Omega_\alpha = kR_L\sqrt{(1 - \mu^2)(1 - \eta^2)} \). This yields

\[
\int_{z_{\min}}^{\infty} dz \, z^{-(1+\eta)} J_n(z)J_{n+1}(z) = \int_{0}^{\infty} dz \, z^{-(1+\eta)} J_n(z)J_{n+1}(z) - \int_{0}^{z_{\min}} dz \, z^{-(1+\eta)} J_n(z)J_{n+1}(z)
\]

(6.26)

with \( z_{\min} = k_{\min} R_L \sqrt{(1 - \mu^2)(1 - \eta^2)} \) being the lower integration boundary. The first integral can be solved by using formula (6.574.2) of Gradshteyn and Ryzhik [1965]. By using the approximation (4.82), the second integral can be performed asymptotically for \( z_{\min} \ll 1 \). One readily obtains

\[
\int_{0}^{\infty} dz \, z^{-(1+\eta)} J_n(z)J_{n+1}(z) = \frac{2^{2\eta} \Gamma(1 + q/2)\Gamma(n + q/2)\Gamma(2n - q)}{\sqrt{\pi} \Gamma((3 + q)/2)\Gamma(n - q/2)\Gamma(2n + q)} \frac{1}{2n + q + 1}
\]

(6.27)

\[
\int_{0}^{z_{\min}} dz \, z^{-(1+\eta)} J_n(z)J_{n+1}(z) \approx \frac{2^{-(2n+1)}z^{2n-q+1}}{\Gamma(1+n)\Gamma(2+n)(2n-q+1)} \bigg|_{0}^{z_{\min}}
\]

(6.28)

with \( 2n + 1 - q > 0 \). A closer inspection of (6.27) and (6.28) and their dependences on \( n \) leads, for the limit \( n \to \infty \), to the following result:

\[
\lim_{n\to\infty} \int_{z_{\min}}^{\infty} dz \, z^{-(1+\eta)} J_n(z)J_{n+1}(z) = 0
\]

(6.29)
Therefore, the second term in equation (6.25) vanishes completely. An alternative proof of this result is more general and elegant than the first one. For this, the second contribution in the brackets of equation (6.22) is considered once more. Including now, for generality, a finite maximum wavenumber, \( k_{\text{max}} \), the corresponding integral can be written as

\[
\sum_{n=-\infty}^{\infty} \int_{z_{\text{min}}}^{z_{\text{max}}} dz \, z^{-(1+q)} J_n(z) J_n'(z) = \frac{1}{2} \sum_{n=-\infty}^{\infty} J_n^2(z) \left|_{z_{\text{min}}}^{z_{\text{max}}} \right| + \frac{1+q}{2} \sum_{n=-\infty}^{\infty} J_n^2(z)
\]

(6.30)

where a partial integration has been applied to the left-hand side. The upper integration boundary is given by \( z_{\text{max}} = k_{\text{max}} R_L \sqrt{(1-\mu^2)(1-\eta^2)} \). On the basis of the addition theorem (6.23), the sum over the Bessel functions yields unity. The integration in the second term of (6.30) can then readily be carried out. One immediately arrives at

\[
\sum_{n=-\infty}^{\infty} \int_{z_{\text{min}}}^{z_{\text{max}}} dz \, z^{-(1+q)} J_n(z) J_n'(z) = 0
\]

(6.31)

In contrast to the first proof, this approach is valid for arbitrary integration boundaries and does not require an asymptotic representation for the Bessel functions. Equation (6.22) is then expressible as

\[
N = 2\pi \frac{v}{B_0^2} \sum_{j=\pm 1} g_0^j \int d\eta \int dk k^{-(1+q)} \sigma^j b
\]

(6.32)

The non-resonant term \( N \) is solely determined by the magnetic helicity \( \sigma^j \). Furthermore, it depends on the dispersion relation \( \omega_j \) entering \( N \) via the function \( b \) (see equation (4.39) for its definition). Restricting the further treatment to fast magnetosonic waves, i.e. \( \omega_j = j v_A k \), and assuming a constant magnetic helicity \( \sigma^j \), equation (6.32) then reads, after having performed the integration with respect to \( k \), as follows:

\[
N = \frac{2\pi v}{q B_0^2 k_{\text{min}}} \left[ 1 - \left( \frac{k_{\text{min}}}{k_{\text{max}}} \right)^q \right] \sum_{j=\pm 1} g_0^j \sigma^j \int d\eta \left( j \eta v_A / v - \mu \right)
\]

(6.33)

Taking into account the corresponding normalization constant (4.58) and performing the \( \eta \) integration, one readily arrives at

\[
N = -l_{\text{max}} v \mu \frac{(q-1)}{4\pi q} \left[ 1 - (l_{\min}/l_{\max})^q \right] \sum_{j=\pm 1} \left( \frac{\delta B^j}{B_0} \right)^2 \sigma^j
\]

(6.34)

where the minimum and the maximum wavenumbers \( k_{\text{min}} \) and \( k_{\text{max}} \) were expressed by the outer and inner correlation scale lengths \( l_{\text{max}} = 2\pi / k_{\text{min}} \) and \( l_{\min} = 2\pi / k_{\text{max}} \), respectively. So far, the calculations were performed only on the Fokker-Planck level. However, the diffusion-convection equation (3.24) requires the transport parameter \( \kappa_T = \kappa_X Y = -\kappa_Y X \). According to the representation (3.23b), \( \kappa_T \) is given as a simple pitch angle average of the non-resonant term \( N \). On substituting equation (6.34) into the representation (4.116), one readily obtains

\[
\kappa_T = 0
\]

(6.35)

for an isotropic turbulence consisting of undamped fast magnetosonic waves. After all the laborious calculations, this is a rather surprising result. An interpretation for this astonishing finding is given in the next section.
6.2.2 The Drift Coefficient for a Slab Turbulence

The calculations presented above were performed for an isotropic turbulence consisting of undamped fast magnetosonic waves. However, it is worthy to consider also a simple Alfvénic slab turbulence. The non-resonant term \( N \), equation (4.114), has then to be evaluated for the power spectrum for slab geometry given by equation (4.59). As it was shown above, the second term in the brackets of equation (4.114) vanishes completely. The use of the addition theorem (6.23) and the substitution of the power spectrum (4.59) immediately yields

\[
N = 2\pi \frac{v}{B_0} \sum_{j=\pm 1} k_{\max} g_j \int_{k_{\min}}^{k_{\max}} dk \| k \|^{-1} (1+q) \sigma^j b \tag{6.36}
\]

This expression agrees, in principle, with the non-resonant term (6.32) derived for fast magnetosonic waves in the isotropic turbulence. However, equation (6.32) has, in contrast to (6.36), an additional integration with respect to \( \eta \), i.e. an averaged cosine of the wave propagation angle. This is a consequence of the isotropic geometry of the turbulence for which \( \kappa_T \) vanishes. This is not the case for slab geometry. As it is shown below, the dimensionality of the turbulence is of fundamental importance for understanding the drift parameter \( \kappa_T \). In order to make further progress, dispersionless Alfvén waves, i.e. \( \omega_j = j v_A k \| \), are assumed for the following calculations. Equation (6.36) can then be cast into the form

\[
N = l_{\max} v_A \frac{(q-1)}{4\pi q} \frac{[1 - (l_{\min}/l_{\max})^q]}{[1 - (l_{\min}/l_{\max})^{q-1}]} \sum_{j=\pm 1} \left( \frac{\delta B^j}{B_0} \right)^2 j \sigma^j 
\]

\[
- l_{\max} v_\mu \frac{(q-1)}{4\pi q} \frac{[1 - (l_{\min}/l_{\max})^q]}{[1 - (l_{\min}/l_{\max})^{q-1}]} \sum_{j=\pm 1} \left( \frac{\delta B^j}{B_0} \right)^2 \sigma^j \tag{6.37}
\]

where the normalization (4.60) for slab geometry was used and \( k_{\min} \) and \( k_{\max} \) were expressed by the maximum and minimum correlation scale lengths \( l_{\max} \) and \( l_{\min} \), respectively. The second contribution of (6.37) corresponds to equation (6.34) derived for fast magnetosonic waves. Again, this term does not contribute to \( \kappa_T \), due to the pitch angle average given in equation (4.116). However, since a slab model is considered and, therefore, no average of the wave propagation angle occurs, the first term arises in equation (6.37). Hence, equation (4.116) yields

\[
\kappa_T = - l_{\max} v_A \frac{(q-1)}{2\pi q} \frac{[1 - (l_{\min}/l_{\max})^q]}{[1 - (l_{\min}/l_{\max})^{q-1}]} \sum_{j=\pm 1} \left( \frac{\delta B^j}{B_0} \right)^2 j \sigma^j \tag{6.38}
\]

For the further treatment, it is convenient to recall the definition of the magnetic helicity for both the forward and backward propagation direction, i.e.

\[
\sigma^+ = \frac{(\delta B_L^+)^2 - (\delta B_R^-)^2}{(\delta B_L^+)^2 + (\delta B_R^-)^2} \quad \text{and} \quad \sigma^- = \frac{(\delta B_L^-)^2 - (\delta B_R^+)^2}{(\delta B_L^-)^2 + (\delta B_R^+)^2} \tag{6.39}
\]

where the superscripts (+) and (−) denote forward \( (j = +1) \) and backward \( (j = -1) \) propagating wave intensities, respectively. Furthermore, it is instructive to introduce the left- and right-handed normalized cross helicities \( h_L \) and \( h_R \) by the expressions

\[
h_L = \frac{(\delta B_L^+)^2 - (\delta B_L^-)^2}{(\delta B_L^+)^2 + (\delta B_L^-)^2} \quad \text{and} \quad h_R = \frac{(\delta B_R^+)^2 - (\delta B_R^-)^2}{(\delta B_R^+)^2 + (\delta B_R^-)^2} \tag{6.40}
\]
varying between $-1$ and $+1$. These values are attained if only left- or right-handed polarized waves are present moving backward or forward along the background magnetic field. Equation (6.38) can then be cast, after some algebra, into the following form:

$$\kappa_T = -\kappa_{T,0} \left[ \frac{(1 + h_L)(1 + \sigma^-)\sigma^+}{(1 + h_L)(1 + \sigma^-) + (1 - h_L)(1 + \sigma^+)} - \frac{(1 - h_R)(1 - \sigma^-)\sigma^-}{(1 - h_R)(1 - \sigma^-) + (1 + h_R)(1 - \sigma^+)} \right]$$

(6.41)

where the normalization is given by

$$\kappa_{T,0} = l_{\text{max}} v_A \frac{(q - 1)}{2\pi q} \frac{[1 - (l_{\text{min}}/l_{\text{max}})^q]}{[1 - (l_{\text{min}}/l_{\text{max}})^{q-1}]} \left( \frac{\delta B}{B_0} \right)^2$$

(6.42)

with $(\delta B)^2$ being the total turbulence intensity. A closer inspection of the drift coefficient (6.41) results in the following findings: first, $\kappa_T$ is entirely determined by the normalized cross helicities $h_L$ and $h_R$ and, furthermore, by the magnetic helicities $\sigma^+$ and $\sigma^-$ of forward and backward propagating wave fields, respectively. Second, $\kappa_T$ depends on neither the charge nor the mass of the particle; $\kappa_T$ is only completely independent of particle properties. On the other hand, the derivation of $\kappa_T$ is valid only for particles with nonzero charge. In view of these facts, one immediately recognizes before in the context of heliospheric physics.

In order to get more physical insight into the drift coefficient (6.41) and its dependence on the helicities and, therefore, on the states of polarization, a variety of wave fields with different propagation directions is considered. To do so, it is convenient to introduce the abbreviations LHP and RHP for the left- and right-handed polarization states, respectively:

1. **LHP wave in forward direction.** This implies $\sigma^+ = +1$ and $h_L = +1$. The helicities $\sigma^-$ and $h_R$ are undetermined. The evaluation of equation (6.41) yields $\kappa_T = -\kappa_{T,0}$.

2. **RHP wave in forward direction.** This implies $\sigma^+ = -1$ and $h_R = +1$. The helicities $\sigma^-$ and $h_L$ are not determined. The evaluation of equation (6.41) yields $\kappa_T = +\kappa_{T,0}$.

3. **LHP wave in backward direction.** This implies $\sigma^- = +1$ and $h_L = -1$. The helicities $\sigma^+$ and $h_R$ are undetermined. The evaluation of equation (6.41) yields $\kappa_T = +\kappa_{T,0}$. This corresponds to case (2). A LHP wave in backward direction can be replaced by a RHP wave propagating in forward direction.

---

2 Actually, this finding is not surprising. As it was shown in section 4.6, all resonant contributions vanish exactly. The transport parameter $\kappa_T$ for particle drifts is characterized only by $N$ and, therefore, by the (imaginary) non-resonant part of the resonance function (4.8). But, however, it should be noted that particle properties enter $\kappa_T$ for damped waves. This will be the subject of forthcoming studies.
(4) RHP wave in backward direction. This implies \( \sigma^- = -1 \) and \( h_R = -1 \). The helicities \( \sigma^+ \) and \( h_L \) are not determined. The evaluation of equation (6.41) yields \( \kappa_T = -\kappa_{T,0} \). This corresponds to case (1). A LHP wave in forward direction can be replaced by a RHP wave propagating in backward direction.

(5) LHP and RHP waves propagating forward and backward to the background magnetic field, respectively. This implies \( \sigma^+ = +1, \sigma^- = -1, h_L = 1 \) and \( h_R = -1 \). The evaluation of equation (6.41) yields \( \kappa_T = -2\kappa_{T,0} \). Note that this is the sum of case (1) and (4). The drift becomes two times stronger.

For the case of equal net polarization for forward and backward propagating wave fields, i.e. \( \sigma^+ = \sigma^- = \sigma \), implying \( h_L = h_R = h \), one finds

\[
\kappa_T = -\kappa_{T,0} h \sigma
\]  

(6.43)

It immediately becomes clear that \( \kappa_T \) becomes zero for a vanishing net polarization. Furthermore, \( \kappa_T \) changes its sign for the cases that predominantly LHP or RHP wave fields are present.

6.3 Transport Parameters in the Slab Model

In contrast to the first two sections of this chapter it is here assumed that diffusive particle transport takes place in a slab turbulence consisting of dispersionless Alfvén waves. It has already been shown in section 4.5.2 that \( \kappa_L \) vanishes for a turbulence having a slab geometry (see equation (4.91) and the comments following it). Furthermore, the drift coefficient \( \kappa_D \) is zero for an Alfvénic slab turbulence if forward and backward streaming waves consist of left- and right-handed polarized modes of equal intensity, i.e. \( \sigma^j = 0 \) for linearly polarized waves. This was shown in the previous section. Therefore, the only non-vanishing transport parameters are the parallel diffusion coefficient, \( \kappa_\parallel \), the rate of turbulent adiabatic deceleration, \( a_1 \), and the momentum diffusion coefficient \( a_2 \). The appropriate FPCs for an Alfvénic slab turbulence with \( \sigma^j = 0 \) were already derived in section 4.5.2. They are given by equations (4.87), (4.88) as well as (4.89) and provide, in combination with the representations (3.23a), (3.23c) and (3.23d) as well as the power spectrum normalization (4.60), for the mathematical basis of this section.

6.3.1 The Coefficient for Parallel Spatial Diffusion

Considering first the transport parameter for parallel spatial diffusion, equation (3.23a), the corresponding FPC \( D_{\mu\mu} \), given by (4.87), can easily be cast into the form

\[
D_{\mu\mu} = \frac{\pi^2 g_0^+ \Omega_A^2 q}{2 B_0^2 c_A^{1-q}} (1 - \mu^2) \epsilon^{1-q} f_{\mu\mu}(\mu)
\]  

(6.44)

where the resonance function \( f_{\mu\mu} \) is given by

\[
f_{\mu\mu}(\mu) = (1 - \mu \epsilon)^2 |\mu - \epsilon|^{q-1} + \delta (1 + \mu \epsilon)^2 |\mu + \epsilon|^{q-1}
\]  

(6.45)

Here, the abbreviation \( \delta = \frac{g_0^-}{g_0^+} = (\delta B^-/\delta B^+)^2 \), i.e. the ratio of the intensities of forward \( (j = +1) \) to backward \( (j = -1) \) propagating Alfvén waves, has been introduced. Note that the superscripts (+) and (−) denote, throughout this section, the quantities related to \( j = +1 \) and \( j = -1 \), respectively. The ratio \( \delta \) can also be expressed by the normalized cross helicity \( h_c = (g_0^+ - g_0^-)/(g_0^+ + g_0^-) \), so that

\[
\delta = \frac{1 - h_c}{1 + h_c}
\]  

(6.46)
where $h_c$ varies between $-1$ and $+1$. These values are attained if only backward or forward moving waves are present, respectively. According to equation (3.23a) of chapter 3, the parallel diffusion coefficient $\kappa_{||}$ is defined by an pitch angle integral over the inverse of equation (6.44). Upon substituting (6.44) into the representation (3.23a), the corresponding transport parameter reads

$$\kappa_{||} = \frac{v^2}{8} \int_{-1}^{1} d\mu \frac{(1 - \mu^2)^2}{D_{\mu\mu}} = \frac{v_A^{3-q} B_0^2}{8\pi^2 \Omega_{\perp}^2 q^2 y_0^3} \int_{-1}^{1} d\mu \frac{1 - \mu^2}{f_{\mu\mu}(\mu)} \tag{6.47}$$

Expressing the gyrofrequency $\Omega_\perp$ by its non-relativistic counterpart $\Omega_{\perp,0}$, i.e. $\Omega_\perp = \Omega_{\perp,0}/\gamma$ with $\gamma$ being the Lorentz factor, using the relation $\epsilon = v_A/v = \gamma p_A/p$, where $p_A$ denotes the momentum of a particle having the speed $v_A$, and inserting the appropriate normalization $g_0^+$, equation (4.60), the transport coefficient (6.47) can be written as

$$\kappa_{||} = \kappa_0 (1 + \delta) \gamma^{-1} \left( \frac{p}{p_A} \right)^{3-q} \int_{-1}^{1} d\mu \frac{1 - \mu^2}{f_{\mu\mu}(\mu)} \tag{6.48}$$

Here, the normalization constant of the diffusion coefficient is

$$\kappa_0 = \frac{v_A^{3-q} f_{\mu\mu}^{-1} g_{\perp,0}^{-2}}{\pi (q - 1)} \left( \frac{B_0}{\delta B} \right)^2 \tag{6.49}$$

where the relations $(\delta B)^2 = (\delta B^-)^2 + (\delta B^+)^2$ and $(B_0/\delta B^+)^2 = (B_0/\delta B)^2 (1 + \delta)$ were used.

In order to consider the influence of the ratio $\delta$ or, alternatively, normalized cross helicity $h_c$, on the parallel transport of charged particles in an Alfvénic slab turbulence, equation (6.48) is solved numerically by using Gauss’ method. In figure 6.5 computational results are presented for the parallel diffusion coefficient (6.48) for three different intensity ratios $\delta = 10^{-3}$, 1 and $10^3$. The figure shows the normalized coefficient $\kappa_{||}/\kappa_0$ as a function of the normalized momentum $p/p_A$. The spectral index $q$ and the Alfvén speed $v_A$ are assumed to be $5/3$ and $50$ km s$^{-1}$, respectively. A closer inspection of figure 6.5 reveals in following findings: $\kappa_{||}$ is not affected by the plasma wave intensity ratio in the intermediate and high momentum regime over a broad range of $\delta$-values. Only a slight variation is noticeable for the case $\delta = 1.0$ (dotted line) at very low energies. This feature will be discussed below in more detail. The essential independence of $\kappa_{||}$ on the ratio $\delta$ is not surprising and can easily be explained as follows: the Alfvénic turbulence intensity $(\delta B)^2 = (\delta B^+)^2 + (\delta B^-)^2$ provides for an energy pool supporting the particle diffusion. Both the forward as well as the backward propagating wave modes contribute to this energy pool and represent an individual rate of wave-particle interactions. Since $(\delta B)^2$ is fixed it does not matter whether the particles interact only with the forward or the backward propagating modes. The rate of wave-particle interactions is the same for all values of $\delta$.

To investigate the diffusive particle transport in a turbulent plasma for a certain physical scenario, it may be useful to obtain an analytical expression for the diffusion coefficient $\kappa_{||}$. A classical example for such a scenario will be considered in more detail in chapter 8. There, the solar modulation of anomalous as well as galactic cosmic rays in the interplanetary background medium is discussed. The appropriate equation of particle transport is then Parker’s equation (see equation (8.1) and the comments following it). This equation may be considered as a simplified version of the fundamental diffusion-convection equation (3.24) and includes, via the diffusion tensor (3.25), the parallel diffusion coefficient $\kappa_{||}$. The main result of chapter 8 is an analytical solution of Parker’s equation and its application to solar modulation. Therefore, it is desirable, and actually inevitable, to employ an analytical representation of the diffusion coefficient (6.48). Furthermore, an analytical expression for (6.48) allows to check the numerical results presented above.
Chapter 6 On Transport Parameters

Figure 6.5: Numerical solutions of the normalized diffusion coefficient \((6.48)\) as a function of the normalized momentum \(p/p_A\) for three different values of 
\(δ = (\delta B^-/\delta B^+)^2\) (see the legend for the values) and a fixed \((\delta B)^2\). The spectral index \(q\) and the Alfvén speed \(v_A\) were chosen to be as \(5/3\) and \(50\ \text{km s}^{-1}\), respectively.

In order obtain a treatable representation of \(κ∥\), it is convenient to split the \(µ\)-integral in equation \((6.48)\) in two parts from \(µ = −1\) to \(µ = 0\) and \(µ = 0\) to \(µ = 1\) and to substitute in the first integral \(t = −µ\). Subsequently, the new variable \(t\) can be renamed with \(µ\). Equation \((6.48)\) can then be expressed by

\[
κ∥ = κ_0 γ^{-1} \left( \frac{p}{p_A} \right)^{3-q} (1 + δ)^2 \int_0^1 dµ (1 - µ^2) \left[ \frac{f_{µµ}(µ, δ = 1)}{f_{µµ}(-µ) f_{µµ}(µ)} \right] \tag{6.50}
\]

where \(f_{µµ}(µ, δ = 1)\) is given by equation \((6.45)\) including the condition \(δ = 1\). By using \((6.45)\), the denominator of the integrand becomes

\[
f_{µµ}(-µ) f_{µµ}(µ) = (1 + δ)^2 (F_1 + F_2) \tag{6.51}
\]

where the auxiliary functions \(F_1\) and \(F_2\) are defined by the following expressions:

\[
F_1 = (1 - µ^2 \epsilon^2)^2 |µ - \epsilon|^{q-1} |µ + \epsilon|^{q-1} \tag{6.52}
\]

\[
F_2 = \frac{δ}{(1 + δ)^2} \left[ (1 + µ \epsilon)^2 |µ + \epsilon|^{q-1} - (1 - µ \epsilon)^2 |µ - \epsilon|^{q-1} \right] \tag{6.53}
\]

With this notation the parallel diffusion coefficient \(κ∥\) reads

\[
κ∥ = κ_0 γ^{-1} \left( \frac{p}{p_A} \right)^{3-q} \int_0^1 dµ (1 - µ^2) \left[ \frac{f_{µµ}(µ, δ = 1)}{F_1 + F_2} \right] \tag{6.54}
\]

At a first glance, it immediately becomes clear that the ratio \(δ\) enters the calculations only via the auxiliary function \(F_2\). A further treatment of \((6.54)\) requires, therefore, a closer inspection of the function \(F_2\) and, additionally, of \(F_1\). For illustrative purposes and to understand the properties of the function \(F_2\), figure 6.6 displays two computational calculations of equation \((6.53)\).

The left panel shows \(F_2\) as a function of the pitch angle \(µ\) and the intensity ratio \(δ\). The computed values are shown as a surface over the \(µ-δ\)-plane for \(q = 5/3\) and \(ε = 10^{-3}\). The right panel presents, in difference to the left panel of figure 6.6, a computation for \(q = 5/3\) and \(δ = 1\). In this case,
the values of $F_2$ form a surface over the $\mu$-$\epsilon$-plane, i.e. $F_2$ is a function depending on $\mu$ and $\epsilon$. An inspection of both panels shows the following properties of equation (6.53): first, $F_2$ has two maxima with respect to $\mu$ for a given $\delta$-value. This can nicely be seen by considering the left panel of figure 6.6. Apparently, equation (6.53) reveals a resonance character. It is strong at $\mu = \epsilon$ and $\mu = 1$ for $\delta = 1$ but less appreciable within a relative small increment of increasing and decreasing $\delta$ for both $\mu$-values. Furthermore, considering the maximum at $\mu = \epsilon$, the function $F_2$ drops off relatively quickly within a relative small increment of increasing and decreasing $\mu$. Second, the maximum values of $F_2$ become less important for decreasing $\epsilon$-values. This is shown in the right panel of figure 6.6. The maximum of $F_2$ at $\mu = \epsilon$ is shifted to smaller values of $\mu$, whereas the maximum value at $\mu = 1$ drops off faster than that at $\mu = \epsilon$. Note that the right panel shows the $\epsilon$-behavior of $F_2$ for $\delta = 1$. On the basis of the left panel it immediately becomes clear that the $F_2$-values in the right panel would even be much smaller than shown.

(a) The function $F_2$ for $q = 5/3$ and $\epsilon = 10^{-3}$.

(b) The function $F_2$ for $q = 5/3$ and $\delta = 1$.

Figure 6.6: Illustrations of the auxiliary function $F_2$ as given by equation (6.53). The left panel shows $F_2$ as a function depending on the pitch angle $\mu$ and the intensity ratio $\delta$. The values are given as a surface over the $\delta$-$\mu$-plane for the spectral index $q = 5/3$ and $\epsilon = 10^{-3}$. For illustrative purposes, the pitch angle $\mu$ and the ratio $\delta$ are indicated by a logarithmic scaling, respectively. The right panel displays a computation for $q = 5/3$ and $\delta = 1$. Here, $F_2$ is a function of $\mu$ and $\epsilon$. In this case, the values of $F_2$ produce a surface over the $\epsilon$-$\mu$-plane.

In order to estimate the importance of $F_2$ on the denominator of the integral in equation (6.54), it is necessary to compare $F_2$ with the function $F_1$. Figure 6.7 illustrates $F_1$, equation (6.52), as a function of $\mu$ for three different values of $\epsilon$. The solid, dotted and dashed curves correspond to $\epsilon = 10^{-1}$, $10^{-2}$ and $10^{-3}$, respectively. It can immediately be seen that $F_1$ reveals null points at the corresponding values of $\mu = \epsilon$. Of course, this is also clear from equation (6.52). These gaps are filled up by the maximum values of $F_2$ at the corresponding pitch angles $\mu = \epsilon$ for $\delta = 1$ (compare the left panel of figure 6.6 with the dashed curve shown in figure 6.7, and the corresponding $F_2$
and $F_1$ values, respectively). For pitch angles less than $\epsilon$, equation (6.52) is given by a constant. At $\mu > \epsilon$, the function $F_1$ behaves as a power law in $\mu$ and becomes more appreciable.

In view of these findings, it may be useful to obtain approximations for $F_1$ and $F_2$ for the pitch angle intervals $\mu \in [0, \epsilon]$ and $\mu \in [\epsilon, 1]$. Considering first equation (6.52), one easily finds that the auxiliary function $F_1$ is approximately expressible (in accordance with figure 6.7) as $F_1 \simeq \epsilon^{2q-2}$ and $F_1 \simeq \mu^{2q-2}$ for the intervals $\mu \in [0, \epsilon]$ and $\mu \in [\epsilon, 1]$, respectively. Similarly, equation (6.53) yields $F_2 \simeq 16\delta\mu^2 \epsilon^{2q}/(1 + \delta)^2$ for the interval $\mu \in [0, \epsilon]$ and $F_2 \simeq 16\delta\mu^2 \epsilon^2/(1 + \delta)^2$ if $\mu \in [\epsilon, 1]$. Hence, one readily arrives at

$$\frac{F_2}{F_1} = \frac{16\delta\mu^2 \epsilon^2}{(1 + \delta)^2} \quad \text{or} \quad \frac{F_2}{F_1} = 4(1 - \kappa)^2 \mu^2 \epsilon^2$$  \hspace{1cm} (6.55)

for both pitch angle intervals. Evidently, the function $F_2$ is much less than $F_1$ if $\epsilon \ll 1$. It is noteworthy that the condition $\epsilon \ll 1$ has already been assumed for the diffusion approximation and, therefore, for the derivation of the diffusion-convection transport equation $(3.24)$ presented in chapter 3. It is also worthy to recall now the numerical calculation of equation (6.48) for $\delta = 1$ visualized by the dotted curve in figure 6.5. The reason for the relative small variation of this curve from the solid ($\delta = 10^{-3}$) or the dashed ($\delta = 10^3$) line becomes clear now. For $\delta = 1$ and relative small values of $p/p_A \propto \epsilon^{-1}$, the function $F_2$ becomes most appreciable and provides, additionally to $F_1$, for a small contribution. Accordingly, the denominator of the integrand in equation (6.54) becomes larger and, therefore, forces the diffusion coefficient to decrease slightly by a small amount. Since $F_2$ becomes negligible with increasing $p/p_A \propto \epsilon^{-1}$, the dotted curve approaches quickly the solid and dashed lines.

In view of the explanations presented above and, furthermore, the fact that $F_2$ becomes important only for $\epsilon$-values being incompatible with the diffusion approximation and, therefore, with the representation (3.23a), one may neglect without any bother for the following calculations the function $F_2$. The coefficient (6.54) can then be written as

$$\kappa = \kappa_0 \gamma^{-1} \left( \frac{P}{p_A} \right)^{3-q} \int_0^1 d\mu (1 - \mu^2) \frac{f_{\mu\mu}(\mu, \delta = 1)}{F_1}$$  \hspace{1cm} (6.56)
By considering the pitch angle intervals from above, the integral in equation (6.48) can be split in two parts from $\mu = 0$ to $\mu = \epsilon$ and $\mu = \epsilon$ to $\mu = 1$, so that
\[
\int_{0}^{1} d\mu (1 - \mu^2) \frac{f_{\mu\mu}(\mu, \delta = 1)}{F_1} = \frac{\epsilon}{\epsilon} \int_{0}^{\epsilon} d\mu (1 - \mu^2) \frac{f_{\mu\mu}(\mu, \delta = 1)}{F_1} + \int_{\epsilon}^{1} d\mu (1 - \mu^2) \frac{f_{\mu\mu}(\mu, \delta = 1)}{F_1} \quad (6.57)
\]
Since $\epsilon \ll 1$ and $\mu \in [0, 1]$, the cyclotron-resonance function (6.45) may be written as
\[
f_{\mu\mu}(\mu, \delta = 1) \simeq |\mu - \epsilon|^{q-1} + |\mu + \epsilon|^{q-1} \quad (6.58)
\]
This expression can be subjected to a further approximation and yields $f_{\mu\mu} \simeq 2\epsilon^{q-1}$ for the pitch angle interval $0 \leq \mu < \epsilon$ and $f_{\mu\mu} \simeq 2\mu^{q-1}$ if $\epsilon < \mu \leq 1$. By using for these intervals the appropriate approximations of $F_1$, i.e. $\epsilon^{2q-2}$ and $\mu^{2q-2}$ for $\mu \in [0, \epsilon]$ and $\mu \in [\epsilon, 1]$, respectively, the integration can readily be carried out to obtain
\[
\int_{0}^{1} d\mu (1 - \mu^2) \frac{f_{\mu\mu}(\mu, \delta = 1)}{F_1} \simeq \frac{4}{(2 - q)(4 - q)} + \frac{2q - 1}{q - 2} \epsilon^{2q - 2} \quad (6.59)
\]
where the spectral index has to satisfy the conditions $1 < q < 4$ and $\mu \neq 2$. For values of $q < 2$, one finally arrives at
\[
\kappa_{||} = \kappa_{||,0} \gamma^{-1} \left( \frac{p}{p_A} \right)^{3-q} \quad \text{with} \quad \kappa_{||,0} = \frac{4\nu_A^{3-q}k_{\min}^{1-q}Q_{\alpha,0}^{q-2}}{\pi(q - 1)(2 - q)(4 - q)} \left( \frac{B_0}{\delta B} \right)^2 \quad (6.60)
\]
where the appropriate $q$-dependence of the integration has been shifted into the rearranged normalization constant $\kappa_{||,0}$. The coefficient (6.60) agrees, despite from a dimensionless constant in $\kappa_{||,0}$, with equation (74) of Schlickeiser [1989a] for $q < 2$. He has calculated $\kappa_{||}$ for forward and backward propagating Alfvén waves streaming with the same intensity, i.e. $\delta = 1$. Here, however, it is shown that the diffusion coefficient for parallel particle transport is not affected by $\delta$ or, alternatively, the normalized cross helicity $h_c$. In this sense, the considerations presented above generalize the earlier result derived by Schlickeiser [1989a].

6.3.2 The Rate of Turbulent Adiabatic Deceleration

After having considered $\kappa_{||}$ in more detail, this section turns the attention to the second transport parameter, i.e. $a_1$. It is given by equation (3.23c) and describes adiabatic deceleration due to the turbulence. In combination with the FPCs (4.88) and (6.44), the representation (3.23c) can readily be written as
\[
a_1 = \frac{1}{\epsilon} \int_{1}^{\epsilon} d\mu (1 - \mu^2) \frac{D_{\mu\mu}}{D_{\mu\mu}} = \frac{1}{\epsilon} \int_{-1}^{1} d\mu (1 - \mu^2) \frac{f_{\mu\mu}(\mu)}{f_{\mu\mu}(\mu)} \quad (6.61)
\]
with $a_{1,0} = p_A$ being the corresponding normalization constant. The function $f_{\mu\mu}$ was already introduced by equation (6.45), while the resonance function $f_{\mu\mu}$ is given by
\[
f_{\mu\mu}(\mu) = (1 - \mu^2)|\mu - \epsilon|^{q-1} - (1 + \mu^2)|\mu + \epsilon|^{q-1} \quad (6.62)
\]
Similar to the case of parallel particle diffusion presented above, equation (6.61) was solved numerically for different values of the ratio $\delta$. Figure 6.8 displays numerical solutions of equation (6.61) and shows the normalized rate $a_1/a_{1,0}$ as a function of the normalized momentum $p/p_A$. For the spectral index $q$ and the Alfvén speed $v_A$ typical values such as 5/3 and 50 km s$^{-1}$ are chosen. Going from the top to the bottom, equation (6.61) was solved for the ratios $\delta = 0, 0.5, 1, 2, 5, 10$.
and $10^3$. On the first view it immediately becomes obvious that $a_1$ changes its sign for values above $\delta = 1$ (horizontal line). For $\delta$ values less (greater) than unity $a_1$ is positive (negative). For forward and backward propagating waves having the same intensity ($\delta = 1$) the rate of adiabatic deceleration, due to turbulence, has no influence on the particle transport, i.e. $a_1 = 0$. This numerical intermediate result is in accordance with equation (67) of Schlickeiser [1989a]. For the limit $\delta \to \infty$, i.e. $(\delta B^+)^2 \to 0$, equation (6.61) approaches the same limit as for $\delta = 0$, i.e. $(\delta B^-)^2 \to 0$, but with a reversed sign.

**In order to obtain an analytical, i.e. treatable, expression for the rate of adiabatic deceleration $a_1$, equation (6.61) can also be considered on the basis of symmetry conditions of the functions $f_{\mu p}$ and $f_{\mu \mu}$. In analogy to the $\kappa_{\parallel}$-calculations performed before (see equation (6.50) and the comments following it), the rate (6.61) becomes**

$$a_1 = 2a_{1,0}\gamma \frac{(1-\delta^2)}{(1+\delta)^2} \int_0^1 d\mu \frac{(1-\mu^2)(1-\mu^2\epsilon^2)|\mu-\epsilon|^{q-1}|\mu+\epsilon|^{q-1}}{F_1 + F_2}$$

(6.63)

where the functions $F_1$ and $F_2$ are given by the equations (6.52) and (6.53), respectively. It has already been shown above that the function $F_2$ has its maximum contribution for $\delta = 1$ and becomes negligible with decreasing $\epsilon$ (see equation (6.55) and the comments following it). Therefore, one may make use of $F_1 \gg F_2$ for $\epsilon \ll 1$. In combination with (6.52) the rate (6.63) may then be written as

$$a_1 = 2a_{1,0}\gamma \frac{(1-\delta^2)}{(1+\delta)^2} \int_0^1 d\mu \frac{(1-\mu^2)}{(1-\mu^2\epsilon^2)}$$

(6.64)

where the integral can readily be calculated to result in

$$\int_0^1 d\mu \frac{(1-\mu^2)}{(1-\mu^2\epsilon^2)} = \frac{1}{\epsilon^3} \left[ \epsilon - (1-\epsilon^2) \arctanh(\epsilon) \right] \approx \frac{2}{3}$$

(6.65)

Figure 6.8: Numerical solutions of the normalized rate (6.61), being a function of the normalized momentum $p/p_A$, for the ratios $\delta = 0, 0.5, 1, 2, 5, 10$ and $10^3$ (from the top to the bottom). The spectral index $q$ and the Alfvén speed $v_A$ were assumed to be 5/3 and 50 km s$^{-1}$, respectively.
Here, \( \text{arctanh} \) denotes the hyperbolic arc tangent. The approximation holds for \( \epsilon \ll 1 \). With the latter relation the rate of adiabatic deceleration, due to the Alfvénic turbulence, reads as follows:

\[
a_1 = \frac{4}{3} a_{1,0} \gamma \frac{(1 - \delta)}{(1 + \delta)} = \frac{4}{3} a_{1,0} \gamma h_c
\]  

(6.66)

with \( h_c = (1 - \delta)/(1 + \delta) \) being the normalized cross helicity. Apparently, the rate of adiabatic deceleration is directly proportional to \( h_c \) and is independent of the spectral index \( q \) and the particle charge. However, \( a_1 \) is proportional, via the normalization \( a_{1,0} = p_A \), to the mass \( m_\alpha \) so that it becomes more appreciable for heavier particles. Including the restrictions \( \delta = 0 \) and \( \delta = 1 \), equation (6.66) agrees exactly with equations (61) and (67) of Schlickeiser [1989a], respectively. He calculated \( a_1 \) for Alfvén waves propagating only in parallel direction (\( \delta = 0 \)) and for waves streaming in both directions with the same intensity (\( \delta = 1 \)). Furthermore, it confirms the considerations performed earlier by Dung and Schlickeiser [1990] (cf. their formula 31).

The analytical expression (6.66) can nicely be compared with the numerical calculations presented in figure 6.8. For instance, the choice \( \delta = 2 \) leads to \( a_1/a_{1,0} = -0.44\gamma \). Then, a closer inspection of figure 6.8 (the fourth curve from the top) results in the finding that the amplitude (−0.44) of \( a_1 \) is indeed given by equation (6.66). Note that this agreement of the numerical and the analytical approach holds for all values of \( \delta \).

### 6.3.3 The Coefficient for Stochastic Acceleration

The last standard transport parameter is the momentum diffusion coefficient \( a_2 \). It provides, according to the representation (3.23d), for the diffusive energization of charged particles, i.e. Fermi-II acceleration. By using the FPCs (4.87), (4.88) and (4.89) one finds that (3.23d) is expressible as

\[
a_2 = \frac{1}{2} \int_{-1}^1 d\mu \left[ \frac{D_{pp}D_{p\mu} - D_{p\mu}^2}{D_{p\mu}} \right] = a_{2,0}(1 + \delta)^{-1} \gamma \left( \frac{p}{p_A} \right)^{q-1} \int_{-1}^1 d\mu \left[ \frac{f_{pp}(\mu)f_{p\mu}(\mu) - f_{pp}^2(\mu)}{f_{p\mu}(\mu)} \right]
\]  

(6.67)

where several physical parameters have been summarized in the normalization constant

\[
a_{2,0} = \frac{\pi(q-1)\Omega_{a,0}^2 p_A^{2-q} k_m B_A^2}{16\epsilon v_A^{1-q}} \left( \frac{\delta B}{B_0} \right)^2
\]  

(6.68)

The functions \( f_{\mu\mu} \) and \( f_{pp} \) are given by (6.45) and (6.62), while the resonance function \( f_{pp} \), stemming from the FPC \( D_{pp} \), takes the form

\[
f_{pp} = |\mu - \epsilon| q^{-1} + \delta |\mu + \epsilon| q^{-1}
\]  

(6.69)

In analogy to the numerical calculations performed for the transport coefficients \( \kappa_\parallel \) and \( a_1 \), equation (6.67) was solved by applying Gauss’s method. Figure 6.9 shows computational results, representing the normalized coefficient \( a_2/a_{2,0} \) as a function of the normalized momentum \( p/p_A \), for six different \( \delta \)-values. The ratios are \( \delta = 10^{-3} \) (solid line), 1 (dotted line) and 10, 100, 2000 and \( 10^4 \) (dashed curves going from the top to the bottom). The spectral index \( q \) and the Alfvén speed \( v_A \) are chosen to be 5/3 and 50 km s\(^{-1}\), respectively. Solutions for \( \delta = 0 \) and the limit \( \delta \to \infty \) (e.g. \( 10^{10} \)) are not shown in the figure. For these two cases the code yields \( a_2 = 0 \): no stochastic acceleration occurs. By considering figure 6.9 in more detail the following scenario becomes clear: for very small \( \delta \)-values (\( \delta = 10^{-3} \), solid line) the coefficient increases with increasing momentum and becomes steeper around \( p/p_A \approx 7 \cdot 10^4 \). The change in the slope is due to relativistic effects. With increasing values of \( \delta \) the coefficient (6.67) approaches a maximum for \( \delta = 1 \). For this specific ratio, stochastic
acceleration becomes most effective. A further increase in $\delta$ leads to a less efficient momentum diffusion (see the dashed curves and their corresponding values given above). The quintessence of figure 6.9 is that stochastic acceleration is most effective if forward and backward propagating plasma waves have the same intensity ($\delta = 1$).

Figure 6.9: Numerical solutions of equation (6.67) showing the momentum diffusion coefficient $a_2/a_{2,0}$ as a function of the momentum $p/p_A$. The $\delta$-values were assumed to be $10^{-3}$ (solid line), 1 (dotted line) and 10, 100, 2000 and $10^4$ (dashed curves going from the top to the bottom). The spectral index $q$ and the Alfvén speed $v_A$ were set to $5/3$ and 50 km s$^{-1}$, respectively.

For later applications, it is useful to derive an analytical expression for the momentum diffusion coefficient $a_2$. Chapter 7 is related to stochastic acceleration of PUIs in the heliosphere. There, an appropriate version of the fundamental diffusion-convection equation (3.24) will be solved analytically. Accordingly, a tractable expression for equation (6.67) is required. Similarly to the $\kappa_\parallel$-calculations performed above (see equation (6.50) and the comments following it), equation (6.67) may be subjected to a further treatment by using the symmetry properties of the functions $f_{\mu\mu}$, $f_{\mu p}$ and $f_{pp}$. After some algebra, the momentum diffusion coefficient (6.67) becomes

$$a_2 = 4a_{2,0} \frac{\delta}{(1 + \delta)^2} \gamma \left( \frac{p}{p_A} \right)^{q-1} \int_0^1 d\mu \frac{(1 - \mu^2)[\mu - \epsilon |q-1|\mu + \epsilon |q-1|f_{\mu\mu}(\mu, \delta = 1)}{F_1 + F_2}$$

(6.70)

where the functions $F_1$ and $F_2$ are given by equations (6.52) and (6.53) again, respectively. The quantity $f_{\mu\mu}(\mu, \delta = 1)$ denotes the resonance function (6.45) for $\delta = 1$. Since $F_2$ is negligible for $\epsilon \ll 1$ (see equation (6.55) and the comments following it), one finds that equation (6.70) is expressible as

$$a_2 = 4a_{2,0} \frac{\delta}{(1 + \delta)^2} \gamma \left( \frac{p}{p_A} \right)^{q-1} \int_0^1 d\mu \frac{(1 - \mu^2)^2}{(1 - \mu^2\epsilon^2)^2} f_{\mu\mu}(\mu, \delta = 1)$$

(6.71)

where equation (6.52) has been used. The integral in (6.71) may be split into two parts from $\mu = 0$ to $\mu = \epsilon$ and $\mu = \epsilon$ to $\mu = 1$. It has already been shown on page 101 that the function $f_{\mu\mu}$ can then be approximated by the expressions $2\epsilon^{q-1}$ and $2\mu^{q-1}$ for $0 \leq \mu < \epsilon$ and $\epsilon < \mu \leq 1$, respectively. Furthermore, the denominator in the integrand of equation (6.71) may be replaced by 1, because $\epsilon \ll 1$ and $\mu \in [0,1]$. Then, the $\mu$-integration yields

$$\int_0^1 d\mu \frac{(1 - \mu^2)^2}{(1 - \mu^2\epsilon^2)^2} f_{\mu\mu}(\mu, \delta = 1) \simeq 2 \int_0^\epsilon \mu (1 - \mu^2)\epsilon^{q-1} + 2 \int_{\epsilon}^1 (1 - \mu^2) \mu^{q-1} \simeq \frac{4}{q(2 + q)}$$

(6.72)
Expressing the ratio $\delta$ by the normalized cross helicity $h_c$ and using the latter relation, the momentum diffusion coefficient takes the form

$$a_2 = a_{2.0} (1 - h_c^2)^{\gamma} \left( \frac{P}{p_A} \right)^{q-1}$$

(6.73)

where the $q$-dependence of the integration has been shifted into the normalization constant, so that

$$a_{2.0} = \frac{\pi (q - 1)}{4q(2 + q)} v_A^{1-q} k_{mn}^{1-q} \left( \frac{\delta B}{B_0} \right)^2$$

(6.74)

Equation (6.73) agrees for $h_c = 0$, apart from a dimensionless constant, with equation (75) of Schlickeiser [1989a]. He has calculated $a_2$ for waves streaming in both directions with the same intensity ($\delta = 1$ or $h_c = 0$). Furthermore, it reveals for $\delta = 0$ or, alternatively, $h_c = \pm 1$ a vanishing stochastic acceleration, i.e. $a_2 = 0$. This is in accordance with equation (62) of Schlickeiser [1989a]. The result (6.73) also supports the calculations carried out earlier by Dung and Schlickeiser [1990], who even included a non-vanishing magnetic helicity $\sigma^j$.

### 6.3.4 Radial Dependences

The transport parameters derived above depend not only on the particle momentum $p$ but also on the coordinates of the spatial configuration space. In general, the background plasma moves with an arbitrary bulk velocity $V$ where each velocity component acts according to an individual profile in all three spatial coordinates. Due to the high conductivity of the streaming plasma, the background magnetic field is frozen into the plasma flow (see equation (C.21) in appendix C) and, therefore, experiences variations in spatial configuration space too. In other words, the turbulence carrier called plasma transfers its spatial dependence on a turbulence parameter such as the Alfvén speed $v_A$ and, therefore, on the transport parameters (6.60), (6.66) and (6.73). The turbulence parameters themselves presumably contain additional dependences on spatial coordinates describing intrinsic evolutions in space within the embedding plasma flow. In this sense, the spatial behavior of the turbulence in all three dimensions does not render completely the spatial evolution of the bulk velocity.

Under certain circumstances, however, the streaming of the background medium can be directed relative to a certain position in spatial space. The motion of the plasma and, therefore, of the turbulence and its properties is then aligned along a distinguished axis. Examples for such scenarios are the solar wind plasma in the heliosphere, expanding to larger heliocentric distances in radial direction, and the galactic wind. In the latter case, the distinction of streaming may roughly be defined by an axis perpendicular to the galactic plane.

Having in mind the two applications mentioned in the previous sections, the following considerations are restricted to the case of the outward expanding solar wind plasma. For the following, it is assumed that the radial bulk speed $V$ is expressible as a simple power law in the heliocentric distance $r$, i.e.

$$V = V_0 \left( \frac{r}{r_E} \right)^{\gamma_W}$$

(6.75)

On the basis of the continuity equation (C.15) introduced in appendix C, one readily finds, for spherical symmetry, the following relation between the speed $V$ and the mass density $\rho$ of the solar wind: $r^2 \rho V = const$. This immediately yields $\rho = \rho_0 (r/r_E)^{\gamma_W - 2}$. Assuming that the background magnetic field evolves radially as $B_0 = B_E (r/r)^a$, the Alfvén speed can be written as $v_A = B_0/\sqrt{4\pi \rho} = v_{A,0} (r/r_E)^{1-a-\gamma_W/2}$. Taking now the plasma wave intensity to vary as a power law in heliocentric distance, i.e. $(\delta B)^2 = (\delta B_0)^2 (r_E/r)^{\eta}$, the standard transport parameters (6.60),
(6.66) and (6.73) can be expressed as follows:

\[
\kappa_\parallel = \frac{4v^{3-q}Ω^{q-2}_q}{\pi(q-1)(2-q)(4-q)} \left( \frac{B_E}{δB_E} \right)^2 \left( \frac{Z}{A} \right)^{q-2} \left( \frac{l_c}{2π} \right)^{q-1} \left( \frac{p}{p_{A,0}} \right)^{3-q} \left( \frac{r}{r_E} \right)^{η-aq} \tag{6.76}
\]

\[
a_1 = \frac{4A^2}{3}p_{A,0}\gamma \left( \frac{r}{r_E} \right)^{1-a-γ/2} \tag{6.77}
\]

\[
a_2 = \frac{π(q-1)}{4q(2+q)} \frac{A^2Ω^{2-q}_{q,0}p_{A,0}^2}{B_E} \left( \frac{δB_E}{B_E} \right)^2 \left( \frac{Z}{A} \right)^{2-q} \left( \frac{l_c}{2π} \right)^{1-q} \left( 1-h_c^2 \right)^γ \left( \frac{p}{p_{A,0}} \right)^{q-1} \left( \frac{r}{r_E} \right)^{a(q-2)+2-γw-η} \tag{6.78}
\]

Here, the minimum wavenumber \( k_{min} \) is expressed by the turbulence correlation scale length \( l_c \).

The quantities \( Z \) and \( A \) denote the particle’s charge and mass number, respectively.

### 6.4 Summary and Conclusions

This chapter introduces transport parameters for both the isotropic and slab turbulence. In the first section, the transport parameter for particle diffusion perpendicular to the background magnetic field is calculated. For this, an isotropic turbulence is assumed consisting of undamped fast magnetosonic waves contributing, for simplicity, only via transit-time damping. The calculations and illustrations point out that the inclusion of a finite turbulence correlation length is of importance for the perpendicular transport of charged particles, in particular for high-energy particles having large Larmor radii. For very long correlation lengths, earlier results developed on the concept of field line random walk are retained. It is discussed that a magnetic fluctuation spectrum with broken power laws may also be important for understanding perpendicular diffusion of charged particles. This is demonstrated by a comparison of the new representation with simulation results obtained previously by Giacalone and Jokipii [1999].

Calculations of the drift coefficient \( \kappa_T \) are presented in the second section. For slab geometry, \( \kappa_T \) is completely determined by both the cross and magnetic helicity of forward and backward propagating wave fields. Furthermore, the particle drift coefficient \( \kappa_T \) is independent of particle properties for undamped wave modes. This is explained by the \( \mathbf{E} \times \mathbf{B} \) drift. The acting force is caused by the electric field of the plasma wave turbulence. Since the polarization states have preferred directions in the slab turbulence with its reduced dimensionality, the drift coefficient changes its sign with the reversal of the turbulence net polarity. The states of polarization exhibit no preferred directions for an isotropic turbulence, and no drift occurs.

In the last section of this chapter, transport parameters for parallel particle diffusion, adiabatic cooling and momentum diffusion are derived for an Alfvénic slab turbulence. Numerical and analytical calculations for different normalized cross helicities generalize and confirm earlier results available so far.
Chapter 7

Stochastic Acceleration of Pick-Up Ions

The primary aim of this chapter is to introduce a first application of the fundamental diffusion-convection transport equation (3.24) and its associated transport parameters derived in chapter 6. For spherical symmetry and steady-state, the diffusion-convection equation is considered in the limit of negligible spatial particle diffusion and drift effects. It then represents a transport equation being appropriate for the consideration of Fermi II acceleration of charged particles in a turbulent plasma. Under the assumption that the momentum diffusion coefficient and the bulk speed of the background plasma may be expressed as arbitrary power laws in particle momentum and distance, a general solution of this transport equation is derived here for the first time. This general solution is given in terms of an arbitrary particle source function. Furthermore, it depends sensitively on the arbitrary exponents of the momentum diffusion coefficient and the bulk speed power laws. The general solution is therefore beneficial not only for the investigation of stochastic acceleration of pick-up ions (PUIs) in the heliosphere, which is the focus of this chapter, but also for other heliospheric as well as astrophysical problems. By using a specific momentum diffusion coefficient for slab Alfvén waves (see section 6.3) and a standard PUI source function, describing the injection of interstellar neutral atoms into the heliosphere and their subsequent ionization, the general solution is applied to the isotropic transport of PUIs and their stochastic acceleration. The new solution generalizes significantly those results available so far for both the case of a vanishing and non-vanishing momentum diffusion coefficient. For large heliocentric distances, several asymptotic phase space integrals are presented and solved analytically for the standard PUI source function. Numerical calculations for interstellar neutral hydrogen yield phase space distributions depending sensitively on parameters of the underlying turbulence and its dependence on heliocentric distance.

7.1 The PUI Transport Equation and Its Solution

The first attempt to investigate PUI velocity distributions from a mathematical point of view was made by Vasyliunas and Siscoe [1976]. Taking into account only adiabatic cooling, due to the expanding solar wind with convection speed $u$, they derived a solution of a simple transport equation describing the isotropic transport of PUIs in the limit of a vanishing momentum diffusion and were able to give the PUI distribution as a function of heliocentric distance and phase space velocity. However, since Vasyliunas and Siscoe [1976] ignored energy diffusion, their analysis resulted in a box-shaped phase space distribution in a comoving frame cutting off sharply above the solar wind speed. In view of a vanishing momentum diffusion, the results of Vasyliunas and Siscoe [1976] cannot explain the observed energetic tails, which have been recognized later in PUI velocity distributions. Besides this strong restriction, le Roux et al. [2000] recently pointed out, that the solution by Vasyliunas and Siscoe [1976] is valid for constant solar wind speed only.

The only analytical solution of the PUI transport problem including the effect of stochastic acceleration has been presented by Isenberg [1987]. He generalized the considerations of Vasyliunas and Siscoe [1976] by extending the corresponding transport equation by an additional term describing
diffusion in velocity space. The generalized transport equation being appropriate to describe the
temporal evolution of the PUI phase space distribution under the influence of adiabatic deceleration
and convection as well as energy diffusion, reads
\[
\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial r} = \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 D \frac{\partial f}{\partial p} \right) + \frac{p}{3r^2} \frac{\partial}{\partial r} \left( r^2 u \right) \frac{\partial f}{\partial p} + S
\]  
(7.1)
where the first term on the right-hand side represents momentum diffusion characterized by the
scalar momentum diffusion coefficient \( D(r, p, t) \). The second contribution describes adiabatic decel-
eration, arising because of the outward convection of PUIs with the steady spherically symmetric
expanding solar wind plasma. The source function \( S(r, p, t) \) represents a local injection rate, i.e.
a continuous ionization of neutral particles. Equation (7.1) has been used in several numerical
studies carried out in the past concerning the isotropic transport of PUIs in the heliosphere (see, e.g., Bogdan, 1991; Chalov et al., 1995; Fichtner et al., 1996).

It is instructive to note that equation (7.1) is only a special simplified version of the diffusion-
convection equation (3.24), which was already introduced in chapter 3. Since PUIs are coupled to the
frozen-in heliospheric magnetic background field, it is usually assumed that effects resulting from
particle drifts and spatial diffusion are small and, therefore, negligible (see Rucinski et al., 1993). The
fundamental diffusion-convection equation (3.24) can then be considered for a vanishing spatial
diffusion tensor (3.25), i.e., \( K = 0 \). Introducing \( a_2 = D \) and \( V = u \), ignoring any loss terms in
equation (3.24) and, furthermore, using for the rate of adiabatic deceleration, due to the turbulence,
the condition \( a_1 = 0 \), equation (3.24) indeed reduces to (7.1).

In the case that the turbulence consists of plasma waves, the restriction \( a_1 = 0 \) implies equal
intensities of parallel and antiparallel propagating waves. As it has been shown in chapter 6 (see
figure 6.9 and the corresponding explanations given in the text), quasi-linear diffusion in momentum
space requires wave power moving essentially in opposite directions with respect to the background
magnetic field. Stochastic acceleration approaches its maximum efficiency if the ratio of forward and
backward moving wave power is equal to unity. However, in the case of heavier interstellar neutral
atoms, in particular helium, due the greater mass and higher ionization potentials and, furthermore,
lower ionization rates such particles get closer to the Sun \(^1\). At heliocentric distances less than 1 AU
waves are commonly found to preferentially propagate away from the Sun (see, e.g., Marsch, 1991),
so that diffusion in momentum space and, therefore, stochastic acceleration is substantially reduced.
This requires a solution of a less simplified diffusion-convection equation (3.24), for which the
restriction \( a_1 = 0 \) has to be dropped. So far, no analytical theory exists including the influence of
\( a_1 \neq 0 \), i.e. forward and backward propagating waves having different wave power, on the phase
space distribution of particle populations. For the case \( a_1 \neq 0 \), a solution of the fundamental
diffusion-convection equation (3.24) or associated simplified versions have not been derived so
far, neither in the case of PUIs and GCRs/ACRs in the heliosphere nor for the transport of
GCRs in the interstellar medium \(^2\). Here, it is also assumed that the rate of turbulent adiabatic
deceleration satisfies the condition \( a_1 = 0 \). Then, the relevant transport equation, which will be
solved analytically, is indeed given by (7.1).

On the basis of equation (7.1), Isenberg [1987] derived a time-dependent solution for the phase
space distribution function of PUIs interacting with a quasi-linear Kolmogorov plasma wave turbu-
lence embedded in a steady solar wind plasma. In order to derive his solution, he used a simplified
source function and restricted his calculations to a specific case of the ambient wave field power
with radial dependence and to a special wave power spectral index. Considering asymptotic limits,
hhe presented several expressions describing the PUI velocity distribution in the limits of large radii
or particle energies.

\(^1\) See figure 7.1 in section 7.2. There, the ionization scale length of helium is estimated with 0.06 AU.
\(^2\) Based on the consideration presented in this chapter, the author recently derived solutions for the case \( a_1 \neq 0 \).
Detailed investigations will be the subject of forthcoming studies.
The “classical” analytical solutions for a vanishing and a non-vanishing momentum diffusion, which have been used in the past concerning the isotropic transport of PUIs, are those derived by Vasyliunas and Siscoe [1976] as well as Isenberg [1987], respectively. As stated above, both solutions are subjected to strong restrictions. It is, therefore, desirable to derive new solutions, which generalize these previous solutions with regard to different plasma parameters and yield more physical insight concerning the underlying turbulence.

Since wave-particle interactions, which take place in the streaming solar wind plasma having the bulk speed $u$, are characterized by the momentum diffusion coefficient $D$, the speed $u(r)$ and the coefficient $D(r,p)$ have to be specified as functions in the heliocentric distance $r$ and the particle momentum $p$. Assuming for further calculations a diffusion coefficient and a plasma speed of the form $D(r,p) = D_0 \left(\frac{p}{p_E}\right)^\alpha (r/r_E)^\beta$ and $u(r) = u_0 (r/r_E)^\gamma$, respectively, where the exponents $\alpha$, $\beta$ and $\gamma$ are arbitrary constants and the subscript $E$ refers to reference values of the corresponding quantities, the steady-state version of equation (7.1) can be manipulated by introducing the new variables

$$y := \frac{\nu p^2 u}{(2 - \alpha)^2 r D} \quad \text{and} \quad \tau := r$$

Here, $\alpha = 2$ is excluded and, furthermore, the abbreviation $\nu = 1 + \beta + \gamma + (2 - \alpha)(2 - \gamma)/3$ is introduced. The new variable $y$ describes, apart from a constant, the ratio of the acceleration and the convection time scale, which will be a fundamental quantity for later applications. Since the exponents $\alpha$, $\beta$ and $\gamma$ are arbitrary, the parameter $\nu$ and, therefore, the variable $y$ can be negative or positive. Besides this, the case $\nu = 0$ is also possible but requires another mathematical approach. This is not considered here. Thus, further considerations will depend on how to handle $\alpha$, $\beta$ and $\gamma$. Restricting the calculations to the case $\nu \neq 0$, one can take into account the different signs of $\nu$ by introducing a parameter $\epsilon = \pm 1$, i.e. $\nu = \epsilon |\nu|$. The transformed version of equation (7.1) can then be written as

$$y \frac{\partial^2 f}{\partial y^2} + [b + \epsilon y] \frac{\partial f}{\partial y} - \frac{\tau}{\nu} \frac{\partial f}{\partial \tau} = -\frac{\tau}{\nu u(\tau)} S(\tau, p[y, \tau])$$

(7.3)

where $\nu$ now has to satisfy the condition $\nu > 0$. Furthermore, the abbreviation $b = 3/(2 - \alpha)$ is introduced. The advantage of having isolated the sign of $\nu$ into the parameter $\epsilon = \pm 1$ is obvious: first, the variable $y$ reflecting the ratio of the acceleration and the convection time scale, is a positive definite quantity, as it should be for time scales. Second, the solution of the transport equation (7.1) will be more general, i.e. one has to apply the solution to the specific case of the PUI transport by choosing a special set of exponents $\alpha$, $\beta$ and $\gamma$ being typical for the heliosphere. Based on standard methods, the general solution related to the partial differential equation (7.3) is given as an integral over Green’s function $G(y, y_0, \tau, \tau_0)$ folded with the source function:

$$f(y, \tau) = \int dy_0 \int d\tau_0 G(y, y_0, \tau, \tau_0) \left[ \frac{\tau S(\tau, p[y, \tau])}{\nu u(\tau)} \right]$$

(7.4)

where the Green’s function obeys the differential equation

$$y \frac{\partial^2 G}{\partial y^2} + [b + \epsilon y] \frac{\partial G}{\partial y} - \frac{\tau}{\nu} \frac{\partial G}{\partial \tau} = -\delta(y - y_0) \delta(\tau - \tau_0)$$

(7.5)

The latter equation corresponds to equation (7.3), for which the source function on the right-hand side is represented by two Dirac’s delta distributions, describing a peak-like particle injection at the coordinates $y = y_0$ and $\tau = \tau_0$. Obviously, the solution (7.4) is characterized by the Green’s function satisfying the differential equation (7.5). Therefore, a further treatment requires a solution of (7.5). Detailed calculations concerning the derivation of the Green’s function as a solution of equation (7.5) are given in appendix B. There, the more general differential equation (B.1) is considered and
its solution, equation (B.20), is derived. The solution of the transformed PUI transport equation (7.3) can be simply obtained by comparing it with the general differential equation (B.1). The PUI phase space distribution function can then be derived as a special case of the general solution (B.20).

In order to derive the solution of (7.3) one has to compare (7.5) with the generalized differential equation (B.1). This results in \( \alpha_1 = b \), \( \alpha_2 = 1/\nu \), \( \xi(\tau) = \tau \) and \( Q(y, \tau) = \tau S(\tau, p[\tau, y])/(\nu u(\tau)) \). Following the considerations in appendix B, one obtains, by using equation (B.5), for the auxiliary variable \( t \) the expression \( t = \nu \ln(\tau/\tau_0) \). A simple transformation of the integral (7.4) to the initial variable set \( r_0 \) and \( v_0 \) results in the solution of the steady PUI transport equation (7.1). According to the general solution (B.20), the PUI phase space distribution function is given by

\[
f(r, p) = \frac{3}{|b|} \int_{r_-}^{r_+} \int_{p_-}^{p_+} \frac{\gamma_0 S(r_0, p_0)}{p_0 u(r_0)} \exp \left( -\frac{\gamma_0}{2\zeta} h_1 \right) I_{b-1} \left( \frac{2\gamma_0}{\zeta} h_1 h_2 \right)
\]

where \( I_{b-1} \) denotes the modified Bessel function of the first kind with the index \( b - 1 \). The auxiliary functions \( h_1 \) and \( h_2 \) are given by the relations

\[
h_1 = \left( \frac{v}{v_0} \right)^{2(\alpha-\gamma)/2} \left( \frac{r_0}{r} \right)^{(1+\beta+\gamma)/2} \quad \text{and} \quad h_2 = \left( \frac{r_0}{r} \right)^{\nu/2}
\]

while \( h_3 \) and \( \zeta \) depend on \( h_1 \) and \( h_2 \) via the equations (B.23) and (B.24), respectively. Equation (7.6), which is expressed in terms of an arbitrary source function, is the general solution of the differential equation (7.3), where the bulk speed \( u(r) \) and the coefficient \( D(r, p) \) evolve as arbitrary power laws in the heliocentric distance and particle momentum.

As described above, this solution is characterized by \( \epsilon \). In order to distinguish between \( \epsilon = +1 \) and \( \epsilon = -1 \), the exponents \( \alpha, \beta \) and \( \gamma \), being typical for a problem of interest, have to be specified. Suitable values of these exponents will further simplify the solution (7.6). In order to adapt it to the case of the PUI transport in the heliosphere, the momentum diffusion coefficient \( D(r, p) \) has to be determined. Then, the exponents \( \alpha \) and \( \beta \) can be specified. Assuming, for instance, that the turbulence consists of (anti-)parallel propagating Alfvén waves with which particles of the PUI population interact, one can use equation (6.56)

\[
D(r, p) = D_0 \left( \frac{p}{p_{A,0}} \right)^{q-1} \left( \frac{r}{r_E} \right)^{a(q-2)+2-\gamma-\eta}
\]

This yields, after comparing it with the ansatz for the momentum diffusion coefficient used to derive (7.3), the conditions \( \alpha = q - 1, \beta = a(q - 2) + 2 - \gamma - \eta, p_E = p_{A,0} \) and

\[
D_0 = \Omega_0^{2-q} v_{A,0}^{1+q} \frac{\pi(q - 1)}{q(q + 2)} \left( \frac{1}{2\pi} \right)^{1-q} \left( \frac{Z}{A} \right)^{2-q} \left( \frac{\delta B_0}{B_0} \right)^{2}
\]

Restricting further considerations to a purely azimuthal background magnetic field, i.e. \( a = 1 \), one readily obtains

\[
\nu = |1 + q - \eta + (3 - q)(2 - \gamma)/3|
\]

Since \( \nu \) distinguishes between \( \epsilon = +1 \) and \( \epsilon = -1 \) we have to consider values for the plasma wave power spectrum index \( q \) and the exponent \( \eta \) being typical for the heliosphere. Regarding \( q \), an often used value is \( 5/3 \) corresponding to the Kolmogorov spectral index of solar wind magnetic fluctuations in the inertial range (see, e.g., Marsch and Tu, 1990; Chalov and Fahr, 1996). Considering \( \eta \), describing the spatial evolution of the turbulence intensity, two values are often used in the literature: first, \( \eta = 3 \), which can be derived by the WKB theory of the evolution of non-dissipative Alfvén waves in a constant solar wind (see Hollweg, 1974) and, second, \( \eta = 8/3 \), which was assumed
by Isenberg [1987]. An uncertain parameter is \( \gamma \) determining the simple spatial evolution of the solar wind speed. Assuming, for example, a solar wind which is decelerated at \( r = 80 \) AU by 20 percent, one estimates \( \gamma \simeq 0.08 \). In view of these values, \( \nu \) can be estimated resulting in the condition \( \epsilon = +1 \). Substitution of \( \epsilon = +1 \) in equation (7.6) finally yields

\[
 f(r, p) = \frac{3}{|b|} \int dr_0 \int dp_0 \frac{y_0 S(r_0, p_0)}{p_0 u(r_0) \zeta} \left( \frac{p_0}{p} \right)^{q/2} \left( \frac{r_0}{r} \right)^{(3+\beta+\nu)/2} 
\]

\[ \times \exp \left[ -\frac{y_0}{\zeta} (h_1^2 + h_2^2) \right] I_{\frac{1}{2}} \left( \frac{2y_0}{\zeta} h_1 h_2 \right) \]  

(7.10)

Equation (7.10) describes the phase space evolution of PUIs for an arbitrary source function with which the freshly ionized PUIs are injected into the heliosphere at the position \( r_0 \) with the momentum \( p_0 \). In generalization of the solution derived by Isenberg [1987], which is just a special case of the PUI phase space function (7.10) by introducing strong restrictions with regard to the quantities \( q, \eta \) and \( u(r) \) as well as to the source function, the solution (7.10) has the advantage to be valid for arbitrary source functions \( S(r_0, p_0) \) and is not restricted to a special choice of the spatial evolution of the underlying wave power. Furthermore, the new solution permits to consider a non-constant solar wind speed. Since PUIs exhibit a non-relativistic nature, particle speeds will be considered in the following instead of momenta.

### 7.1.1 The PUI Source Function

Since equation (7.10) is still formulated in terms of a source function, the injection rate, describing the local injection of a PUI species into the expanding solar wind, has to be specified. For the further treatment, the standard source function

\[
 S(r_0, v_0) = \frac{Q(r_0)}{4\pi u^2(r_0)} \delta(v_0 - u(r_0)) 
\]

(7.11)

will be used. Here, for simplicity, the particles are injected locally into the solar wind plasma with the solar wind speed \( u(r_0) \). In general, the speed of a freshly produced PUI species will depend on the peculiar velocity of the parent atoms leading to a more complicated speed dependence of the source function. Neglecting this intrinsic peculiar motion, one can assume that the PUI speeds at the event of their production coincide with the local solar wind speed. This has been used in most of the afore-mentioned studies performed before.

The production rate of PUIs due to ionization is given by \( Q = n \nu_i \), where \( n \) is the number density of interstellar neutral atoms of different chemical elements entering the heliosphere from the local interstellar medium. According to the cold neutral gas model by Thomas [1978], we use for the number density of interstellar neutral atoms \( n(r_0) = n^\infty \exp(-\lambda/r_0) \), where \( n^\infty \) represents the interstellar number density of neutral atoms at large heliocentric distances. The scale length \( \lambda \) characterizes the ionization cavity surrounding the Sun. The process of ionization of the neutral species is modeled with the ionization frequency \( \nu_i \), which will, in general, consist of various processes. Based on sensitive in-situ measurements of interstellar gas and its ionization products within the heliosphere (see Ruciński et al., 1996), it is possible to estimate the relative effectiveness of these processes. Since charge exchange of the neutral species with solar wind protons and photoionization are the most significant reaction channels for the ionization, we consider the total ionization frequency \( \nu_i \) as a superposition of the charge exchange frequency \( \nu_{ce} \) and the photoionization frequency \( \nu_{ph} \), i.e. \( \nu_i(r_0) = \nu_{ph}(r_0) + \nu_{ce}(r_0) \). Concerning the radial dependences of \( \nu_{ph} \) and \( \nu_{ce} \), it is mostly assumed in the literature, that the ionization profiles evolve according to the same power law behavior, i.e. \( \nu_{ph}(r_0) = \nu_{ph,0}(r_E/r_0)^2 \) and \( \nu_{ce}(r_0) = \nu_{ce,0}(r_E/r_0)^2 \), in a solar wind having
a constant speed. Since $v_{ph,0}$ and $v_{ce,0}$, which are usually related to the Earth’s orbit $r_E$, will be different for various neutral elements, we have to distinguish between the two most important elements:

**Pick-Up Hydrogen**

Considering first the case of photoionization being the minor contribution to the ionization of neutral hydrogen (see table 3 in Ruciński et al., 1996), a number of ionization profiles have been published in the last three decades. Here, we use $v_{ph,0}^H = 1.1 \cdot 10^{-7}$ s$^{-1}$, which agrees with $v_{ph,0}^H = (8.8 \pm 3.3) \cdot 10^{-8}$ s$^{-1}$ taken from Siscoe and Mukherjee [1972] and with the values given by Ruciński et al. [1996].

Regarding the charge exchange frequency, the considerations will involve solar wind protons with number density $n_p$. Considering the flow and thermal speed of the neutral hydrogen as well as the thermal speed of the solar wind to be much lower than the solar wind speed $u$, the ionization frequency is $\nu^H = \sigma^H n_p u$. The cross section for charge exchange is given by $\sigma^H = [7.5 - 2.1 \log(u \text{ km s}^{-1})] \cdot 10^{-15}$ cm$^2$ (Fite et al., 1962; Isenberg, 1986).

Since the solar wind speed scales with distance as $r_0^{-\gamma}$, the thermal proton density can be formulated, according to the continuity equation, by the expression $n_p(r) = n_0(r_E/r_0)^{2-\gamma}$ leading to the above mentioned $r_0^{-2}$ behavior. Substitution of $u(r_0) = u_0(r_E/r_0)^{\gamma}$ yields $\nu_{ce,0}^H = [7.5 - 2.1 \log u(r_0)] n_0 u_0 \cdot 10^{-15}$ cm$^2$ s$^{-1}$ revealing, for a non-constant solar wind speed, a weak dependence in $r_0$. Inserting, for instance, $n_0^H = 8$ cm$^{-3}$ and a constant solar wind speed with $u_0 = 400$ km s$^{-1}$, one would evaluate $\nu_{ce,0}^H \approx 6.5 \cdot 10^{-7}$ s$^{-1}$ agreeing with Ruciński et al. [1996].

**Pick-Up Helium**

Considering, again, first the case of photoionization, which is the dominant ionization process regarding neutral helium, we use $v_{ph,0}^{He} = 1.1 \cdot 10^{-7}$ s$^{-1}$. This agrees with $v_{ph,0}^{He} = (8.0 \pm 4.0) \cdot 10^{-8}$ s$^{-1}$ (Siscoe and Mukherjee, 1972) and the time-averaged value in table 1a by Ruciński et al. [1996].

The case of charge exchange of neutral helium with solar wind protons is not considered here in such detail as in the case for neutral hydrogen. Following Ruciński et al. [1996], the time-averaged charge exchange frequency $\nu_{ce,0}^{He}$ is about $2.5 \cdot 10^{-10}$ s$^{-1}$, which is much less than that of photoionization.

### 7.1.2 Asymptotic Shapes for Large Radii

Since energized PUls, which have been subjected to stochastic acceleration on their way to the solar wind termination shock in the outer heliosphere, are important for a direct conversion to ACR by diffusive shock acceleration, it might be interesting to obtain asymptotic expressions of the solution (7.10) for large heliocentric distances where the heliospheric shock is located. The justification of restricting the following considerations to large heliocentric distances is given in the next section, where numerical solutions of equation (7.10) are presented. There, it is demonstrated that PUI velocity distributions reach asymptotic conditions within several astronomical units.

In order to consider the asymptotic behavior, the following calculations are separated into two different approaches, namely into the low and the high energy regime:

### Low Particle Energies

Turning first the attention to low particle energies, one can make use of an asymptotic form representing the modified Bessel function in equation (7.10) for a small argument. The argument of the Bessel function will be small if the function $h_1$ is small for low particle energies. This requires the condition $q < 3$. Furthermore, restricting the asymptotic approach to large $r$, which leads to a
vanishing function $h_2$, so that $\zeta \approx 1$, one readily obtains

$$f \approx \frac{3}{b \Gamma(b)} \int dr_0 \int dv_0 S(r_0, v_0) \frac{y_0^b}{v_0 u(r_0)} \left( \frac{r_0}{r} \right)^{\nu b}$$

(7.12)

The integral can be solved analytically by using the source function (7.11). Since the ionization profiles scale with distance as $r_0^{-2}$, apart from the additional weak dependence in the case of charge exchange concerning neutral hydrogen, the details of $\nu$ for each chemical element and ionization channels are ignored. This enables one to use the ansatz $\nu(r_0) = \nu_0 (r_E/r_0)^s$, where $s$ is an arbitrary number. The case $s = 2$ corresponds, with $\nu_0 = \nu_{ph,0}$ or $\nu_0 = \nu_{ph,0}$, to photoionization of hydrogen or helium, respectively, and to charge exchange, with $\nu_0 = \nu_{ce,0}$, in the case of hydrogen in a constant solar wind. Substitution of $y_0 = y(r_0, v_0)$ and the injection rate (7.11) yields

$$f \approx \frac{3 n^\infty \nu_0 r_e^{s-2}}{4 \pi b \Gamma(b) u_0 v_A^b} \left[ \frac{\nu v_A^2 u_0}{(3-q)^2 r_E D_0} \right]^b \left( \frac{r_E}{r} \right)^{\nu b} \int_0^r r_0^{2-s} \exp(-\lambda/r_0)$$

(7.13)

Evaluation of the integral results in the following two solutions:

$$f \approx \frac{3 n^\infty \nu_0 r_E^{s-2}}{4 \pi b \Gamma(b) u_0 v_A^b} \left[ \frac{\nu v_A^2 u_0}{(3-q)^2 r_E D_0} \right]^b \left( \frac{r_E}{r} \right)^{\nu b} \left\{ \begin{array}{l}
\left( \frac{\lambda}{r_E} \right)^{3-s} \Gamma(s-3, \frac{\lambda}{r}) \quad \text{for } \lambda \neq 0 \\
\frac{1}{(3-s)} \left( \frac{r_E}{r} \right)^{s-3} \quad \text{for } \lambda = 0
\end{array} \right.$$  

(7.14)

where $\Gamma(a, x)$ denotes the incomplete Gamma function. Obviously, the asymptotic shape of low energetic PUIs at large heliocentric distances depends on the parameter $b = 3/(3 - q)$ as well as $\nu$, given by equation (7.9), and, therefore, on the behavior of the underlying turbulence, it means on the spectral index $q$ and the exponent $\eta$ describing the radial evolution of the plasma wave power. Furthermore, it depends on the ionization profile characterized by the quantity $s$. In the case of a non-vanishing ionization scale length $\lambda$, the asymptotic shape is a decreasing function with increasing $\lambda$. Since the PUI distributions reach asymptotic conditions within several astronomical units, it should be possible to draw useful conclusions concerning $\lambda$ if one would fit the asymptotic shape (7.14) with regard to observations of low energetic PUIs at intermediate heliocentric distances. In the case $\lambda = 0$, which was also considered by Isenberg [1987], the spatial dependence can be summarized to yield $f \propto r^{-7+q}/(3-q)/(3-q)$. Using the same set of parameters as Isenberg [1987], i.e. $q = 5/3$, $\eta = 8/3$, $s = 2$ and $\gamma = 0$, one immediately obtains $f \propto r^{-1}$. This agrees indeed with the asymptotic result derived by Isenberg [1987].

**High Particle Energies**

Considering the second case, i.e. the high energy regime, one can approximate equation (7.10) similarly for high particle speeds by using an asymptotic expression representing the modified Bessel function for large arguments. Requiring $q < 3$, one obtains for the PUI velocity distribution at large heliocentric distances the expression

$$f \approx \frac{3}{2 \sqrt{\pi b}} \exp[-y(r, v)] \int dr_0 \int dv_0 S(r_0, v_0) \frac{y_0^b}{v_0 u(r_0)} \left( \frac{v_0}{v} \right)^{(3+q)/4} \left( \frac{r_0}{r} \right)^{(1+q-\eta)/2+(9-q)(2-\gamma)/12}$$

(7.15)

or, after having inserted $y_0$ and the source function (7.11), the asymptotic integral

$$f \approx \frac{n^\infty \nu_0 r_E^{s-2}}{8 \pi \beta v_A u_0} \left( \frac{v_A}{u_0} \right)^{(q-1)/2} \left[ \nu \frac{v_A^2 u_0}{r_E D_0} \right]^{1/2} \exp[-y(r, v)] \left( \frac{u_0}{v} \right)^{(3+q)/4} \left( \frac{r_E}{r} \right)^{\delta_1} \int_0^r r_0^{\delta_2} \exp(-\lambda/r_0)$$

(7.16)
where the abbreviations \( \delta_1 = (1 + q - \eta)/2 + (9 - q)(2 - \gamma)/12 \) and \( \delta_2 = (9 - q)/6 + \gamma(3 + q)/3 - s \) were introduced. Carrying out the integration leads to the following asymptotic shapes for the distribution of energized PUIs at large heliocentric distances:

\[
f \simeq \frac{v^\infty v_0 v_E}{8\pi^{3/2}v_A u_0^3} \left( \frac{v_A}{u_0} \right)^{(q-1)/2} \left[ \frac{v^2 u_0}{v_E D_0} \right]^{1/2} \exp[-y(r, v)] \left( \frac{u_0}{v} \right)^{(3+q)/4} \left( \frac{r_E}{r} \right)^{\delta_1} \times \begin{cases} \\
\left( \frac{\lambda}{r_E} \right)^{1+\delta_2} \Gamma \left( -1+\delta_2, \frac{\lambda}{r} \right) & \text{for } \lambda \neq 0 \\
\frac{1}{(1+\delta_2)} \left( \frac{r_E}{r} \right)^{-(1+\delta_2)} & \text{for } \lambda = 0
\end{cases}
\] (7.17)

As in the case of the low energy approach, both asymptotic distributions depend again strongly on the behavior of the plasma wave turbulence. In contrast to the asymptotic shapes (7.14) reflecting PUI distributions at small energies where adiabatic deceleration is the dominant process, the asymptotic expressions (7.17) for higher particle energies are sensitively characterized by the variable \( y(r, v) \) containing the ratio of the acceleration and the convection time scale. The case of a non-vanishing \( \lambda \) gives, as expected, again an asymptotic distribution decreasing with increasing ionization scale length \( \lambda \). In order to compare eq. (7.17) with previous solutions, the case \( \lambda = 0 \) can be considered in more detail. The important contribution in terms of \( r \) and \( v \) is then

\[
f \propto r^{1-s-(1+q-\eta)/2+\gamma(7+q)/4} \nu^{-(3+q)/4} \exp \left[ -\nu \frac{v_A^2 u_0}{v} \left( \frac{v}{v_A} \right)^{3-q} \left( \frac{r_E}{r} \right)^{1+q-\eta} \right]
\] (7.18)

The expression (7.18) can nicely be compared with equation (24) by Isenberg [1987]. Restricting equation (7.18) to \( \eta = 8/3 \), \( s = 2 \) and \( \gamma = 0 \), one obtains for the important dependences \( f \propto r^{-1+3q/6} \nu^{-(3+q)/4} \exp(\sim \nu^{3-q} r^{(5-3q)/3}) \), which still contains the spectral index being in the range \( 1 < q < 2 \). Remember that the calculations carried out by Isenberg [1987] are valid only for \( q = 5/3 \). So, in order to enable a comparison, one has to choose \( q = 5/3 \). This leads to the same radial asymptotic behavior, i.e. \( f \propto r^{-1} \), and to the same exponential expression exhibiting no spatial dependence. However, in contrast to this, the energy dependence in front of the exponential expression is quite different. In the next section, the influence of a varying spectral index \( q \) and a varying spatial behavior of the plasma wave power by using computational results will be presented for the first time. The considerations made in this section and, furthermore, the following numerical calculations show that Isenberg’s [1987] solution is strongly restricted and can be classified as a special case of the solution (7.10).

### 7.2 Numerical Calculations

In order to get more physical insight into the isotropic PUI transport related to the solution (7.10) and its associated asymptotic shapes and, furthermore, to demonstrate the potential provided with the general solution (B.20), equation (7.10) is solved numerically for pick-up hydrogen by using the diffusion coefficient (7.8) and the source function (7.11).

For the numerical computation, the diffusion coefficient (7.8) has to be evaluated for physical parameters being typical for the heliosphere. For the purely azimuthal background magnetic field \( B_0 \) and the Alfvén speed \( v_{A,0} \) typical values such as \( 4 \cdot 10^{-5} \) Gauss and 50 km s\(^{-1} \), respectively, are chosen. Furthermore, for the correlation length \( l_c \) of the plasma wave turbulence a typical value of \( 3.2 \cdot 10^5 \) km (Chalov and Fahr, 1996) is used. The wave power spectral index \( q \) and the ratio \( (\delta B_0/B_0)^2 \) are assumed to be, until noted otherwise, 5/3 and 0.01, respectively.
In order to evaluate the source function (7.11), it is assumed that \( u_0 = 400 \text{ km s}^{-1} \). Concerning the ionization profiles of section 7.1.1, interstellar neutral hydrogen is injected into the heliosphere with a number density of \( n_H^\infty = 0.115 \text{ cm}^{-3} \) (Gloeckler et al., 1996). Furthermore, for the solar wind proton number density at Earth’s orbit \( n_0 = 8 \text{ cm}^{-3} \) is assumed. Concerning the ionization scale length \( \lambda \), an often used value for hydrogen is \( \lambda_H \approx 6 \text{ AU} \). This value results from the cold gas model by Thomas [1978]. It is based on the assumption that the solar gravitational force acting on the interstellar neutral atoms is balanced by the solar radiation pressure. In order to illustrate the influence of \( \lambda \) on the number density of the interstellar neutral hydrogen, figure 7.1(a) shows the number densities of neutral hydrogen as functions of the heliocentric distance \( r \). For completeness, figure 7.1(b) illustrates the helium density as a function of the heliocentric distance \( r \). The solid lines correspond to number densities given by Ruciński et al. [1993]. In order to estimate \( \lambda \) for hydrogen and helium, the number density \( n(r) = n^\infty \exp(-\lambda/r) \) is given as a function of \( r \) for different ionization scale lengths. Considering first the case of hydrogen, figure 7.1(a), the dotted line represents \( \lambda_H = 6 \text{ AU} \), according to the cold gas model by Thomas [1978]. The dashed line shows the case \( \lambda_H = 3 \text{ AU} \). Regarding helium, figure 7.1(b), the dotted line corresponds to \( \lambda_{He} = 0.06 \text{ AU} \) whereas the case \( \lambda_{He} = 0.03 \text{ AU} \) is represented by the dashed curve. Obviously, \( \lambda_{He} \) is much smaller than \( \lambda_H \), due to the higher ionization potential of helium. Until noted otherwise, \( \lambda_H = 6 \text{ AU} \) is used.

**Figure 7.1:** The number densities of hydrogen (left panel) and helium (right panel) as functions of the heliocentric distance \( r \). The solid lines represent densities as calculated by Ruciński et al. [1993]. The dotted and the dashed curves illustrate hydrogen and helium number densities for \( \lambda_H = 6 \) and 3 AU as well as \( \lambda_{He} = 0.06 \) and 0.03 AU, respectively.
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(a) Radial evolution for $\delta B_0/B_0 = 0.01$.

(b) Radial evolution for $\delta B_0/B_0 = 0.05$.

(c) Radial evolution for $\delta B_0/B_0 = 0.10$.

(d) Radial evolution for $\delta B_0/B_0 = 0.50$.

Figure 7.2: Numerical solution of equation (7.10): the radial evolution of the velocity distribution of pick-up hydrogen injected into a solar wind with constant speed for different heliocentric distances and different values of the turbulence level $\delta B_0/B_0$. The phase space distribution $f(v/u_0)$ is scaled with $r/r_E$. The particle speed $v$ is measured in the solar wind rest frame. For all four panels, the plasma parameters are used as given in panel (a).
7.2 Numerical Calculations

Figure 7.2 shows numerical results for the pick-up hydrogen velocity distribution function (7.10) for four different turbulence levels $\delta B_0/B_0 = 0.01, 0.05, 0.1$ and 0.5. They are considered separately in the panels 7.2(a) through 7.2(d). Each panel gives the radial evolution for the heliocentric distances $r = 1.001, 1.05, 1.2, 1.5, 2, 4, 8$ and 100 AU (from bottom to top). Close to the injection distance $r_0$ and independent of the magnitude of the turbulence level, the distributions have only spread slightly from the thin shell of pick-up hydrogen injected at the inner boundary; the distributions still appear as spikes around $v/u_0 \simeq 1$. The steep change in the slope around $v/u_0 = 1$ results from a continuous injection of particles counteracting the energy diffusion at this speed.

All four panels show that some particles fill in the region at lower energies, i.e. $v/u_0 < 1$, with increasing heliocentric distance, due to adiabatic deceleration. Within several AU, the distributions approach their asymptotic shapes, resulting from the competition of adiabatic cooling, which tends to transport the particles to higher energies, i.e. $v/u_0 > 1$. In the case of a small turbulence level, i.e. $\delta B_0/B_0 = 0.01$ considered in panel 7.2(a), adiabatic cooling is the dominant process and overwhelms the stochastic acceleration at all heliocentric distances. Therefore, the particles pile up at low energies and no pick-up hydrogen has been energized. This can nicely be seen by considering the high-energy regime of figure 7.2(a): no tail occurs. It reflects the limit of a vanishing particle momentum diffusion, which has been studied by Vasyliunas and Siscoe [1976]. With an increasing turbulence level $\delta B_0/B_0$, panels 7.2(b) through 7.2(d), the plasma wave power leads to a larger fluctuation energy pool. This supports the particles with turbulent energy required for the acceleration process. An increased wave power, as it is represented in figure 7.2(b) for the turbulence level $\delta B_0/B_0 = 0.05$, leads to a higher turbulent energy pool providing the same number of particles with much more acceleration energy. In view of this, an increased wave power flattens the distribution in the low energy regime, so that the phase space distributions approach, with increasing wave power, a plateau-like shape. This distribution evolution in turbulence energy can be seen in panels 7.2(c) and 7.2(d) for the low-energy range $v < u_0$. Furthermore, the increased fluctuation energy accelerates substantially more particles to speeds several times the solar wind speed. With increasing turbulence level, this leads to a more distinct energetic tail in the high energy regime.

The influence of a changing ionization scale length $\lambda = \lambda_H$, i.e. $\lambda_H = 0, 3, 6$ and 9 AU, on pick-up hydrogen velocity distributions is shown in figure 7.3. Here, the panels (a) and (b) illustrate the evolution in $\lambda$ at the heliocentric distances $r = 10$ AU and $r = 100$ AU, respectively, for $\delta B_0/B_0 = 0.05$. The panels (c) and (d) display distributions for different ionization scale lengths at $r = 10$ AU and $r = 100$ AU, respectively, but for the turbulence level $\delta B_0/B_0 = 0.5$. As expected, according to the asymptotic shapes (7.14) and (7.17), the number densities decrease for both turbulence levels and heliocentric distances with increasing $\lambda$, resulting from the less efficient production rate of pick-up hydrogen. Farther out, at $r = 100$ AU, the influence of $\lambda$ becomes smaller and, therefore, does not substantially change the asymptotic shape of the distribution, independent of what the magnitude of the turbulence level is (compare panel (b) with (d)).

Figure 7.4 illustrates the effect of changing the parameter $\eta$, representing the spatial evolution of the plasma wave intensity, for $r = 10$ AU and $r = 100$ AU as well as the turbulent levels $\delta B_0/B_0 = 0.05$ (upper panels) and $\delta B_0/B_0 = 0.1$ (lower panels). This was not done by Isernberg [1987]. His derivation required the specific case $\eta = 8/3$. Panels 7.4(a) through 7.4(d) reflect a significant, not negligible, effect of the parameter $\eta$ on pick-up hydrogen distribution functions for both turbulence levels. For a given heliocentric distance, the panels show that the tails of the distributions increase with decreasing $\eta$. In contrast to this, cooled particles pile-up at low energies, due to a longer acceleration time scale with increasing $\eta$. Increasing simultaneously the turbulence power (compare panel (a) with (c) and panel (b) with (d)), the distributions exhibit substantially extended tails, reaching up to higher numbers times the solar wind speed.
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Figure 7.3: Numerical results: pick-up hydrogen velocity distributions for different ionization scale lengths (given in the legends), two different heliocentric distances and turbulence levels. Upper panels (a) and (b): distributions at $r = 10$ AU and $r = 100$ AU for $\delta B_0/B_0 = 0.05$, respectively. Lower panels (c) and (d): distributions at $r = 10$ AU and $r = 100$ AU for $\delta B_0/B_0 = 0.5$, respectively. The plasma parameters used are listed in panel (a).
7.2 Numerical Calculations

(a) \( \delta B_0 / B_0 = 0.05 \) at \( r = 10 \) AU

(b) \( \delta B_0 / B_0 = 0.05 \) at \( r = 100 \) AU

(c) \( \delta B_0 / B_0 = 0.1 \) at \( r = 10 \) AU

(d) \( \delta B_0 / B_0 = 0.1 \) at \( r = 100 \) AU

Figure 7.4: Numerical results: solutions of equation (7.10) representing velocity distributions for a changing spatial evolution of the turbulence intensity characterized by the power \( \eta \). Upper panels (a) and (b): distributions at \( r = 10 \) and \( r = 100 \) AU for \( \delta B_0 / B_0 = 0.05 \), respectively. Lower panels (c) and (d): distributions at \( r = 10 \) and \( r = 100 \) AU for \( \delta B_0 / B_0 = 0.1 \), respectively. The ionization scale length \( \lambda \) is assumed to be 6 AU. The used plasma parameters are listed in panel (a).
Figure 7.5 presents numerical results for $r = 100$ AU with regard to the interplay of three parameters and their influence on pick-up hydrogen phase space distributions. First, a varying $\gamma$ is considered in each panel of figure 7.5 for two distinguished turbulence levels. The parameter $\gamma$ describes the simple radial power law behavior of the solar wind via the relation $u \sim r^{-\gamma}$. In order to take into account this spatial dependence, three different values of $\gamma$ are considered for each turbulence level. $\gamma = 0$ is used for a constant solar wind speed, while $\gamma = 0.04$ and $\gamma = 0.08$ represent a solar wind decelerated by 15 and 30 percent, respectively, at $r = 100$ AU. In order to illustrate the complex interrelation of $\delta B_0/B_0$ and the parameter $\eta$, which also determines, besides $\delta B_0/B_0$, the magnitude of the momentum diffusion coefficient (7.8), the cases $\eta = 3.0$ and $\eta = 2.6$ are presented separately in the left panels (a) and (c) as well as in the right panels (b) and (d), respectively. The influence of a varying turbulence power amplitude or, simultaneously, a changing $\eta$ was already investigated in figures 7.2 and 7.4: an increasing turbulence intensity $\delta B_0^2$ leads to a higher acceleration rate resulting in more distinct tails of energized particles and substantial lower particle numbers at low energies, where adiabatic cooling takes place and dominates the stochastic acceleration. For both values of $\eta$, the stronger the deceleration of the solar wind the narrower is the plateau-like region of the distribution functions in the low-energy regime $v < u$. This can be recognized in panels (a)-(d) for each turbulence level. This means that the energetic tails are also shifted to lower speeds. Turning the attention first to the upper panels (a) and (b), representing the turbulence levels $\delta B_0/B_0 = 0.5$ and $\delta B_0/B_0 = 0.05$, and considering the change in $\eta$ in more detail, the shift to lower velocities cannot compensate for the acceleration process if the turbulence level is sufficient high enough for both values of $\eta$ (see the curves for the case $\delta B_0/B_0 = 0.5$ in the panels (a) and (b)). However, for a lower turbulence power, the velocity shift changes the ordering of the curves for both values of $\eta$. This is shown in the panels (a) and (b) of figure 7.5 for the case of $\delta B_0/B_0 = 0.05$. The lower panels (c) and (d) of figure 7.5 show, again, the effect of a changing $\gamma$ on a distribution function for the case $\delta B_0/B_0 = 0.5$, but in contrast to the upper panels the second turbulence level is here given by $\delta B_0/B_0 = 0.1$. As it is shown in panel (c), the ordering of the curves is changed for $\delta B_0/B_0 = 0.1$ and $\eta = 3.0$. This feature also arises in the case of $\delta B_0/B_0 = 0.05$ given in panel (a). However, keeping the intermediate turbulence level $\delta B_0/B_0 = 0.1$ and decreasing $\eta$ leads to what is illustrated in the panel (d). The change of the curve ordering is removed for $\delta B_0/B_0 = 0.1$ and $\eta = 2.6$. This is not the case for $\delta B_0/B_0 = 0.05$ (compare panel (a) with (b) and then (c) with (d)). This can be explained by considering the momentum diffusion coefficient (7.8): an decreasing $\eta$ leads, for a constant $r$ and a constant ratio $(\delta B_0/B_0)^2$, to an increasing $D(r,v)$, so that the shift to lower velocities cannot be compensated again by the acceleration effect, even for moderate turbulence levels such as $\delta B_0/B_0 = 0.1$.

The panels of figure 7.6 show pick-up hydrogen distribution functions for different values of the wave power spectral index $q$, varying only slightly from the often observed Kolmogorov spectral index $q = 5/3$. Each individual panel illustrates phase space distributions calculated for $q = 1.4, 1.6$ and 1.8 as well as two different heliocentric distances, $r = 2$ and 100 AU. Furthermore, calculations for $\eta = 3.0$ and $\eta = 2.6$ are shown separately in the left and right panels, respectively. Moreover, the panels (a) and (b) reflect the moderate turbulence level $\delta B_0/B_0 = 0.1$, while (c) and (d) illustrate calculations for a low turbulence level, i.e. $\delta B_0/B_0 = 0.01$. The latter case is presented for illustrative purposes. The investigation of the $q$ dependence was not done by Isenberg [1987]. His calculations are restricted to the special case $q = 5/3$. Considering first panels (a) and (b), figure 7.6 shows, for both $\eta$ values, the following scenario: at small distances, the tails of the distributions become steeper with increasing spectral index $q$. Since the turbulence power scales with $\delta B_0^2 k^{-q}$ the fluctuation energy decreases, for a given total amount of wave power $\delta B_0^2$, with increasing $q$. This leads to a smaller fluctuation energy pool supporting the particles with turbulent energy required for the stochastic acceleration. At larger distances, the ordering of the curves is reversed for both $\eta$ values; it means the tails become steeper with decreasing spectral index $q$. 
7.2 Numerical Calculations

(a) Distributions for $\eta = 3.0$

(b) Distributions for $\eta = 2.6$

(c) Distributions for $\eta = 3.0$

(d) Distributions for $\eta = 2.6$

Figure 7.5: Computational model results illustrating in each panel, first, the effect of two different turbulence levels $\delta B_0/B_0 = 0.05$ and $\delta B_0/B_0 = 0.5$ and, second, the influence of a weakly varying solar wind speed $u \sim r^{-\gamma}$ for different values of $\gamma$ at $r = 100$ AU. Panel (a): velocity distributions for $\eta = 3.0$. Panel (b): velocity distributions for $\eta = 2.6$. See figure 7.4(a) for the used plasma parameters. Panel (c) and (d): same as panel (a) and (b) but for $\delta B_0/B_0 = 0.1$ and $\delta B_0/B_0 = 0.5$. 


Chapter 7 Stochastic Acceleration of Pick-Up Ions

Figure 7.6: Numerical results showing phase space distribution functions at the heliocentric distances $r = 2$ AU and $r = 100$ AU for different wave power spectral indices $q$ (see the legends for $q$ values), turbulence levels and two different values of $\eta$. Panel (a) and (b): velocity distributions for the moderate turbulence level $\delta B_0/B_0 = 0.1$ and $\eta = 3.0$, panel (a), as well as $\eta = 2.6$, panel (b). Panel (c) and (d): same as panel (a) and (b), but for the turbulence level $\delta B_0/B_0 = 0.01$. 

(a) Distributions for $\delta B_0/B_0 = 0.1$ and $\eta = 3.0$.
(b) Distributions for $\delta B_0/B_0 = 0.1$ and $\eta = 2.6$.
(c) Distributions for $\delta B_0/B_0 = 0.01$ and $\eta = 3.0$.
(d) Distributions for $\delta B_0/B_0 = 0.01$ and $\eta = 2.6$. 
7.3 Summary and Conclusions

Ignoring the influence of \( \lambda_H \), the switch can be explained by the asymptotic expression (7.18). It contains, via the variable \( \gamma \), the ratio of the acceleration to the convection time scale. This ratio evolves in distance with \( r^{\eta - q - 1} \). The exponential expression in (7.18) then dominates all other contributions beyond a definite distance \( r_s \). This results in changing the \( q \) dependence for \( r > r_s \). The switching radius \( r_s \) can be estimated by setting the ratio of the acceleration and the convection time scale for two different spectral indices \( q_1 \) and \( q_2 \) to unity, yielding

\[
r = r_E \left[ \frac{q_1(q_1 + 2)(q_2 - 1)}{q_2(q_2 + 2)(q_1 - 1)} \right]^{1/(q_1 - q_2)} \frac{Z \Omega_0 l_c}{A u_0 2\pi v}
\]  

(7.19)

Using, for instance, \( q_1 = 1.4 \) and \( q_2 = 1.8 \) as well as the parameters as given in figure 7.6 leads to \( r_s \sim 20 \) AU. Checking the computational model by using this value would indeed lead to identical distributions. Furthermore, increasing the mass number \( A \) results in a decreasing \( r_s \), which is for pick-up helium at \( 5 \) AU. Turning the considerations now to the remaining panels (c) and (d) presenting distributions for \( \delta B_0 / B_0 = 0.01 \) and \( \eta = 3.0 \) and \( \eta = 2.6 \), respectively, no significant changes can be recognized for a decreased value of the exponent \( \eta \). This means, the turbulence level has to be sufficiently high so that variations of the spectral index \( q \) become more important for the isotropic transport of PUIs in the heliosphere.

7.3 Summary and Conclusions

In this chapter a first application of the fundamental diffusion-convection equation and its transport parameters is given. For steady-state conditions and spherical symmetry, the diffusion-convection equation is restricted to the limit of vanishing spatial diffusion and particle drifts so that diffusion in momentum space represents the dominating process. The simplified version of the diffusion-convection equation serves as a transport equation which enables one to consider Fermi II acceleration of charged particles in a turbulent plasma. Assuming a momentum diffusion coefficient and a background plasma bulk speed which vary as power laws in particle momentum and distance with arbitrary exponents, this transport equation is subjected to a mathematical transformation allowing a treatment from a more general point of view. It is shown that the transformed transport equation can be expressed as a specific case of a generalized differential equation. The latter equation is solved analytically by constructing the corresponding Green’s function and making use of the Laplace transformation. This is demonstrated in appendix B. By using the solution of this generalized differential equation, it is shown that the general solution of the transport equation can readily be obtained in a simple manner, namely by comparing it with the generalized differential equation. The solution of the generalized differential equation derived in appendix B and, therefore, the solution of the transport equation considered in this chapter are presented here for the first time. The solution of the transport equation should be beneficial for a variety of heliospheric as well as astrophysical problems, because it possesses the following advantages:

- First, the solution includes, via the integrand, an arbitrary source function. This allows to consider very complicated particle injection rates for the system of interest, because the corresponding integrals can easily be solved numerically by using the semi-analytical approach presented in this chapter.
- Second, presumed that the momentum diffusion coefficient is expressible as a power law in particle momentum and radius, different turbulence models may be taken into account.
- Third, since the bulk speed is assumed to vary as an arbitrary power law in distance, the new solution enables one to consider stochastic particle acceleration not only in an expanding background plasma flow, but also in a contracting streaming plasma, as it is the case in accretion flows.
The general solution is then applied to stochastic acceleration of PUIs in the heliosphere. In this context, it substantially generalizes the two “classical” solutions derived earlier by Vasyliunas and Siscoe [1976] and Isenberg [1987] for a vanishing and a non-vanishing momentum diffusion, respectively. It is assumed that the turbulence consists of parallel and antiparallel propagating Alfvén waves with the same intensity. Several asymptotic phase space integrals are derived, representing the PUI phase space distribution for large heliocentric distances and low and high particle energies. These integrals are solved analytically by using a standard PUI source function including ionization profiles of interstellar neutral atoms. Numerical calculations for PUI-hydrogen demonstrate the potential and flexibility of the new semi-analytical approach. It allows to investigate the influence of several turbulence parameters on the phase space distributions of PUIs. For the numerical calculations, parameters being typical for the heliosphere are used. The results illustrate the sensitive dependence of PUI distributions, in particular of the high-energy tail, on all parameters of the underlying turbulence.
Chapter 8

On Solar Modulation of Galactic and Anomalous Cosmic Rays

The fundamental diffusion-convection transport equation (3.24) was already considered in the previous chapter as a seed equation for the isotropic transport of PUIs and their stochastic acceleration within the heliospheric background plasma. This chapter introduces a second application of the diffusion-convection equation and its associated transport parameters described in chapter 6. In order to illustrate the complex interplay of plasma waves and particles and its effects on the transport of charged particles in a plasma wave turbulence, the isotropic transport of anomalous and galactic cosmic rays and their associated solar modulation are the focus here. For spherical symmetry and steady-state, the diffusion-convection equation is considered in the limit of a vanishing momentum diffusion and a negligible drift coefficient. It then represents a standard transport equation which is appropriate for the investigation of the spatial diffusion of charged particles in a turbulent plasma. This standard transport equation, also called Parker’s equation, finds comprehensive applications in heliospheric and astrophysical problems. Under the assumption that the spatial diffusion coefficient in radial direction and the bulk speed of the background plasma are expressible as arbitrary power laws in particle momentum and distance, a general solution of Parker’s equation is presented for spherical symmetry. The new general solution depends sensitively on arbitrary exponents of the spatial diffusion coefficient and the bulk speed power laws and is given in terms of an arbitrary particle injection function. Therefore, the general solution is useful not only for the investigation of spatial diffusion of cosmic rays in the heliosphere but also for other heliospheric as well as astrophysical problems. Assuming a slab Alfvénic turbulence (see section 6.3) and using appropriate source functions describing the injection of both ACRs and GCRs into the heliosphere, the general solution is applied to the problem of solar modulation of both particle populations. The new solution generalizes earlier analytical more limited results derived in the context of solar modulation. Numerical calculations for protons of both particle species yield energy spectra depending sensitively on parameters of the underlying turbulence and its dependence on heliocentric distance.

8.1 The Transport Equation and Its Solution

It was Parker [1965] who first investigated extensively the propagation of cosmic rays in the heliosphere. He established the fundamental cosmic ray transport equation, the so-called Parker equation, taking into account the processes of spatial diffusion, convection and adiabatic cooling in the expanding solar wind plasma flow. It was already mentioned in section 3.3.2 that Jokipii et al. [1977] first recognized the importance of large-scale drifts of cosmic rays in the heliospheric magnetic field and supplemented Parker’s equation by an appropriate term, namely $K_D$ entering the diffusion coefficient (3.25). Drift effects are indispensable and have to be incorporated into the transport equation if multi-dimensional large-scale modulation of particles is considered. However,
in order to reproduce observations it has been demonstrated that drift effects do not have to be included in all cases (see, e.g., Reinecke et al., 1993; le Roux and Fichtner, 1997).

Similarly to the considerations performed in chapter 7, the appropriate transport equation, i.e. Parker's equation, can readily be deduced on the basis of the diffusion-convection equation (3.24) by introducing several simplifications. First, according to equation (3.25), the tensor of spatial diffusion is only determined by $K_S$ if drift effects are neglected. Second, any loss terms in equation (3.24) are ignored. Third, assuming that diffusion in spatial configuration space is the dominant process, the scalar momentum diffusion coefficient $a_2$ is usually set to zero. Introducing for the bulk velocity the notation $u = \mathbf{V}$ and using for the rate of turbulent adiabatic deceleration the condition $a_1 = 0$, the simplified diffusion-convection equation (3.24) results in Parker’s equation, which reads

$$\nabla \cdot (K_S \cdot \nabla f) - u \cdot \nabla f + \frac{p}{3} \nabla \cdot u \frac{\partial f}{\partial p} = \frac{\partial f}{\partial t} - S(x, p, t) \tag{8.1}$$

It can be seen that cosmic rays are subjected to the processes of spatial diffusion and, as in chapter 7, convection and adiabatic cooling, respectively, when propagating through the outward expanding solar wind plasma. The spatial diffusion tensor $K_S$ includes contributions describing parallel and perpendicular particle diffusion with respect to the heliospheric background magnetic field (see equation (3.25) and the comments following it). The quantity $S(x, p, t)$ represents a source function with which ACRs and GCRs are injected into the heliosphere. Equation (8.1) has been used, with or without drift terms, in a variety of numerical studies (see, e.g., Potgieter et al., 1993; le Roux et al., 1996; Burger and Hattingh, 1998; Fichtner, 2001). Under the assumption of spherical symmetry and, furthermore, if the considerations are restricted to the heliospheric equatorial plane, i.e. the heliographic co-latitude satisfies $\theta = \pi/2$, the steady-state version of equation (8.1) can be written as

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \kappa_{rr} \frac{\partial f}{\partial r} \right) - u \frac{\partial f}{\partial r} + \frac{p}{3r^2} \frac{\partial}{\partial r} \left( r^2 u \right) \frac{\partial f}{\partial p} = -S(r, p) \tag{8.2}$$

Here, diffusion in space is determined only by the spatial diffusion coefficient in radial direction, $\kappa_{rr}$ given by equation (3.32). Already Parker [1965] has given a variety of analytical solutions of equation (8.2) for simplified cases of $\kappa_{rr}$ and $u$. Many more analytical or semi-analytical solutions were presented in the context of solar modulation during the last decades (see, e.g., Fisk and Axford, 1969; Gleeson and Webb, 1974; Cowisk and Lee, 1977; Gross et al., 1977; Zhang, 1999). Although these solutions describe the basic effects of solar modulation, they are not exact in a strict mathematical sense as they are employing various approximations and asymptotic expansions or assumptions about source functions. Exact solutions for the Green’s function of Parker’s equation were presented by several authors in a quite different context, namely the acceleration of particles in accretion flows (see Schneider and Bogdan, 1989; Becker, 1992). However, it is shown below that these solutions are not applicable to the case of solar modulation. Therefore, it is desirable and inevitable to derive a new solution of Parker’s equation. This solution generalizes and adapts those earlier results published in the context of solar modulation and particle energization in accretion flows, respectively.

In order to derive a more general solution of the transport equation (8.2), first the radial diffusion coefficient $\kappa_{rr}$ has to be considered in more detail. According to equation (3.32), $\kappa_{rr}$ can be expressed in terms of the diffusion coefficients parallel and perpendicular to the heliospheric background magnetic field,

$$\kappa_{rr} = \kappa_\perp \sin^2 \psi + \kappa_\parallel \cos^2 \psi \tag{8.3}$$

\footnote{One the basis of the considerations presented here, the author recently derived solutions of Parker’s equation for the case $a_1 \neq 0$. Detailed investigations of the influence of $a_1$ on solar modulation will be the subject of forthcoming studies.}
where $\psi$ denotes the angle between the radial direction and the average magnetic field. $\kappa_\perp$ corresponds to $\kappa_{XY}$ in equation (3.32). Under the assumption that the large-scale heliospheric magnetic field can be represented by the Parker field, equation (3.31), the heliospheric termination shock is quasi-perpendicular, so that $\psi \approx \pi/2$ beyond a few astronomical units in the equatorial plane inside the termination shock. Therefore, the radial diffusion coefficient $\kappa_{rr}$ is mainly determined by the diffusion coefficient perpendicular to the Parker field, i.e. by $\kappa_\perp$. Combining this with the standard assumption of long standing that the perpendicular and parallel diffusion coefficients are coupled via $\kappa_\perp = h \kappa_\parallel$ (with $h \approx 0.01 - 0.05$), the radial diffusion coefficient can be readily written as $\kappa_{rr} \approx h \kappa_\parallel$. Based on comprehensive magnetohydrodynamic turbulence models including turbulence generation by stream interactions as well as by PUIs, recent studies (see, e.g., Burger and Hattingh, 1998; Zank et al., 1998) pointed out that the dependence of the radial diffusion coefficient on heliocentric distance can be expressed as $\kappa_{rr} \propto r^\beta$, with $\beta < 1$. Also, recent analyses of ACR and GCR data (Steenberg, 1998; Moraal et al., 1999) resulted in the finding that $\beta \in [0, 0.5]$. In view of this knowledge it is assumed that the diffusion coefficient in radial direction can be written as $\kappa = \kappa_{rr} = \kappa_0 (p/p_E)^\alpha (r/r_E)^\beta$. Here the exponents $\alpha$ and $\beta$ are arbitrary. As in chapter 7, the subscript $E$ refers to reference values of the corresponding quantities. Since the wave-particle interactions take place in the expanding solar wind plasma, the bulk speed $u(r)$ has to be specified as a function in the heliocentric distance $r$. Assuming for further calculations a bulk speed of the simple form $u(r) = u_0 (r/r_E)^{-\gamma}$, where $\gamma$ is an arbitrary number, the steady-state version of equation (8.2) can be manipulated by introducing the new variables

$$y := \frac{\nu}{(1 - \gamma - \beta)^2} \frac{r u}{\kappa} \quad \text{and} \quad \tau := p$$

(8.4)

Here, the condition $\gamma + \beta = 1$ is excluded and the abbreviation $\nu = 1 - \gamma - \beta + \alpha(2 - \gamma)/3$ is introduced. In contrast to the transformed transport equation of PUIs in chapter 7, the variable $y$ represents here the ratio of the diffusion to the convection time scale, the so-called modulation parameter. Since $\alpha$, $\beta$ and $\gamma$ are arbitrary, the parameter $\nu$ and, therefore, the variable $y$ can be negative or positive. As in chapter 7, further considerations are substantially determined by the magnitudes of $\alpha$, $\beta$ and $\gamma$. Restricting the calculations to the case $\nu \neq 0$, the different signs of $\nu$ can be taken into account by introducing the parameter $\epsilon = \mp 1$, so that $\epsilon = -1$ and $\epsilon = +1$ correspond to the cases $\nu > 0$ and $\nu < 0$, respectively. This is quite opposite to the relation between $\epsilon$ and $\nu$ in the case of the PUI transport. Making use of the new variables (8.4) and the parameter $\epsilon$, the transformed version of equation (8.2) can be cast into the form

$$y \frac{\partial^2 f}{\partial y^2} + [b + \epsilon y] \frac{\partial f}{\partial y} + \frac{(2 - \gamma) - \epsilon}{3\nu} \frac{\partial f}{\partial \tau} = - \frac{r[\tau, y]}{\nu u(r[\tau, y])} S(r[\tau, y], \tau)$$

(8.5)

with $\nu > 0$ and $b = (2 - \gamma)/(1 - \gamma - \beta)$. The advantage of having decoupled the sign of $\nu$ into the parameter $\epsilon = \mp 1$ is obvious: first, the variable $y$ is a positive definite quantity, as it should be for the ratio of the diffusion and the convection time scales. Second, the final solution of the transport equation (8.2) will be more general. In order to apply this general solution to solar modulation of ACRs and GCRs in the heliosphere, a specific set of exponents $\alpha$, $\beta$ and $\gamma$ being typical for the heliosphere has to be chosen.

At this stage of the calculations it is necessary to consider the aforementioned solutions derived by Schneider and Bogdan [1989] and Becker [1992] in the context of particle energization in accretion flows and, furthermore, to justify the statement that their Green’s functions of Parker’s equation are not applicable to the case of solar modulation. These previously published solutions for the Green’s function require $\gamma > -2$ and can be classified using the auxiliary parameter $\eta_{aux} = (1 - \gamma - \beta)/(2 - \gamma)$. This quantity is exactly the inverse of the parameter $b$ introduced above. The solution derived by Schneider and Bogdan [1989] is valid for $\eta_{aux} < 0$, i.e. $\beta + \gamma > 1$. 

8.1 The Transport Equation and Its Solution

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(see their equation (2.6)). Noticing this and generalizing to an arbitrary dependence of \( \kappa \) on momentum, Becker [1992] presented the solution for the cases \( \eta_{aux} < 0 \) and \( \eta_{aux} > 1 \) (see his equation (A7) and his comments following it). For the case of modulation in the heliosphere, one has \( 2 > \gamma \geq 0 \), but \( \beta < 1 \), so that \( 0 \leq \eta_{aux} < 1 \). This, however, means that these previously published solutions are not applicable to the case of solar modulation.

In order to derive the solution of the transformed ACR/GCR transport equation, (8.5) has to be compared with the differential equation

\[
y'' \frac{\partial f}{\partial y^2} + (\alpha_1 + \epsilon y) \frac{\partial f}{\partial y} - \alpha_2 \xi(\tau) \frac{\partial f}{\partial \tau} = -Q(y, \tau)
\]

(8.6)

which is the generalized differential equation (B.1) established in appendix B. This readily leads to \( \alpha_1 = b, \alpha_2 = (2 - \gamma)/[3\nu] \) and \( \xi(\tau) = -\tau \). Additionally, one obtains for the source function the relation \( Q(y, \tau) = r[\tau, y]S(r[\tau, y], \tau)/[\nu u(r[\tau, y])] \). Following the considerations in appendix B, the auxiliary variable \( t \), which has been introduced to apply the Laplace transformation (B.7) to equation (8.6), can be evaluated according to the integral relation (B.5). This yields \( t = \ln(\tau_0/\tau)/\alpha_2 \).

In view of these relations, equation (B.20) results in the following solution of (8.2):

\[
f(r, p) = \frac{3}{b} \int dr_0 \int dp_0 \frac{y_0 S(r_0, p_0)}{p_0 u(r_0)} h_1^{1-b} h_2^{1+b} \exp \left[ \frac{y_0}{2\zeta} \right] I_{b-1} \left[ \frac{2y_0}{\zeta} h_1 h_2 \right]
\]

(8.7)

Here, the solution (8.7) includes the following auxiliary functions \( h_1 \) and \( h_2 \), which are defined by equations (B.21) and (B.22) in appendix B, respectively:

\[
h_1 = \left( \frac{p_0}{p} \right)^{\alpha/2} \left( \frac{r}{r_0} \right)^{(1-\gamma-\beta)/2} \quad \text{and} \quad h_2 = \left( \frac{p}{p_0} \right)^{3\nu/[2(2-\gamma)]}
\]

(8.8)

The auxiliary functions \( h_3 \) and \( \zeta \) depend on \( h_1 \) and \( h_2 \) via the equations (B.23) and (B.24), respectively. Equation (8.7), which is expressed in terms of an arbitrary source function, is the general solution of the transport equation (8.2) for the case that the bulk speed \( u(r) \) and the spatial diffusion coefficient \( \kappa \) evolve as arbitrary power laws in heliocentric distance and particle momentum. Note that (8.7) has the same structure as equation (7.6) derived in the context of stochastic acceleration of PUs. Although revealing the same structure in the auxiliary functions, both solutions are quite different in so far as they include different \( h_1, h_2, h_3 \) and \( \zeta \) and different underlying diffusion processes. Furthermore, the physics of the source function is different.

In order to adapt the solution (8.7) to the heliospheric transport of ACRs and GCRs and their associated solar modulation, the spatial diffusion coefficient \( \kappa \) evolve as arbitrary power laws in heliocentric distance and particle momentum. Then, the exponents \( \alpha \) and \( \beta \) can be specified. Assuming, for instance, that the turbulence consists of forward and backward propagating Alfvén waves having the same intensities, the parallel diffusion coefficient (6.76) can be used. The diffusion coefficient in radial direction then reads

\[
\kappa = \kappa_0 \left( \frac{p}{p_{A,0}} \right)^{3-q} \left( \frac{r}{r_E} \right)^{\eta-aq}
\]

(8.9)

Therefore, the exponents \( \alpha \) and \( \beta \) and the reference momentum \( p_E \) are given by \( \alpha = 3-q, \beta = \eta-aq \) and \( p_E = p_{A,0} \), respectively, where \( p_{A,0} \) denotes the momentum of a particle having the Alfvén speed \( v_{A,0} \). Furthermore, the comparison yields for the normalization

\[
\kappa_0 = \frac{h \Omega_0^{q-2} \nu_{A,0}^{3-q}}{\pi (q-1)(2-q)(4-q)} \left( \frac{l_c}{2\pi} \right)^{q-1} \left( \frac{Z}{A} \right)^{q-2} \left( \frac{B_0}{\delta B_0} \right)^2
\]

Equation (8.7) describes the phase space evolution of both ACRs and GCRs for arbitrary source functions with which particles of both populations are injected with the momentum \( p_0 \) into the heliosphere at the position \( r_0 \).
8.2 The Boundary Spectra of ACRs and GCRs

Since equation (8.7) is still formulated in terms of a particle injection function \( S(r_0, p_0) \), appropriate source functions have to be determined for ACRs and GCRs such that correct boundary spectra result at the heliocentric distance \( r_{sh} \) where the heliospheric termination shock is located.

8.2.1 The ACR Source Function

According to the theory of diffusive shock acceleration (e.g. Drury, 1983), it is expected that the accelerated ACR spectrum at the solar wind termination shock is related to the scattering center compression ratio \( s \) (see also, e.g., Vainio and Schlickeiser, 1999). Formally, the theory of diffusive shock acceleration gives the spectrum downstream of a planar shock as \( f \propto p^{-\mu} \), where the spectral index \( \mu \) is determined by the compression ratio \( s \) via \( \mu = 3s/(s-1) \). Therefore, the ACR spectrum at the distance \( r_{sh} \) is chosen as

\[
S(r_0, p_0) = S_{ACR,0}(p_0/p_n)^{-\mu} \exp(-p_0/p_c)\delta(r_0 - r_{sh}) \tag{8.10}
\]

Here, the factor \( S_{ACR,0} \) allows for suitable normalization. The Kronecker’s delta distribution indicates that the ACR population is injected at a certain radial position \( r_{sh} \). Furthermore, an exponential cut-off is introduced. The latter allows to include the effect of a finite radius of the termination shock, so that the spectrum cuts off in the energy range 0.1–1.0 GeV. This is controlled by the constant momentum \( p_c \). Furthermore, a normalization momentum \( p_n \) is introduced.

8.2.2 The GCR Source Function

Many studies have been carried out to derive the interstellar spectrum with which GCRs arrive at the heliospheric termination shock. For instance, Webber et al. [1987] found for the local interstellar spectrum of protons the form \( j = p^2f \propto (v/c)(E + 0.5E_0)^{-2.6} \). Here, \( v \) and \( c \) denote the speeds of the particle and light, respectively. \( E \) is the kinetic energy of a proton and \( E_0 \) the rest energy. Therefore, for the calculation of the galactic component of cosmic rays the source function of GCR protons is specified as follows:

\[
S(r_0, p_0) = S_{GCR,0} \frac{v}{p_0^n} [E(p_0) + 0.5E_0]^{-2.6} \delta(r_0 - r_{sh}) \tag{8.11}
\]

Analogously to the ACRs, the factor \( S_{GCR,0} \) is a normalization factor. The delta distribution indicates that the solar wind termination shock defines the modulation barrier for GCR protons. This barrier might actually be located farther out beyond the distance \( r_{sh} \) (see, e.g., Gurnett et al., 1993) but this is not of importance for the numerical solutions of equation (8.7).

8.3 Numerical Calculations

In order to demonstrate the potential and flexibility provided with the new solution, equation (8.7) is solved numerically for both ACR and GCR protons (\( Z = 1, A = 1 \)) by using the corresponding source functions (8.10) and (8.11), respectively. Furthermore, it is assumed that the coefficient (8.9) provides for the spatial diffusion of both particle populations inside the termination shock. For all calculations the termination shock is located at \( r_{sh} = 100 \) AU. The value 400 km s\(^{-1}\) is used for a constant (\( \gamma = 0 \)) solar wind speed \( u_0 \). For the numerical computation, the spatial diffusion coefficient (8.9) has to be evaluated for physical parameters being typical for the heliosphere.
Chapter 8 On Solar Modulation of Galactic and Anomalous Cosmic Rays

Figure 8.1: Calculated differential intensities in particles m$^{-2}$s$^{-1}$sr$^{-1}$[MeV/nucleon]$^{-1}$ as functions of kinetic energy in MeV nucleon$^{-1}$ for different heliocentric distances in the equatorial plane. The dotted and dashed curves in the left panel denote ACRs and GCRs, respectively. From the bottom to the top, curves are shown for the radial distances 1, 25, 50, 75 and 100 AU for each proton species, respectively. The solid curves in the right panel displays the sum of both particle populations at each heliocentric distance. The compression ratio $s$ and the turbulence parameters are used as listed in the left panel.

For the purely azimuthal ($a = 1$) heliospheric background magnetic field $B_0$ and the Alfvén speed $v_{A,0}$ at the Earth’s orbit typical values such as $4 \cdot 10^{-5}$ Gauss and 50 km s$^{-1}$ are chosen, respectively. Until noted otherwise, the spectral index $q$ is assumed to be $5/3$. The turbulence level $δB_0/B_0$ is 0.1 throughout all calculations. Considering the plasma wave correlation length $l_c$, a value of $5 \cdot 10^7$ km is used until noted otherwise. The reference momentum $p_n$ and the cut-off momentum $p_c$, entering the calculations by the ACR source function (8.10), are assumed to be $60m_pν_0$ and $500m_pν_0$, respectively, where $m_p$ is the proton mass. The latter momentum corresponds to the cut-off energy $E_c = 200$ MeV nucleon$^{-1}$. This values is used until noted otherwise. The parameter $ε = ±1$ is taken into account by its definition by cases. This means the code switches to $ε = -1$ or $ε = +1$ if $ν > 0$ or $ν < 0$, respectively. Instead of presenting the phase space distribution $f$, the following figures show differential intensities, i.e. $j = p^2f$, in particles m$^{-2}$s$^{-1}$sr$^{-1}$[MeV/nucleon]$^{-1}$ as functions of the particle’s kinetic energy in MeV nucleon$^{-1}$.

Figure 8.1 shows computational model results of performing the integration in equation (8.7) numerically for both the ACR and the GCR particle population. The dotted and dashed lines in the left panel represent ACRs and GCRs, respectively, for five different heliocentric distances. Going from the bottom to the top, equation (8.7) is solved for the heliocentric distances $r = 1, 25, 50, 75$ and 100 AU. For the ACR source function (8.10), the compression ratio $s$ is assumed to be 2.5. The top curves of both particle species are the spectra at the termination shock. A closer inspection of the left panel results in the finding that all characteristic features of spherical solar modulation of ACRs and GCRs in the heliosphere are clearly visible. At the lowest energies, where adiabatic cooling overwhelms spatial diffusion, the spectral slopes of both populations converge to a slope around 1. This is a typical consequence of the convection limit for which $y \propto T_{Dif}/T_{Con} \gg 1$, with $T_{Dif}$ and $T_{Con}$ being the diffusion and the convection time scales, respectively. Since adiabatic deceleration becomes less efficient with increasing heliocentric distance, the slopes become larger.
than 1 and increase with increasing radial distance at higher energies, but still below the spectral peaks. In this energy regime, modulation is determined by the competition between diffusion and convection. Except at 1 AU, this can be seen at all radial distances. At the peaks, the radial gradient of the ACR population is larger than for GCRs. Beyond these spectral peaks, the ACR spectra exhibit cut-offs at higher energies around $10^2 - 10^3$ MeV. This is a result of the exponential expression in the ACR source function (8.10). It reflects the influence of a spherical shock curvature and is controlled with the cut-off energy $E_c = 200$ MeV nucleon$^{-1}$. This is in contrast to the GCR spectra having no cut-offs. Checking the slopes of the GCR population at high energies leads, indeed, to the same slope as it is given in the expression by Webber et al. [1987], namely 2.6. The right panel of figure 8.1 shows a compilation of the computed ACR and GCR differential intensities presented in the left panel. Here, the solid lines denote the sum of the corresponding ACR and GCR intensities at the heliocentric distances used to compute the left panel. It roughly shows what a detector on a spacecraft sees at different heliocentric distances in the heliospheric equatorial plane.

The influence of a varying turbulence correlation length $l_c$ is shown in figure 8.2. As in figure 8.1 the dotted and dashed curves denote ACRs and GCRs, while the solid lines represent their sum. Going from the bottom to the top, both panels show modulated spectra calculated for the heliocentric distances 1, 25, 50, and 75 AU, respectively. Since modulation does not substantially affects the ACR and GCR boundary spectra at the radial distance where the heliospheric shock of termination is located, spectra for $r = 100$ AU are not shown here.

**Figure 8.2:** Numerical model results: modulated spectra of ACRs (dotted curves) and GCRs (dashed curves) as well as the sum of both (solid lines) at $r = 1$, 25, 50 and 75 AU (from the bottom to the top), respectively. The differential intensities shown in panel (a) and (b) are calculated for the correlation lengths $l_c = 3.2 \cdot 10^7$ km and $l_c = 3.2 \cdot 10^8$ km, respectively.
The energy spectra presented in panel (a) are computed for the correlation length $l_c = 3.2 \cdot 10^7$ km, while the right panel displays numerical solutions for the case $l_c = 3.2 \cdot 10^8$ km. A closer inspection of panels (a) and (b) leads to the following results: the intensities of both particle populations increase at low energies for a larger value of the correlation length $l_c$. This can be easily understood on the basis of the relation $\kappa \propto l_c^{-q-1}$. An increasing correlation length implies a more efficient spatial diffusion. This means, particles of both populations are subjected to a less efficient wave-particle interaction rate. This results in the fact that more particles can propagate to smaller heliocentric distances so that the differential intensities of both species increase for a larger $l_c$. As a consequence of the increased spatial diffusion coefficient all spectral peaks are located at lower energies in panel (b). Furthermore, the convection limit is also shifted to lower energies for an increased correlation length and for both particle species. This can nicely be seen by comparing panel (a) with (b).

Figure 8.3 shows spectra of ACRs (dotted lines) and GCRs (dashed lines) for two different values of the spectral index $q$ and for the heliocentric distances 1, 25, 50 and 75 AU (from the bottom to the top). The solid curves denote the sum of both differential intensities. Panel (a) illustrates numerical solutions for $q = 1.4$, while the case $q = 1.8$ is considered in panel (b). Both panels reflect the following scenario: the intensities of ACRs and GCRs increase with increasing spectral index $q$. Since the turbulence power scales with $\delta B_0^2 k^{-q}$ the fluctuation intensity decreases, for a given total amount of $\delta B_0^2$, with increasing $q$. This leads to a smaller fluctuation energy pool supporting the wave-particle interaction rate required for spatial diffusion. This means, spatial diffusion becomes more efficient with increasing $q$. As a result of the smaller turbulence energy pool, the differential intensities increase in the low energy regime and adiabatic cooling sets in at lower energies.
8.3 Numerical Calculations

Figure 8.4 illustrates the effect of changing the parameter \( \eta \), which represents the spatial evolution of the turbulence power, for the heliocentric distance \( r = 75 \) AU. Panel (a) shows the differential intensity of ACRs for four different values of \( \eta \), namely 2.6, 2.8, 3.0 and 3.2, and for the correlation length \( 5 \cdot 10^7 \) km. Panel (b) presents GCR spectra for the same values of \( \eta \). The sum of both the ACR and GCR contributions presented in the first two panels is shown in panel (c) for each value of \( \eta \) at 75 AU. Considering first the panels (a) and (b) in more detail, a significant effect of the parameter \( \eta \) on the differential intensities of both particle populations can be seen. Both ACR and GCR spectra exhibit the feature that particle intensities increase with increasing \( \eta \). This can be easily understood by considering the relation \( \kappa \propto r^\eta \). Since the turbulence power decreases for larger values of \( \eta \) at a certain heliocentric distance, the spatial diffusion becomes more efficient. This means, adiabatic cooling affects the particles at lower energies so that the spectral peaks are shifted to lower energies with increasing \( \eta \). As a consequence of this the intensities increase, due to a less efficient wave-particle interaction rate. The effect of changing \( \eta \) becomes more obvious by considering the sum of the ACR and GCR differential intensities. As it is shown in panel (c), the anomalous component is suppressed for low values of \( \eta \). Since the turbulence power provides for the energy pool required for wave-particle interactions, the spatial diffusion process is strong for low \( \eta \) values such as 2.6. For an increasing \( \eta \), the spatial diffusion of particles is more efficient, so that the anomalous component piles up and dominates the sum at lower energies.

Panels (a) and (b) of figure 8.5 show numerical model results for two different compression ratios, namely \( s = 2 \) and \( s = 4 \), respectively. The dotted and dashed curves denote ACRs and GCRs, while the solid lines represent their sum. From the bottom to the top, the heliocentric distances \( r = 1, 25, 50, 75 \) and 100 AU are used. Assumed values of the correlation length, \( \eta \) and \( q \) are listed in both panels. As a consequence of having increased the compression ratio, the slope of the ACR termination shock spectrum at 100 AU decreases. This agrees with what is expected on the basis of diffusive shock acceleration, since the spectral index \( \mu \) decreases with increasing \( s \). Due to the more efficient shock acceleration the intensity of ACRs increases so that the anomalous component piles up and becomes more important with increasing \( s \).

Figure 8.6 illustrates the effect of changing the cut-off momentum \( p_c \) or, alternatively, cut-off energy \( E_c \) on the differential intensity of ACRs for two different radial distances. The dashed and dotted lines correspond to \( r = 75 \) AU and \( r = 100 \) AU. From the left to the right, the dashed and dotted curves represent the cut-off energies \( E_c = 2, 200 \) and 2000 MeV nucleon\(^{-1} \) for each heliocentric distance. The solid lines at 75 and 100 AU show the asymptotic shapes for the limit \( E_c \to \infty \). Considering first the distance 75 AU, figure 8.6 reflects the following scenario: coming from the asymptotic shape (solid line for 75 AU), the ACR spectra cuts off at lower energies with decreasing cut-off energy, as it is expected. As a consequence, the spectral peaks and the convection limits are shifted to lower energies. Furthermore, the particle intensity decreases for lower cut-off energies. At 100 AU, where the termination shock is located, the spectra behave as expected. Considering, for instance, the case \( E_c = 20 \) MeV nucleon\(^{-1} \) (the outer left-handed dotted and dashed curves), the spectrum at 100 AU (dotted curve) experiences a strong cut-off and becomes more steeper. With increasing energy it approaches around 100 MeV nucleon\(^{-1} \) the dashed curve, which corresponds to the ACR spectrum at 75 AU. With increasing \( E_c \) the cut-off sets in at higher energies until it approaches the asymptotic shape for 100 AU. Although the ACRs are injected by a power law function, which can be nicely recognized at energies below \( 10^2 \) MeV nucleon\(^{-1} \), both asymptotic shapes become steeper around \( 10^3 \) MeV nucleon\(^{-1} \), where the solid lines merge. The change in slope at 100 AU results from the exponential expression in the solution (8.7).
Figure 8.4: Numerical model results showing the influence of a changing radial behavior of the underlying turbulence power, i.e. $(δB)^2 \propto r^{-\eta}$, on ACR and GCR spectra. Panel (a): Differential intensities of ACRs at 75 AU for four different values of $\eta$. Panel (b): Same as (a), but for the GCR population. Panel (c): The sum of the ACR and GCR data shown in panels (a) and (b).
8.3 Numerical Calculations

(a) Spectra for the compression ratio $s = 2$.

(b) Spectra for the compression ratio $s = 4$.

Figure 8.5: Numerical model results: differential intensities of ACRs (dotted lines), GCRs (dashed lines) and their corresponding sums (solid lines) calculated for the heliocentric distances 1, 25, 50, 75 and 100 AU (from the bottom to the top). Panel (a) and (b) show calculations for the compression ratios $s = 2$ and $s = 4$, respectively.

Figure 8.6: Numerical results showing the differential intensities of ACRs at the heliocentric distances 75 (dashed lines) and 100 AU (dotted lines) for four different cut-off energies. From the left to the right, the cut-offs are controlled by the energies $E_c = 2, 200$ and $2000$ MeV. The fourth cut-off energy corresponds to the limit $E_c \rightarrow \infty$, which is represented for both heliocentric distances by the asymptotic shapes (solid lines).
8.4 Summary and Conclusions

This chapter introduces a second application of the diffusion-convection equation. It is assumed that diffusion in spatial space is the most important process. The diffusion-convection equation then results in Parker’s equation. For spherical symmetry and steady state conditions, an exact solution of this fundamental transport equation of cosmic rays is presented. The derivation is quite similar to the considerations performed in the previous chapter. There, a general solution of the isotropic PUI transport equation is presented by using the solution of the generalized differential equation established in appendix B. Although different in energy range and physical interpretation, both the PUI transport equation as well as Parker’s equation may be considered as special cases of this generalized differential equation. In order to obtain the solution of Parker’s equation on the basis of this more general approach, it is assumed that the spatial diffusion coefficient in radial direction and the bulk speed of the background plasma can be written as arbitrary power laws in particle momentum and radial distance. Then, Parker’s equation is subjected to a mathematical transformation, enabling one to make use of the generalized differential equation. The solution of Parker’s equation is then obtained by a direct comparison of the transformed Parker’s equation with this generalized differential equation. The new solution generalizes earlier solutions to the parameter range being characteristic for solar modulation and, moreover, supplement those so far available for the spatial diffusion of particles in accretion flows. The new solution has the advantage to be valid not only for arbitrary exponents of the diffusion coefficient and bulk speed power laws, but also for an arbitrary source function. Furthermore, a simple power law behavior of the bulk speed of the background plasma is included. This allows to consider more complicated particle injection rates, for which the integrals can be solved numerically. Furthermore, presumed that the spatial diffusion coefficient can be expressed as a power law in particle momentum and radius, different turbulence models may be taken into account via its arbitrary power law exponents.

The general solution is then applied to solar modulation of galactic as well as anomalous cosmic rays. For this, it is assumed that the turbulence consists of Alfvén waves propagating forward and backward to the heliospheric background magnetic field with the same intensity. Furthermore, specific source functions are introduced, representing the injection of ACRs and GCRs at the heliospheric shock of termination. Numerical calculations for ACR and GCR-hydrogen demonstrate the potential and flexibility of the new semi-analytical approach. For the numerical calculations, parameters being typical for the heliosphere are used. The influence of several turbulence parameters on the differential intensities of both particle populations is discussed. Energy spectra of both GCRs and ACRs depend sensitively on parameters of the underlying turbulence. The effect of a changing wave power spectral index, turbulence correlation length and, furthermore, of the radial evolution of the turbulence intensity on energy spectra is considered in detail. Moreover, the influence of the cut-off energy, entering the ACR source function, and of different termination shock compression ratios is illustrated. This was not done before on such a semi-analytical level.

Besides an easy way to compare observations and theory, in particular with respect to the turbulence model determining the spatial diffusion of cosmic rays, the new solution renders the opportunity to check on modeling results obtained purely numerically with sophisticated computer codes.
Chapter 9
Summary, Conclusions and Outlook

Diffusive particle transport in a magnetized, turbulent plasma is a crucial point in the physics of plasmas being present in both interstellar and interplanetary space. Heliospheric physics enables one to connect theoretical plasma physics with in-situ observations made aboard spacecraft propagating through the solar wind plasma. This has the advantage to measure directly a variety of physical parameters of the solar wind plasma and its embedded turbulence with great precision. Heliospheric physics, therefore, represents a vital link between plasma physics and astronomy and provides for valuable diagnostics of a variety of astrophysical phenomena. In this thesis, various aspects of this important link are studied from a theoretical point of view. They deal with solar wind magnetic fluctuations and their influence on the diffusive transport of charged particles in the heliosphere. The main results and conclusions presented in this thesis can be summarized as follows:

- The fundamental Fokker-Planck coefficients are considered in detail in chapter 4. They represent the physical key for understanding particle diffusion in a three-dimensional plasma wave turbulence. The “classical” Bessel function representation is dropped and new expressions are deduced generalizing those available so far. They yield significantly more insight into the underlying mathematical structure of all coefficients than earlier presentations. The new Fokker-Planck coefficients are written in terms of three contributions containing, via the fluctuating magnetic fields, all information about the underlying plasma wave turbulence and its evolution in three-dimensional wavenumber space. The approach of a dynamical magnetic turbulence is automatically achieved for a vanishing plasma wave dispersion relation. Comparable considerations of Fokker-Planck coefficients on such a general level are not known so far. The new representations should be beneficial for a deeper understanding of particle diffusion in a plasma wave turbulence embedded in a magnetized plasma.

To obtain more tractable expressions, a specific magnetic correlation tensor representation for isotropic geometry is used. The corresponding fluctuating magnetic fields are calculated and it is demonstrated that the magnetic helicity enters the standard and perpendicular Fokker-Planck coefficients in a characteristic manner. Three different restrictive limits of the isotropic model are presented. The Fokker-Planck coefficients for the restriction of an isotropic dynamical magnetic turbulence are derived for the first time. Furthermore, the “classical” transversal slab turbulence limit is investigated. For this, it is shown that particle motion perpendicular to the background magnetic field is suppressed. This leads to the conclusion that a compressional plasma wave component is required to explain particle diffusion across the background magnetic field. This is explained by the reduced dimensionality of the slab model in both the three-dimensional spatial configuration as well as wavenumber space. The calculations find support by the so-called JKG theorem developed by Jokipii et al. [1993].

The Fokker-Planck coefficients providing for particle drift due to the turbulence itself are extensively investigated for the first time. It is shown that resonant wave-particle interactions
vanish completely, even for damped plasma waves. As a consequence of this, these Fokker-Planck coefficients are antisymmetric and depend solely on non-resonant contributions. It is shown that the new drift tensor then indeed reveals an antisymmetric structure, as expected.

- The variation of the turbulence power in wavenumber space is the topic of chapter 5. On the basis of plasma wave damping rates obtained with a linear Vlasov code, numerical solutions of the governing wave transport equation and Kolmogorov diffusion throughout wavenumber space, it is argued that plasma wave damping of both the Alfvén and the magnetosonic/whistler mode cannot explain the steepening of solar wind magnetic turbulence spectra in the dissipation range at intermediate wavenumbers. Generalizing the approach by taking into account dispersion of the magnetosonic/whistler wave mode, a model is constructed which may represent the magnetosonic/whistler mode contributions to solar wind magnetic power spectra. In order to develop this model, it is assumed that the Kolmogorov wavenumber diffusion may not represent the fluctuation energy transfer at wavelengths where dispersion and damping of the magnetosonic/whistler mode becomes important. Numerical calculations for small values of the plasma beta yield magnetic power spectra at intermediate wavenumbers and sharp cut-offs at shorter wavelengths where electron-cyclotron damping dominates. It is, therefore, suggested to call the range of solar wind magnetic fluctuation power spectra at intermediate wavenumber the “dispersion range” rather than “dissipation range”. With increasing $\beta_p$, proton-cyclotron damping of the magnetosonic/whistler mode becomes more appreciable. For relative high values of $\beta_p$, the dispersion range disappears and is replaced by the proton-cyclotron dissipation range.

- In chapter 6 several transport parameters are presented for an isotropic and a slab turbulence. The transport parameter describing particle diffusion perpendicular to the background magnetic field is calculated for isotropic geometry for the first time. It is shown that the inclusion of a finite turbulence correlation length and a more realistic power spectrum behavior in wavenumber space is of importance for the understanding of perpendicular diffusion of charged particles. This is demonstrated by comparing the new diffusion coefficient with simulation results obtained previously by Giacalone and Jokipii [1999]. For the limit of long correlation lengths, it is shown that the new transport parameter includes earlier results deduced on the quite different concept of field line random walk governed by the power spectrum at zero wavenumber. The argument against the presence of waves with infinitely long wavelengths is that the plasmas in both interstellar and, in particular, interplanetary space have only finite extent. The considerations presented in this thesis indicate that a finite plasma wavelength has to be taken into account. The new perpendicular diffusion coefficient then indeed describes cross-field diffusion.

On the basis of the Fokker-Planck coefficients for “turbulent” particle drift, the corresponding transport parameter is derived for both an isotropic and a slab turbulence for the first time. Based on the different dimensionalities of the turbulence geometries, detailed calculations show that no drift occurs for an isotropic turbulence. The particle drift is determined by the normalized cross and magnetic helicity of the transversal slab turbulence. It is shown that the new drift term is related to the $E \times B$ drift via the electric field of the plasma wave turbulence. The sign of the drift term depends sensitively on the net polarization of the plasma wave turbulence. Comparable calculations are not known so far.

Additionally, the transport parameters for parallel particle diffusion, adiabatic deceleration and diffusion in momentum space are recalculated for a slab turbulence. Earlier results available so far are confirmed.
• The first application of the fundamental diffusion-convection transport equation and its associated transport parameters is presented in chapter 7. There, stochastic acceleration of pick-up ions is considered. The governing transport equation is solved analytically. In order to do so, a generalized differential equation is formulated and solved in appendix B. The new solution of this generalized differential equation is beneficial not only for the investigation of stochastic acceleration of pick-up ions in the heliosphere, but also for other astrophysical problems. The new solution is applied to the transport of pick-up ions. It significantly generalizes the two “classical” solutions used during the last three decades. For large heliocentric distances, several asymptotic phase space integrals are presented and solved analytically by using a standard particle source function. Numerical calculations for interstellar neutral hydrogen yield particle distributions depending sensitively on the underlying turbulence and its evolution with heliocentric distance. Comparable calculations performed on such a comprehensive semi-analytical level are not known so far. The new solution enables one to check on modeling results obtained purely numerically with sophisticated computer codes. Furthermore, it renders the opportunity to compare theory and observations in future studies.

• A second application of the particle transport equation is presented in chapter 8. There, solar modulation of anomalous and galactic cosmic rays is discussed. The appropriate equation of transport is solved analytically by making use of the generalized differential equation introduced in appendix B. The new solution generalizes earlier more limited solutions presented in the context of solar modulation. Numerical calculations yields energy spectra depending sensitively on parameters of the underlying turbulence physics and its dependence on heliocentric distance. Furthermore, the influence of different compression ratios of the heliospheric termination shock and cut-off energies, entering the source function of anomalous cosmic rays, is discussed. Comparable considerations on such a semi-analytical level are not known. Besides an easy way to check more complicated computer codes developed for the investigation of solar modulation, the new solution enables one to compare theory and observations in forthcoming investigations.

The issues studied in this thesis suggest a variety of interesting and exciting continuations which will involve not only a more comprehensive mathematical treatment of all topics discussed here, but also comparisons of theory and observations. Some of them are:

• The transport parameter for perpendicular particle diffusion in an isotropic turbulence will be treated in its general form. For this, all values of the integer number $n$ have to be taken into account. It is expected, however, that the contributions to particle scattering from large $|n|$ are small. The wavenumber integration appearing in the corresponding Fokker-Planck coefficients will be solved numerically or, if possible, analytically for a more realistic turbulence power spectrum. For the latter, different power laws in distinguished wavenumber regimes will be considered. This will allow to investigate the influence of both a maximum and minimum turbulence correlation length on the particle motion normal to the background magnetic field. Furthermore, forthcoming studies will also include an anisotropic turbulence. This requires an evaluation of the corresponding Fokker-Planck coefficients for a more realistic magnetic correlation tensor of second-rank. The restriction to a pure slab turbulence will then serve as a crucial test, because it should show a suppressed perpendicular particle diffusion. Here, a widely open field of research of considerable importance for the understanding of perpendicular particle transport in a plasma wave turbulence exists.

• The new drift term derived in chapter 6 will be evaluated for damped Alfvén waves. The appropriate damping rate of this wave mode is presented in chapter 5. In this thesis, only undamped waves are taken into account and no particle properties enter the new “helicity”
drift. The inclusion of the Alfvénic damping rate requires a numerical computation of the wavenumber integration. This will allow to investigate the effect of a changing plasma $\beta_p$ on this particle drift. Beside this, the influence of a turbulence magnetic power spectrum with broken power laws will be investigated. Furthermore, the Fokker-Planck coefficients will be evaluated for an anisotropic turbulence geometry. The restriction to an isotropic turbulence would then be a crucial test for the model, because it should reflect the result of a vanishing drift obtained in chapter 6 of this work.

- The new solution of the pick-up ion transport equation is derived in terms of integrals over the particle source function describing the injection of freshly ionized pick-up ions into the heliosphere. For the sake of simplicity, it is assumed that the particles are injected locally into the solar wind plasma with the solar wind speed. In general, it is expected that the speed of pick-up ions depend on the peculiar velocity of the neutral parent atoms. Presumably, this would lead to a more complicated dependence of the source function on the local particle speed. The new solution and the numerical code render the opportunity to include a more realistic particle injection function in an easy way. Furthermore, since the new solution deals with the evolution of the velocity distribution of pick-up ions after their ionization until they reach the heliospheric shock, it seems to be suitable to investigate the pre-acceleration of pick-up ions by the second-order Fermi process. In this thesis, it is shown that this process leads to an asymptotic shape of the distribution function beyond several astronomical units. For this, an analytical expression is derived describing the pick-up ion behavior at large heliocentric distances. It should be beneficial to apply the new solution or, at least, its asymptotic representation to the process of diffusive shock acceleration. Both, the inclusion of a more realistic source function and the application of the general solution to first-order Fermi acceleration at the heliospheric shock will be the subject of forthcoming studies. Furthermore, the influence of the adiabatic deceleration coefficient $a_1$ will be taken into account. A corresponding analytical solution of the diffusion-convection transport equation for $a_1 \neq 0$ is already derived, but not presented in this thesis.

- Concerning the new solution derived in the context of solar modulation of anomalous and galactic cosmic rays, different particle species will be considered in forthcoming studies. For this, specific source functions have to be taken into account. In particular for the anomalous cosmic ray population, the new solution of the pick-up ion transport equation can be treated, after its application to diffusive shock-acceleration, as an appropriate source function which has to be inserted into the solution of Parker’s equation. This approach will merge both general solutions presented in this thesis.
Appendix A

Fokker-Planck Coefficients and Their Derivations

In order to calculate the fundamental Fokker-Planck coefficients, the temporal variations of the coordinates (3.8) are required. These variations are given by the fluctuating force fields (3.7a) through (3.7f) and include the scattering of energetic particles at electromagnetic fields of the turbulence. Under the assumption of a homogeneous plasma, electromagnetic fluctuations enter the calculations via so-called correlation tensors determining sensitively all Fokker-Planck coefficients and the associated transport parameters (3.23a) to (3.23d). In order to enable a further analytical reduction of the fluctuating force fields (3.7a) through (3.7f), one usually applies a standard perturbation method with which the force fields are calculated for unperturbed particle orbits.

A.1 Unperturbed Particle Orbits

Unperturbed particle orbits can easily be calculated on the basis of equation (3.3). For vanishing turbulence contributions, the latter equation yields

\[ \dot{p}_x = \Omega_\alpha p_y, \quad \dot{p}_y = -\Omega_\alpha p_x, \quad \dot{p}_z = 0 \] (A.1)

After integration with respect to time \( t \), the system of equations (A.1) leads to an unperturbed particle orbit in the ambient magnetic field, given by

\[ \bar{x}(t) = \bar{x}_0 - v_\perp \Omega_\alpha \sin(\phi_0 - \Omega_\alpha t) \] (A.2)
\[ \bar{y}(t) = \bar{y}_0 + v_\perp \Omega_\alpha \cos(\phi_0 - \Omega_\alpha t) \] (A.3)
\[ \bar{z}(t) = \bar{z}_0 + v_\parallel t \] (A.4)

where the set \( (\bar{x}_0, \bar{y}_0, \bar{z}_0, \phi_0) \) represents initial coordinates at \( t_0 \). In order to calculate the fluctuating force fields (3.7a) through (3.7f) for an unperturbed particle orbit, a Fourier representation of the fluctuating electromagnetic field is used, where the true orbit is approximated by the unperturbed orbit, i.e.

\[ \delta B(\mathbf{x}, t) = \int d^3k \mathbf{B}(\mathbf{k}, t) e^{i k \cdot \mathbf{x}(t)} \simeq \int d^3k \mathbf{B}(\mathbf{k}, t) e^{i k_\perp \cdot \bar{x}(t)} \] (A.5)
\[ \delta E(\mathbf{x}, t) = \int d^3k \mathbf{E}(\mathbf{k}, t) e^{i k \cdot \mathbf{x}(t)} \simeq \int d^3k \mathbf{E}(\mathbf{k}, t) e^{i k_\perp \cdot \bar{x}(t)} \] (A.6)

Assuming at time \( t_0 = 0 \) the initial position \( x_0 = y_0 = z_0 = 0 \), introducing the wavenumber representation \( k_x = k_\perp \cos \psi, k_y = k_\perp \sin \psi, k_z = k_\parallel \) and the abbreviation \( W = k_\perp R_L \sqrt{1 - \mu^2} \), where \( R_L = v/\Omega_\alpha \) denotes the Larmor radius of the particle, one can express the exponential function as follows:

\[ e^{i k_\perp \cdot \bar{x}(t)} = \sum_{n=-\infty}^{+\infty} J_n(W) \exp \left( i m(\psi - \phi_0 + \Omega_\alpha t) + i k_\parallel v_\parallel t \right) \] (A.7)
with $J_n(W)$ being a Bessel function of the first kind and of the order $n$. Expressing the electric and magnetic quantities in equations (3.7a) through (3.7f) by their corresponding Fourier counterparts, where the arguments $k$ and $t$ will be dropped for the following considerations, and making use of the identity (A.7), the fluctuating force fields read

$$\bar{g}_p(t) = q_\alpha \sum_{n=-\infty}^{\infty} \int d^3k \exp \left[ i n (\psi - \phi_0) + i (k \parallel v + n\Omega_\alpha) t \right]$$

$$\times \left\{ \mu E\| J_n(W) + \sqrt{(1 - \mu^2)/2} \left[ e^{i\psi} J_{n+1}(W) E_R + e^{-i\psi} J_{n-1}(W) E_L \right] \right\}$$

$$\bar{g}_\mu(t) = \frac{\Omega_\alpha}{B_0} \sum_{n=-\infty}^{\infty} \int d^3k \exp \left[ i n (\psi - \phi_0) + i (k \parallel v + n\Omega_\alpha) t \right] \left\{ \frac{c}{v} (1 - \mu^2) E\| J_n(W) \right\}$$

$$+ \frac{1}{\sqrt{2(1 - \mu^2)}} \left[ e^{i\psi} J_{n+1}(W) \left( B_R + i\mu \frac{c}{v} E_R \right) - e^{-i\psi} J_{n-1}(W) \left( B_L - i\mu \frac{c}{v} E_L \right) \right]$$

$$\bar{g}_\phi(t) = \frac{\Omega_\alpha}{B_0} \sum_{n=-\infty}^{\infty} \int d^3k \exp \left[ i n (\psi - \phi_0) + i (k \parallel v + n\Omega_\alpha) t \right] \left\{ -B\| J_n(W) \right\}$$

$$+ \frac{1}{\sqrt{2(1 - \mu^2)}} \left[ e^{i\psi} J_{n+1}(\xi) \left( \mu B_R + i\mu \frac{c}{v} E_R \right) + e^{-i\psi} J_{n-1}(\xi) \left( \mu B_L - i\mu \frac{c}{v} E_L \right) \right]$$

$$\bar{g}_x(t) = \frac{v}{\sqrt{2B_0}} \sum_{n=-\infty}^{\infty} \int d^3k \exp \left[ i n (\psi - \phi_0) + i (k \parallel v + n\Omega_\alpha) t \right]$$

$$\times \left\{ -\sqrt{(1 - \mu^2)/2} \left[ J_{n+1}(W) e^{i\psi} + e^{-i\psi} J_{n-1}(W) \right] B\| \right\}$$

$$+ \frac{1}{\sqrt{2(1 - \mu^2)}} \left[ e^{i\psi} J_{n+1}(W) \left( E_R - E_L - i\mu \frac{c}{v} \left( B_R + B_L \right) \right) \right]$$

$$\bar{g}_y(t) = \frac{v}{\sqrt{2B_0}} \sum_{n=-\infty}^{\infty} \int d^3k \exp \left[ i n (\psi - \phi_0) + i (k \parallel v + n\Omega_\alpha) t \right]$$

$$\times \left\{ \frac{1}{\sqrt{2}} \left[ J_{n+1}(W) e^{i\psi} - e^{-i\psi} J_{n-1}(W) \right] B\| \right\}$$

$$- \frac{c}{v} J_n(W) \left[ E_R + E_L + i\mu \frac{c}{v} \left( B_L - B_R \right) \right] \right\}$$
A.2 The Bessel Function Representation

Having established the fluctuating force fields for the unperturbed particle orbit, one can now proceed to calculate the Fokker-Planck coefficients. According to equation (3.14), one has to multiply the fluctuating force fields with their complex conjugated counterparts, has to take the ensemble average and, finally, has to integrate the result with respect to \( s \). Although straightforward, the calculations are tedious and result in the following set of Fokker-Planck coefficients:

\[
D_{\mu\nu} = \frac{\Omega_0^2}{B_0^2} (1 - \mu^2) \sum_{n=-\infty}^{\infty} \Re \oint d^3 k \int_0^\infty ds \exp \left[ -i(k_{||}v_{||} + n\Omega_0)s \right] \left\{ \frac{c^2}{v^2} (1 - \mu^2) J_n^2(W) R_{||} \right. \\
+ \frac{1}{2} J_{n+1}^2(W) \left( P_{RR} + \mu \frac{c^2}{v^2} R_{RR} - i\mu \frac{c}{v} [T_{RR} - Q_{RR}] \right) \\
+ \frac{1}{2} J_{n-1}^2(W) \left( P_{LL} + \mu \frac{c^2}{v^2} R_{LL} + i\mu \frac{c}{v} [T_{LL} - Q_{LL}] \right) \\
- \frac{1}{2} J_{n+1}(W) J_{n-1}(W) \\
\left. \times \left[ e^{2i\psi} \left( P_{RL} - \mu \frac{c^2}{v^2} R_{RL} + i\mu \frac{c}{v} [T_{RL} + Q_{RL}] \right) \right. \right. \\
+ e^{-2i\psi} \left( P_{LR} - \mu \frac{c^2}{v^2} R_{LR} - i\mu \frac{c}{v} [T_{LR} + Q_{LR}] \right) \right] \\
+ \frac{c}{v} \sqrt{1 - \mu^2} J_n(W) \\
\left\{ J_{n+1}(W) \left( e^{i\psi} T_{||} - e^{-i\psi} Q_{||} + i\mu \frac{c}{v} \left[ e^{i\psi} R_{||} + e^{-i\psi} R_{||} \right] \right) \right. \\
+ J_{n-1}(W) \left( e^{i\psi} Q_{||} - e^{-i\psi} T_{||} + i\mu \frac{c}{v} \left[ e^{i\psi} R_{||} + e^{-i\psi} R_{||} \right] \right) \left\} \right\} 
\]

\[
D_{\mu\nu} = \frac{c}{v} (1 - \mu^2) \sum_{n=-\infty}^{\infty} \Re \oint d^3 k \int_0^\infty ds \exp \left[ -i(k_{||}v_{||} + n\Omega_0)s \right] \left\{ -\frac{c}{v} \mu J_n^2(W) R_{||} \right. \\
+ \frac{1}{2} J_{n+1}(W) \left( T_{RR} + i\mu \frac{c}{v} R_{RR} \right) - \frac{1}{2} J_{n-1}(W) \left( T_{LL} - i\mu \frac{c}{v} R_{LL} \right) \\
+ \frac{1}{2} J_{n+1}(W) J_{n-1}(W) \left[ e^{2i\psi} \left( T_{RL} + i\mu \frac{c}{v} R_{RL} \right) - e^{-2i\psi} \left( T_{LR} - i\mu \frac{c}{v} R_{LR} \right) \right] \\
- \frac{1}{\sqrt{2(1 - \mu^2)}} J_n(W) \left( i(1 - \mu^2) \frac{c}{v} e^{-i\psi} R_{||} - i\mu e^{i\psi} \left[ T_{||} + i\mu \frac{c}{v} R_{||} \right] \right) \\
+ J_{n-1}(W) \left( i(1 - \mu^2) \frac{c}{v} e^{i\psi} R_{||} + i\mu e^{-i\psi} \left[ T_{||} - i\mu \frac{c}{v} R_{||} \right] \right) \left\} \right\} 
\]
\[ D_{pp} = q_\alpha^2 \sum_{n=-\infty}^{\infty} \Re \int d^3 k \int_0^\infty ds \exp \left[ -i(k_{\parallel} v_{\parallel} + n \Omega_{n})s \right] \left\{ \mu^2 J_n^2(W) R_{\parallel\parallel} \right\} \]

\[ + \frac{1 - \mu^2}{2} \left[ J_{n+1}^2(W) R_{RR} + J_{n-1}^2(W) R_{LL} + J_{n+1}(W) J_{n-1}(W) \left( e^{2i\psi} R_{RL} + e^{-2i\psi} R_{LR} \right) \right] \]

\[ + \mu \sqrt{\frac{1 - \mu^2}{2}} J_n(W) \left[ J_{n+1}(W) \left( e^{i\psi} R_{\parallel L} + e^{-i\psi} R_{L\parallel} \right) + J_{n+1}(W) \left( e^{-i\psi} R_{R\parallel} + e^{i\psi} R_{R\parallel} \right) \right] \]

\[ D_{XX} = \frac{v^2}{4B_0^2} \sum_{n=-\infty}^{\infty} \Re \int d^3 k \int_0^\infty ds \exp \left[ -i(k_{\parallel} v_{\parallel} + n \Omega_{n})s \right] \]

\[ \times \left\{ (1 - \mu^2) P_{\parallel\parallel} \left[ J_{n+1}^2(W) + J_{n-1}^2(W) + J_{n+1}(W) J_{n-1}(W) \left( e^{2i\psi} + e^{-2i\psi} \right) \right] \right\} \]

\[ + \sqrt{2(1 - \mu^2)} \frac{c}{v} J_n(W) \left( J_{n+1}(W) e^{i\psi} + J_{n-1}(W) e^{-i\psi} \right) \left[ T_{\parallel R} - T_{\parallel L} + \mu \frac{v}{c} P_{\parallel R} \right] \]

\[ + \sqrt{2(1 - \mu^2)} \frac{c}{v} J_n(W) \left( J_{n+1}(W) e^{-i\psi} + J_{n-1}(W) e^{i\psi} \right) \left[ Q_{R\parallel} - Q_{L\parallel} - \mu \frac{v}{c} P_{\parallel L} \right] \]

\[ + 2 \frac{c^2}{v^2} J_n^2(W) \]

\[ \times \left[ R_{RR} + \mu \frac{v}{c} (Q_{RR} - T_{RR}) + \mu \frac{v}{c} P_{RR} + R_{LL} + \mu \frac{v}{c} (T_{LL} - Q_{LL}) + \mu \frac{v^2}{c^2} P_{LL} \right. \]

\[ - R_{RL} + \mu \frac{v}{c} (T_{RL} + Q_{RL}) + \mu \frac{v^2}{c^2} P_{RL} - R_{LR} - \mu \frac{v}{c} (T_{LR} + Q_{LR}) + \mu \frac{v^2}{c^2} P_{LR} \left\} \right\} \]

\[ D_{YY} = \frac{v^2}{4B_0^2} \sum_{n=-\infty}^{\infty} \Re \int d^3 k \int_0^\infty ds \exp \left[ -i(k_{\parallel} v_{\parallel} + n \Omega_{n})s \right] \]

\[ \times \left\{ (1 - \mu^2) P_{\parallel\parallel} \left[ J_{n+1}^2(W) + J_{n-1}^2(W) - J_{n+1}(W) J_{n-1}(W) \left( e^{2i\psi} + e^{-2i\psi} \right) \right] \right\} \]

\[ + \sqrt{2(1 - \mu^2)} \frac{c}{v} J_n(W) \left( J_{n+1}(W) e^{-i\psi} - J_{n-1}(W) e^{i\psi} \right) \left[ Q_{L\parallel} + Q_{R\parallel} + \mu \frac{v}{c} (P_{L\parallel} - P_{R\parallel}) \right] \]

\[ - \sqrt{2(1 - \mu^2)} \frac{c}{v} J_n(W) \left( J_{n+1}(W) e^{i\psi} - J_{n-1}(W) e^{-i\psi} \right) \left[ T_{L\parallel} + T_{R\parallel} - \mu \frac{v}{c} (P_{L\parallel} - P_{R\parallel}) \right] \]

\[ + 2 \frac{c^2}{v^2} J_n^2(W) \]
\[ D_{XY} = \frac{v^2}{4B_0^2} \sum_{n=-\infty}^{\infty} \Re \{ \int_{-\infty}^{\infty} \int_{0}^{\infty} ds \exp \left[ -i(k\|v\| + n\Omega_\alpha)s \right] \} \]

\[ \times \left\{ (1 - \mu^2)P_{||} \left[ J_{n+1}^2(W) - J_{n-1}^2(W) - J_n(W)J_{n-1}(W)(e^{2i\psi} - e^{-2i\psi}) \right] \right. \]

\[ - \frac{i\sqrt{2(1 - \mu^2)v}}{c} J_n(W) \left( J_{n+1}(W)e^{i\psi} + J_{n-1}(W)e^{-i\psi} \right) \left[ T_{||} - T_{\perp} + \mu v \left( P_{||} - P_{\perp} \right) \right] \]

\[ - \frac{i\sqrt{2(1 - \mu^2)v}}{c} J_n(W) \left( J_{n+1}(W)e^{-i\psi} - J_{n-1}(W)e^{i\psi} \right) \left[ Q_{||} - Q_{\perp} - \mu v \left( P_{||} + P_{\perp} \right) \right] \]

\[ - 2 \frac{c^2}{v^2} J_n^2(W) \]

\[ \times \left[ R_{RR} + \mu v \left( Q_{RR} - T_{RR} \right) + \mu^2 \frac{v^2}{c^2} P_{RR} - R_{LL} + \mu v \left( Q_{LL} - T_{LL} \right) - \mu^2 \frac{v^2}{c^2} P_{LL} \right. \]

\[ + R_{RL} - \mu v \left( T_{RL} + Q_{RL} \right) - \mu^2 \frac{v^2}{c^2} P_{RL} - R_{LR} - \mu v \left( T_{LR} + Q_{LR} \right) + \mu^2 \frac{v^2}{c^2} P_{LR} \right] \}

\[ D_{YX} = \frac{v^2}{4B_0^2} \sum_{n=-\infty}^{\infty} \Re(-i) \int_{-\infty}^{\infty} \int_{0}^{\infty} ds \exp \left[ -i(k\|v\| + n\Omega_\alpha)s \right] \]
For the derivation of the Fokker-Planck coefficients, it was assumed that the Fourier components are not correlated at different wave vectors, i.e. the turbulence is homogeneous and in a stationary state. For the ensemble averaged electric and magnetic fields of the turbulence, the following correlation tensors were introduced:

\[<B_\alpha(k,t)B^*_\beta(k',t+s)> = \delta(k-k')P_{\alpha\beta}(k,s) \] (A.15a)
\[<E_\alpha(k,t)B^*_\beta(k',t+s)> = \delta(k-k')Q_{\alpha\beta}(k,s) \] (A.15b)
\[<B_\alpha(k,t)E^*_\beta(k',t+s)> = \delta(k-k')T_{\alpha\beta}(k,s) \] (A.15c)
\[<E_\alpha(k,t)E^*_\beta(k',t+s)> = \delta(k-k')R_{\alpha\beta}(k,s) \] (A.15d)

Note that the tensors have, additional to the dependence on the wave vector, a so far unspecified dependence in the time shift \(s\). In order to make further progress, additional assumptions have to be made concerning the nature of the turbulence, in particular with regard to \(s\).

### A.3 Correlation Tensors for Plasma Wave Turbulence

Under the assumption that the turbulence consists of transverse plasma waves, the integration in all Fokker-Planck coefficients (A.8)-(A.14) with respect to \(s\) can easily be performed, resulting in the resonance function (4.8). According to Faraday’s law, equation (4.4), the electric fields can be expressed by magnetic fields so that one obtains for the components of the electromagnetic correlation tensors (A.15b) through (A.15d) the following expressions:

\[Q^j_{RR} = \frac{i \omega^j}{ck^2} (P^j_{R||}k_R - P^j_{||}k_R)\]
\[Q^j_{LL} = \frac{i \omega^j}{ck^2} (P^j_{L||}k_L - P^j_{||}k_L)\]
\[Q^j_{RL} = \frac{i \omega^j}{ck^2} (P^j_{R||}k_L - P^j_{||}k_L)\]
\[Q^j_{LR} = \frac{i \omega^j}{ck^2} (P^j_{L||}k_R - P^j_{||}k_R)\]

\[T^j_{RR} = \frac{i \omega^j}{ck^2} (P^j_{R||}k_R - P^j_{||}k_R)\]
\[T^j_{LL} = \frac{i \omega^j}{ck^2} (P^j_{L||}k_R - P^j_{||}k_R)\]
\[T^j_{RL} = \frac{i \omega^j}{ck^2} (P^j_{R||}k_L - P^j_{||}k_L)\]
\[T^j_{LR} = \frac{i \omega^j}{ck^2} (P^j_{L||}k_R - P^j_{||}k_R)\]
\[T^j_{||} = \frac{i \omega^j}{ck^2} (P^j_{R||}k_R - P^j_{||}k_L)\]
A.4 Helical Intensities in Cartesian Coordinates

In section 4.2 of chapter 4, it is convenient to transform the helical components of \( P^j_{\alpha\beta} \) to their corresponding Cartesian counterparts summarized in the tensor \( P^j_{lm} \). For completeness, this section provides for a list of all transformed helical intensities:

\[
R^j_{RR} = \left( \frac{|\omega_j|}{ck^2} \right)^2 \left[ P^j_{RR} k^2_{||} + P^j_{R||} |k_{R}|^2 - \left( P^j_{R||} k_{L} + P^j_{R||} k_{R} \right) k_{||} \right]
\]

\[
R^j_{LL} = \left( \frac{|\omega_j|}{ck^2} \right)^2 \left[ P^j_{LL} k^2_{||} + P^j_{L||} |k_{L}|^2 - \left( P^j_{L||} k_{L} + P^j_{L||} k_{R} \right) k_{||} \right]
\]

\[
R^j_{RL} = \left( \frac{|\omega_j|}{ck^2} \right)^2 \left[ \left( P^j_{R||} + P^j_{L||} \right) k_{R} k_{||} - P^j_{RL} k^2_{||} - P^j_{LR} k^2_{||} \right]
\]

\[
R^j_{LR} = \left( \frac{|\omega_j|}{ck^2} \right)^2 \left[ \left( P^j_{L||} + P^j_{R||} \right) k_{L} k_{||} - P^j_{LR} k^2_{||} - P^j_{LR} k^2_{||} \right]
\]

\[
R^j_{R||} = \left( \frac{|\omega_j|}{ck^2} \right)^2 \left[ \left( P^j_{RL} k_{L} - P^j_{RR} k_{R} \right) k_{||} - P^j_{R||} |k_{R}|^2 + P^j_{R||} k^2_{R} \right]
\]

\[
R^j_{L||} = \left( \frac{|\omega_j|}{ck^2} \right)^2 \left[ \left( P^j_{LR} k_{R} - P^j_{LL} k_{L} \right) k_{||} - P^j_{R||} |k_{L}|^2 + P^j_{R||} k^2_{L} \right]
\]

\[
R^j_{R} = \left( \frac{|\omega_j|}{ck^2} \right)^2 \left[ \left( P^j_{RL} k_{L} - P^j_{RR} k_{R} \right) k_{||} - P^j_{R||} |k_{R}|^2 + P^j_{R||} k^2_{R} \right]
\]

\[
R^j_{L} = \left( \frac{|\omega_j|}{ck^2} \right)^2 \left[ \left( P^j_{RL} k_{L} - P^j_{LL} k_{R} \right) k_{||} - P^j_{R||} |k_{L}|^2 + P^j_{R||} k^2_{L} \right]
\]

\[
R^j_{R} = \left( \frac{|\omega_j|}{ck^2} \right)^2 \left[ \left( P^j_{RL} + P^j_{LL} \right) |k_{R}|^2 - P^j_{LR} k^2_{R} - P^j_{LR} k^2_{L} \right]
\]

A.4 Helical Intensities in Cartesian Coordinates
Appendix B
Solution of a Generalized Differential Equation

The two transport equations (7.1) and (8.2), describing the isotropic phase space evolution of PUIs and ACRs/GCRs, respectively, can be transformed to a set of two partial differential equations having the same mathematical structure. The only difference are varying signs of different constants appearing in several terms of the transformed equations. The distinguished underlying diffusion processes, i.e. momentum diffusion in the case of PUIs and spatial diffusion in the case of GCRs/ACRs, enter the calculations by the variable $y$ defined by equations (7.2) and (8.4), respectively. Since the transformed transport equations (7.3) and (8.5) are expressed in terms of this dimensionless variable, the background phase space diffusion processes are not important from a mathematical point of view.

In order to solve the transformed transport equations (7.3) and (8.5) analytically, a more generalized differential equation forms here the basis for further calculations. An outstanding feature and advantage of this generalized approach is that the corresponding general solution, may be considered as a seed equation for the specified solutions of the transport equations (7.1) and (8.2). This means that both transport equations have not to be solved separately, they only represent restrictions of this general solution. Therefore, it should also be possible to apply the general solution to other heliospheric as well as astrophysical problems, e.g. such as stochastic acceleration in accretion disks.

B.1 The Generalized Differential Equation

In order to merge both transformed transport equations (7.3) and (8.5) into one partial second-order differential equation, the following equation will be established as the basis of this appendix:

$$ y \frac{\partial^2 f}{\partial y^2} + (\alpha_1 + \epsilon y) \frac{\partial f}{\partial y} - \alpha_2 \xi(\tau) \frac{\partial f}{\partial \tau} = -Q(y, \tau) \quad (B.1) $$

Here $y$ and $\tau$ denote positive variables. The quantities $\alpha_1$ and $\alpha_2$ are arbitrary constants. One parameter of fundamental importance for applications of equation (B.1) is $\epsilon = \pm 1$. The advantage of having introduced the quantity $\epsilon$ will be become clear later. The functions $\xi(\tau)$ and $Q(y, \tau)$ in equation (B.1) are arbitrary. The latter corresponds to a transformed source function describing the injection of particles into a system of interest. Based on standard methods for solving such differential equations, the general solution for the distribution function $f(y, \tau)$ can be expressed by an integral over the Green’s function $G(y, y_0, \tau, \tau_0)$ folded with the source function

$$ f(y, \tau) = \int dy_0 \int d\tau_0 G(y, y_0, \tau, \tau_0) Q(y_0, \tau_0) \quad (B.2) $$

The Green’s function obeys the differential equation

$$ y \frac{\partial^2 G}{\partial y^2} + (\alpha_1 + \epsilon y) \frac{\partial G}{\partial y} - \alpha_2 \xi(\tau) \frac{\partial G}{\partial \tau} = -\delta(y - y_0) \delta(\tau - \tau_0) \quad (B.3) $$
Appendix B Solution of a Generalized Differential Equation

having, apart from the source function on the right-hand side, the same structure as equation (B.1). The source function is now represented by two Dirac’s delta distributions \( \delta(y-y_0) \) and \( \delta(\tau-\tau_0) \), describing a peak-like injection of particles at the coordinates \( y = y_0 \) and \( \tau = \tau_0 \). Obviously, the general solution (B.2) is characterized by the Green’s function, which is still unknown. So, in order to determine the general solution (B.2) of the generalized differential equation (B.1), the inhomogeneous second-order differential equation (B.3) for the Green’s function has to be solved. This will be described in the following section.

B.2 Construction of the Green’s Function

An outstanding and advantageous simplification of equation (B.3) can be achieved by considering it on the basis of a Laplace transformation. In order to apply a Laplace transformation, the variable \( t \) will be introduced by setting

\[
\alpha_2 \xi(\tau) \frac{\partial G}{\partial \tau} = \frac{\partial G}{\partial t} \quad \text{(B.4)}
\]

so that the new variable is given by

\[
t = \frac{1}{\alpha_2} \int_{\tau_0}^{\tau} \frac{d\tau'}{\xi(\tau')} \quad \text{(B.5)}
\]

Here, without loss of generality, it is assumed that \( t_0 = 0 \). Using the new variable \( t \), equation (B.3) can be manipulated to obtain the form

\[
y \frac{d^2 G}{dy^2} + (\alpha_1 + \epsilon y) \frac{dG}{dy} - \frac{dG}{dt} = -\delta(y-y_0)\delta(t) \quad \text{dt} \bigg|_{\tau=\tau_0} \quad \text{(B.6)}
\]

The next step consists of defining a Laplace transformed Green’s function

\[
g = g(y, y_0, s) \equiv \mathcal{L}[G(y, y_0, t)] = \int_0^\infty dt \ e^{-st} G(y, y_0, t) \quad \text{(B.7)}
\]

with \( \mathcal{L} \) and \( s \) being the Laplace transformation and variable, respectively. Using \( \mathcal{L}[\delta(t)] = 1 \) and applying the Laplace transformation with respect to \( t \), one can easily transform the partial differential equation (B.6) into an ordinary differential equation, yielding

\[
y \frac{d^2 g}{dy^2} + (\alpha_1 + \epsilon y) \frac{dg}{dy} - sg = -\delta(y-y_0) \frac{dt}{d\tau} \bigg|_{\tau=\tau_0} \quad \text{(B.8)}
\]

Inserting the ansatz \( g = y^p \exp(\mu y)h(y) \) into the homogeneous part of equation (B.8) and choosing for the free parameters \( \lambda \) and \( \mu \) the relations \( \lambda = 0 \) and \( \mu = -(1 + \epsilon)/2 \), respectively, one readily obtains for the function \( h(y) \) the following differential equation:

\[
y \frac{d^2 h}{dy^2} + (\alpha_1 - y) \frac{dh}{dy} - (d + s)h = 0 \quad \text{(B.9)}
\]

Here, the abbreviation \( d = (1 + \epsilon)\alpha_1/2 \) is introduced. Equation (B.9) is known as the confluent hypergeometric differential equation, also called Kummer’s equation, which has the solution

\[
h = C_1 M(d + s; \alpha_1, y) + C_2 U(d + s; \alpha_1, y) \quad \text{(B.10)}
\]

where \( M, U \) and \( C_1, C_2 \) denote Kummer’s functions and constants, respectively. Following standard methods, the inverse Laplace transformed Green’s function can be constructed as follows:

\[
g = -\frac{K}{p(y_0)W(y_0)} \begin{cases} 
H_1(y)H_2(y_0) & : y \leq y_0 \\
H_1(y_0)H_2(y) & : y \geq y_0
\end{cases} \quad \text{(B.11)}
\]
Here the two partial solutions \( H_1 \) and \( H_2 \) are given by
\[
H_1 = e^{-(1+\epsilon)y/2}M [d + s; \alpha_1, y] \quad \text{and} \quad H_2 = e^{-(1+\epsilon)y/2}U [d + s; \alpha_1, y]
\] (B.12)

To determine \( p(y_0) \) and \( K \), it is convenient to calculate the self-adjoint form of equation (B.8). The result is
\[
\frac{d}{dy} \left( y^{\alpha_1}e^{y} \frac{dg}{dy} \right) - s y^{\alpha_1-1}e^y g = -y_0^{\alpha_1-1}e^{y_0} \delta(y - y_0) \frac{dt}{d\tau} \bigg|_{\tau = \tau_0}
\] (B.13)
from which one readily obtains the relations \( p(y_0) = g_0^{\alpha_1} e^{y_0} \) and \( K = \left| \frac{dt/d\tau}{\tau - \tau_0} p(y_0)/y_0 \right. \). Beside these two quantities, Wronskian’s determinant is also required. Using the formula (13.1.22) by Abramowitz and Stegun [1972], \( W \) can be written as
\[
W(y_0) = H_1(y_0) \frac{d}{dy_0} H_2(y_0) - H_2(y_0) \frac{d}{dy_0} H_1(y_0) = - \frac{1}{p(y_0)} \Gamma(\alpha_1)
\] (B.14)

where \( \Gamma \) denotes the Gamma function. Upon substituting \( p(y_0) \), \( W(y_0) \), \( H_1 \) and \( H_2 \) into equation (B.11), the Laplace transformed Green’s function reads
\[
g(y, y_0, s) = K \frac{\Gamma(d + s)}{\Gamma(\alpha_1)} \exp \left[ -\frac{(1+\epsilon)(y + y_0)}{2} \right] \left\{ M [d + s; \alpha_1, y] U [d + s; \alpha_1, y_0] : y \leq y_0 \\
M [d + s; \alpha_1, y_0] U [d + s; \alpha_1, y] : y \geq y_0
\right\}
\] (B.15)

For the further treatment of this equation, it is constructive to replace Kummer’s functions \( M \) and \( U \) by Whittaker’s functions (see Abramowitz and Stegun [1972], formulas (13.1.32) and (13.1.33)). Then, one obtains for the Laplace transformed Green’s function the equation
\[
g(y, y_0, s) = K \frac{\Gamma(d + s)}{\Gamma(\alpha_1)} \frac{e^{\pi \alpha_1/2}}{(y_0)^{\alpha_1/2}} \exp \left[ -\frac{\epsilon(y + y_0)}{2} \right] \left\{ M_{\kappa-\kappa, \mu} (ye^{\pi \epsilon}) W_{\kappa-s, \mu}(y_0) : y \leq y_0 \\
M_{\kappa-\kappa, \mu} (y_0e^{\pi \epsilon}) W_{\kappa-s, \mu}(y) : y \geq y_0
\right\}
\] (B.16)

where the abbreviations \( \kappa \) and \( \mu \) are given by \( \kappa = -\epsilon \alpha_1/2 \) and \( \mu = (\alpha_1 - 1)/2 \). Having derived the Laplace transformed Green’s function \( g \), the next step consists of applying the inverse Laplace transformation, so that the initial Green’s function \( G(y, y_0, \tau, \tau_0) \) can be derived. To do so, the inverse Laplace transformation can be written as
\[
\mathcal{L}^{-1} \left[ \Gamma(d + s)M_{\kappa-\kappa, \mu}(A)W_{\kappa-s, \mu}(B) \right]
\] (B.17)
including only the relevant terms of equation (B.16), which are subjected to the Laplace transformation. Here, the quantities \( A \) and \( B \) describe, for the interval \( y \leq y_0 \), the variables \( ye^{\pi \epsilon} \) and \( y_0 \), or, for the interval \( y \geq y_0 \), \( ye^{\pi \epsilon} \) and \( y \), respectively. Making use of formula (5.20.27) of Erdélyi et al. [1954], equation (B.17) can be manipulated, so that one arrives at
\[
\mathcal{L}^{-1} \left[ \Gamma(d + s)M_{\kappa-\kappa, \mu}(A)W_{\kappa-s, \mu}(B) \right] = e^{\epsilon t} \mathcal{L}^{-1} \left[ \Gamma(1/2 + \mu + \eta)M_{\kappa, \mu}(A)W_{\kappa, \mu}(B) \right]
\] (B.18)

Taking into account all the remaining terms of equation (B.16), equation (B.18) yields, in combination with (B.16), the following inverse Laplace transformed Green’s function:
\[
G(y, y_0, t) = \frac{K}{2} (y_0)^{(1-\alpha_1)/2} e^{\epsilon t} \sinh(t/2) \exp \left[ -\frac{(y + y_0)}{2} (\epsilon + \coth(t/2)) \right] I_{\alpha_1-1} \left[ \frac{\sqrt{yy_0}}{\sinh(t/2)} \right]
\] (B.19)

Here, the Bessel function of the first kind, \( J_{\mu} \), entering formula (5.20.27) of Erdélyi et al. [1954], is replaced by a modified Bessel function of the first kind, \( I_{2\mu} \).
B.3 The General Solution

The general solution can be obtained by substituting the Green’s function (B.19) into equation (B.2). Expressing the hyperbolic functions in (B.19) by their exponential counterparts, the general solution of the differential equation (B.1) reads, after some algebra, as follows:

\[
f(x, \tau) = \frac{1}{|\alpha|^2} \int dy_0 \int d\tau_0 \frac{Q(y_0, \tau_0)}{\xi(\tau_0) \zeta(t)} h_1^{1-\alpha_1} h_2^{1+\alpha_1} \exp \left( -\frac{y_0}{2\zeta(t)} h_3 \right) I_{\alpha_1-1} \left[ \frac{2y_0}{\zeta(t)} h_1 h_2 \right] \quad (B.20)
\]

Here, for simplicity and clarity, several auxiliary functions were introduced:

\[
\begin{align*}
  h_1 &= \sqrt{y/y_0} \\
  h_2 &= \exp(-t/2) \\
  h_3 &= (1 - \epsilon)(1 + h_1^2 h_2^2) + (1 + \epsilon)(h_1^2 + h_2^2) \\
  \zeta &= 1 - h_2^2
\end{align*}
\]  (B.21-23)

Note that the parameter \( \epsilon = \pm 1 \), being of fundamental importance for the applications presented in the context of the PUI and GCR/ACR transport in the heliosphere, i.e. in chapters 7 and 8, respectively, enters the general solution (B.20) not only in the power of \( h_2 \), but also by the auxiliary function \( h_3 \). The variables \( \tau \) and \( \tau_0 \) are implicitly included via the variable \( t \) defined in equation (B.5).
Appendix C

Theory of Plasma Waves

It is commonly assumed and widely accepted that electromagnetic fluctuations in the interplanetary solar wind plasma can be described by plasma waves. Particles of a non-thermal population interact resonantly with such wave modes propagating within the solar wind background medium. The appropriate mathematical approach for describing the wave-particle interactions is based on the concept of the Fokker-Planck coefficients introduced in chapter 3. An understanding of the fundamental diffusion-convection transport equation (3.24), introduced in chapter 3, and its associated particle transport parameters (3.23a) through (3.23d) is therefore not possible without a knowledge of plasma wave properties. On the one hand, plasma waves form a contribution of an already existing turbulence affecting the phase space behavior of the particles. On the other hand, however, anisotropic particle beams in a plasma can efficiently excite plasma waves. The particles lose energy by providing for the turbulence energy and, subsequently, are scattered by the self-produced waves. The process goes on until the excitation leads to a quasi-isotropic particle beam approaching a stable state. A nice example for such a scenario is the quasi-isotropic state of cosmic rays in our galaxy, resulting from the self-production of Alfvénic turbulence via the stream-instability (Lerche, 1967). Although there is a diversity of studies and books in the literature, several concepts of the standard theory of plasma waves will be reviewed here. For more details, the interested reader is referred to several publications presenting and explaining the following aspects in a substantial as well as comprehensive form (see, e.g., Boyd and Sanderson, 1969; Stix, 1992; Benz, 1993; Gary, 1993; Sturrock, 1994; Schlickeiser, 2002).

C.1 Basic Concepts

Wave phenomena are a fundamental feature in everyday life. We communicate via acoustic waves and the human sense of direction is based on ordinary electromagnetic waves. Due to electromagnetic forces, a plasma exhibits a substantially greater diversity of wave modes. This depends on the components of which the plasma consists, their characteristic plasma frequencies, their thermal motions and two-body collisions as well as on a background magnetic field. The existence of a magnetic background field leads to additional characteristic frequencies, the so-called gyrofrequencies.

Based on these different frequencies, a plasma can roughly be divided into regimes where different assumptions apply. If the collision frequencies of the plasma components exceed the frequency of the wave, the magnetohydrodynamic (MHD) approach is valid, leading to low-frequency MHD waves. The plasma is then considered as a macroscopic fluid (i.e. the individuality of particles is neglected) for which collisions dominate and, therefore, provide for temperature effects. If the mode frequency exceeds the collision rate, a plasma particle reacts to the wave and can oscillate in its own way. Such considerations require a microscopic approach, which involves functions representing the individual behavior of the plasma components in phase space.

The fundamental concept of deriving the properties of plasma wave modes in the macroscopic as well as microscopic approach is based on the combination of both the Maxwell equations and
the Boltzmann equation. In general, they describe a small perturbation of the particle phase space distribution function \( f_\alpha(x, p, t) \), where \( \alpha \) represents a particle species of which the plasma consists. \( f_\alpha(x, p, t) \) describes the probability of finding a particle \( \alpha \) at time \( t \) at a certain position in phase space, i.e. that the particle has a momentum \( p \) at point \( x \). The small variation of the phase space distribution results from electromagnetic fields, which can be either uniform or of irregular nature, excited or produced by external sources. These fields have to be included into the Boltzmann equation via Maxwell’s equation. Assuming that no particles are lost or produced in phase space, the time evolution of \( f_\alpha \) can be considered on the basis of a conservation law, yielding

\[
\frac{df_\alpha}{dt} = \frac{\partial f_\alpha}{\partial t} + \dot{x} \cdot \nabla_x f_\alpha + \dot{p} \cdot \nabla_p f_\alpha = 0 \quad (C.1)
\]

The equations of motion, describing the particle trajectory in phase space under the influence of internal and external electromagnetic fields, result from Lorentz as well as electrical forces acting on the particle. Neglecting further energy losses, e.g. due to synchrotron radiation, the equations of motion are given by

\[
\dot{x} = v = \frac{p}{m_\alpha} \quad \text{and} \quad \dot{p} = q_\alpha \left[ E(x, t) + \frac{1}{c} v \times B(x, t) \right] \quad (C.2)
\]

The electric and magnetic fields can be produced externally by large-scale electric and magnetic background fields, i.e. \( E_0 \) and \( B_0 \), by small-scale electromagnetic components resulting from the plasma waves, i.e. \( \delta E(x, t) \) and \( \delta B(x, t) \), and, furthermore, by microscopic electric and magnetic fields, \( \Delta E \) and \( \Delta B \). The latter micro-scale contributions, which vary rapidly in space and time, are caused by neighboring particles within the so-called Debye sphere. This sphere is defined by the Debye length

\[
\lambda_D = \sqrt{\frac{k_B T_e}{4 \pi n_e e^2}} \quad (C.3)
\]

where \( k_B, T_e, n_e \) and \( e \) denote Boltzmann’s constant, the electron temperature, the electron number density and the elementary charge, respectively. Equation (C.3) represents a distance beyond which elastic two-body collisions, due to Coulomb interactions, of a test ion are shielded by the surrounding free electrons. This means that any electrostatic fields originating from the outside of this Debye sphere are screened by the electrons, so that they cannot contribute to the electric field inside. Based on the assumption of a high plasma conductivity, all large-scale electric fields are neglected, i.e. \( E_0 = 0 \). The total electric and magnetic fields are superpositions of the large-scale background magnetic field, the small-scale electromagnetic plasma wave components and the micro-scale fluctuations. Making use of the superposition concept, rewriting the equation of motion and shifting all fluctuating micro-scale contributions to the right-hand side, equation (C.1) can be rearranged to yield

\[
\frac{\partial f_\alpha}{\partial t} + \dot{x} \cdot \nabla_x f_\alpha + \dot{p} \cdot \nabla_p f_\alpha := \left( \frac{\partial f_\alpha}{\partial t} \right)_{\text{coll}} \quad (C.4)
\]

The effects due to micro-scale fluctuations are summarized in a collision term on the right-hand side. The equation of motion now reads

\[
\dot{p} = q_\alpha \left[ E_T(x, t) + \frac{1}{c} v \times B_T(x, t) \right] \quad (C.5)
\]

with \( B_T(x, t) = B_0 + \delta B(x, t) \) and \( E_T(x, t) = \delta E(x, t) \), where \( \delta B \) satisfies the weak turbulence condition, i.e. \( |\delta B| \ll |B_0| \). Equation (C.4) is the famous Boltzmann equation which finds, in combination with equation (C.5), comprehensive applications in plasma astrophysics. It forms the basis for the afore-mentioned macroscopic point of view, in which the collision term has to be taken into account, and the microscopic or kinetic approach where the collision term is neglected.
Intrinsic interactions of the plasma components are negligible if the time scales defined by the plasma frequency \( \omega_{p,\alpha} \) and the gyrofrequency \( \Omega_\alpha \) are much shorter than time scales representing collisions within the plasma (see, e.g., table 8.1 of Schlickeiser, 2002). Then, the large- and small-scale electromagnetic interactions dominate the micro-scale two-body collisions. The plasma frequency \( \omega_{p,\alpha} \) and gyrofrequency \( \Omega_\alpha \), which are defined by

\[
\omega_{p,\alpha} = \sqrt{\frac{4\pi q_\alpha^2 n_\alpha}{m_\alpha}} \quad \text{and} \quad \Omega_\alpha = \frac{q_\alpha B_0}{m_\alpha c}
\]

respectively, are characteristic for the interactions of the plasma components with the electromagnetic field of the plasma. Here, the particle properties \( q_\alpha, n_\alpha \) and \( m_\alpha \) denote the charge, the number density and the mass of plasma particles of sort \( \alpha \). In the case of an unmagnetized plasma, it is obvious that only electric interactions exist, because the gyrofrequency \( \Omega_\alpha \) vanishes.

A complete description of the plasma can only be achieved if one additionally considers Maxwell’s equations. They provide for the electromagnetic fields entering Boltzmann’s equation on the left-hand side. Considering the total probability of finding a particle at the position \( x \) at time \( t \), the number density is given by

\[
n_\alpha(x, t) = \int d^3p f_\alpha(x, p, t)
\]

Then, the four Maxwell equations read

\[
\nabla \cdot \mathbf{E}(x, t) = 4\pi \rho \\
\nabla \cdot \mathbf{B}(x, t) = 0 \\
\n\nabla \times \mathbf{B}(x, t) = \frac{1}{c} \frac{\partial \mathbf{E}(x, t)}{\partial t} + \frac{4\pi}{c} \mathbf{J}(x, t) \\
\n\nabla \times \mathbf{E}(x, t) = -\frac{1}{c} \frac{\partial \mathbf{B}(x, t)}{\partial t}
\]

where equations (C.10) and (C.11) are referred to as Ampère’s and Faraday’s laws, respectively. Here the total charge and current densities are given by

\[
\rho(x, t) = \sum_\alpha q_\alpha n_\alpha \int d^3p f_\alpha(x, p, t) + \rho_{\text{ext}}(x, t)
\]

\[
\mathbf{J}(x, t) = \sum_\alpha q_\alpha n_\alpha \int d^3p \mathbf{v} f_\alpha(x, p, t) + \mathbf{J}_{\text{ext}}(x, t)
\]

for which it is usual to assume that the external charge and current densities vanish, i.e. \( \rho_{\text{ext}} = 0 \) and \( \mathbf{J}_{\text{ext}} = 0 \), respectively. In general, the coupled set of Boltzmann’s equation and Maxwell’s equations is non-linear and, therefore, not solvable analytically. The streaming plasma components generate, via the charge and current densities (C.12) and (C.13), respectively, the electromagnetic fields \( \mathbf{E}_T \) and \( \mathbf{B}_T \) and, therefore, determine the resulting electromagnetic field according to Maxwell’s equations. Since the treatment of Boltzmann’s equation requires knowledge of exactly the same electromagnetic field that is given by the four Maxwell equations including the charge and current densities and, moreover, the phase space distribution derived from Boltzmann’s equation, it is obvious that the system of equations exhibits a strong non-linear character. In order to derive solutions of Maxwell’s equations and, therefore, to obtain information concerning the properties of plasma waves, it is commonly assumed that small-scale fluctuations perturb the particle phase space distribution slightly from a stable zeroth-order solution. In the case of vanishing disturbances,
the unperturbed distribution represents a static equilibrium solution that fulfills the Boltzmann-Maxwell equation system. This is the test wave approach in which the initial states of particle distribution functions are given, so that electromagnetic fields can be calculated. Vice versa, if initial electric and magnetic fields are given, Boltzmann’s equation enables one to consider the response of particles propagating through the plasma. This is referred to as the test particle approach. If initial conditions are known, the method for solving the equations follows a standard procedure where each equation of the non-linear set of Boltzmann-Maxwell equations is linearized with respect to the variables of the deviation. The method of linearization holds for both the MHD and the kinetic approach.

C.2 MHD Plasma Waves: Macroscopic Approach

It was already mentioned in the previous section that the MHD approach is valid only if the particle collision frequency exceeds the plasma wave frequency. The basic concept of this approach is the assumption that the plasma can be described by a continuous fluid, i.e. the individuality of the particles is lost, only their collective physical properties are retained. Within the framework of a fluid description of a plasma, the information concerning the particle phase space distribution is relinquished and replaced by quantities representing momentum averaged parameters. The macroscopic information on the fluid can be obtained by calculating the moments of the Boltzmann equation. The zeroth-order moment of equation (C.4) results in

\[ \frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{v}) = 0 \]  
(C.14)

where \( \mathbf{v} = \langle \mathbf{v} \rangle \) denotes the averaged velocity with which the fluid streams. This is generally known as the particle continuity equation. Upon multiplying equation (C.14) with \( m_\alpha \) and taking the sum with respect to \( \alpha \), one obtains the mass conservation equation

\[ \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{v}) = 0 \]  
(C.15)

where \( \rho_m = \sum_\alpha n_\alpha m_\alpha \) is the total mass density. The first-order moment of equation (C.4) yields

\[ \rho_m \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \frac{1}{c} (\mathbf{J} \times \mathbf{B}) \]  
(C.16)

where it is assumed that the fluid obeys the condition of charge neutrality. The parameter \( p = \sum_\alpha p_\alpha \) describes the total pressure as the sum of all partial pressures \( p_\alpha = n_\alpha k_BT \), where \( T \) represents the temperature of the fluid. The representation of the total pressure requires that collisions have time to equalize the temperature of all species, yielding an isotropization of the pressure. Together with the adiabatic equation of state, which relates the pressure and the mass density via \( p \rho_m^{\gamma} = \text{const.} \), equations (C.15) and (C.16) form, additionally to the Maxwell equations

\[ \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} \]  
(C.17)

\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]  
(C.18)

the set of basic equations for describing a plasma in the MHD approach where the displacement current in equation (C.17) is usually neglected. In the magnetohydrodynamic description, Ohm’s law reads

\[ \mathbf{J} = \sigma \left( \mathbf{E} + \frac{1}{c} \mathbf{V} \times \mathbf{B} \right) \]  
(C.19)
where the second term results from the relative motion of the plasma with respect to the laboratory system. The electric field can be expressed from Ohm’s law (C.19), in which the current density can be substituted by the magnetic field via Ampère’s law (C.17). Putting them into Faraday’s equation (C.18) and introducing the resistivity \( \eta = 1/\sigma \), one finds for the time evolution of the magnetic field

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) - \frac{c^2}{4\pi} \left[ \nabla \eta \times (\nabla \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \right]
\]

Neglecting the spatial variation of the resistivity, one obtains a diffusion-convection equation for the magnetic field

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) - \frac{c^2 \eta}{4\pi} \nabla^2 \mathbf{B}
\]

The first term on the right-hand side represents convection, while the second term describes diffusion of magnetic field lines through the plasma. Obviously, assuming a conductivity which may be infinite or, equivalently, considering a vanishing resistivity\(^1\), leads to a drastic simplification of equation (C.21), so that

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B})
\]

Equation (C.22) can be interpreted as a frozen-in condition for the magnetic field. Since diffusion is neglected, the magnetic field only convects with the plasma having the fluid velocity \( \mathbf{V} \). The limit of a vanishing resistivity, representing a perfectly conducting plasma, is known as ideal MHD, for which the following set of basic equations is valid:

\[
\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) = 0
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B})
\]

\[
\rho_m \left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = -\nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) (p\rho_m^\gamma) = 0
\]

Since equations (C.23) through (C.26) are non-linear, it is usually assumed that the initial plasma conditions represent a static equilibrium slightly perturbed by small-amplitude disturbances in the plasma velocity. This gives rise to small perturbations of the magnetic field, the fluid pressure and the mass density. Considering for present purposes only one component,

\[
\mathbf{B}(x, t) = \mathbf{B}_0 + \delta \mathbf{B}(x, t)
\]

where \( \mathbf{B}_0 \) and \( \delta \mathbf{B}(x, t) \) describe the initial value and the small perturbation, respectively, one obtains, on substituting (C.27) and corresponding expressions for the other slightly varying quantities \( p, \rho_m \) and \( \mathbf{V} \) into equations (C.23) through (C.26), the following linearized equations

\[
\frac{\partial \delta \rho_m}{\partial t} + \rho_m \mathbf{V} \cdot \nabla (\delta \mathbf{V}) = 0
\]

\[
\frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times (\delta \mathbf{V} \times \mathbf{B}_0)
\]

\(^1\) In a more detailed treatment, it can be shown that the conductivity is inversely proportional to the collision rate. For a fully ionized hydrogen plasma in which electron-electron encounters necessarily have to be taken into account, the conductivity behaves roughly according to \( \sigma \propto T^{3/2} \), so that a hotter plasma is a better conductor.
\[
\rho_{m,0} \frac{\partial \delta V}{\partial t} = -\nabla \delta p + \frac{1}{4\pi} (\nabla \times \delta \mathbf{B}) \times \mathbf{B}_0 \tag{C.30}
\]

\[
\frac{\delta p}{\delta \rho_m} = \gamma \frac{p_0}{\rho_{m,0}} = c_s^2 \tag{C.31}
\]

The quantity \(c_s\) is the sound speed and represents acoustic waves. As indicated by the numerator in equation (C.31), the restoring force is a gradient in pressure. Acoustic or, equivalently, sound waves are longitudinal modes with which density and pressure fluctuations propagate through the plasma. Because of their purely hydrodynamical character, sound waves do not cause any electric or magnetic disturbances, they oscillate like in a neutral gas.

Since equations (C.28) through (C.30) are linear and the unperturbed contributions are constant, it is convenient to use a Fourier transformation. On using the Fourier transformation of the set (C.28)-(C.30) and, combining the correspondingly transformed equations, one finds that

\[
\omega^2 \mathbf{u} = c_s^2 (\mathbf{k} \cdot \mathbf{u}) \mathbf{k} + \frac{1}{4\pi \rho_{m,0}} (\mathbf{k} \times (\mathbf{k} \times (\mathbf{u} \times \mathbf{B}_0))) \times \mathbf{B}_0 \tag{C.32}
\]

where \(\mathbf{u}\) denotes the Fourier transformed \(\delta \mathbf{V}\). Choosing a coordinate system in which the unperturbed background magnetic field is parallel to the z-axis, it means \(B_0 = B_0 \mathbf{e}_z\), and the wave vector lies in the x-z-plane, \(k = k(\sin \theta, 0, \cos \theta)\), equation (C.32) reads

\[
\begin{pmatrix}
  v_{ph}^2 - v_A^2 - c_s^2 \sin^2 \theta & 0 & -c_s^2 \sin \theta \cos \theta \\
  0 & v_{ph}^2 - v_A^2 \cos^2 \theta & 0 \\
  -c_s^2 \sin \theta \cos \theta & 0 & v_{ph}^2 - c_s^2 \cos^2 \theta
\end{pmatrix}
\begin{pmatrix}
  u_x \\
  u_y \\
  u_z
\end{pmatrix} = \hat{\Lambda} \cdot \mathbf{u} = 0 \tag{C.33}
\]

where \(v_{ph} = \omega/k\) and \(v_A = B_0/\sqrt{4\pi \rho_{m,0}}\) denote the phase velocity and the Alfvén speed, respectively. In order to derive the dispersion relation \(\omega\), the determinant of \(\hat{\Lambda}\) can be manipulated to obtain

\[
\text{det}[\hat{\Lambda}] = D_1 \cdot D_2 = \left[ v_{ph}^2 - v_A^2 \cos^2 \theta \right] \cdot \left[ v_{ph}^4 - \left( v_A^2 + c_s^2 \right) v_{ph}^2 + v_A^2 c_s^2 \cos^2 \theta \right] = 0 \tag{C.34}
\]

where each expression in brackets will be treated in turn.

### C.2.1 The Alfvén Wave Mode

Considering first the sub-determinant \(D_1\), one obtains the two dispersion relations

\[
\frac{\omega^2}{k^2} = v_{ph}^2 = v_A^2 \cos^2 \theta \quad \text{or} \quad \omega^2 = v_A^2 k^2_{||} \tag{C.35}
\]

where the phase velocity is independent of the wave number and, therefore, equals the group velocity \(v_g = \partial \omega/\partial k\). Dispersion relations obeying the condition \(v_{ph} = v_g\) are commonly called non-dispersive. The solutions (C.35) represent isotropic low-frequency Alfvén waves in a hot MHD fluid. In contrast to sound waves, they do not depend on the sound speed \(c_s\) and involve, via \(v_A\), only the magnetic pressure of the background magnetic field and the total mass density \(\rho_{m,0}\) of the plasma. With magnetic tension as the restoring force, Alfvén waves exist only if \(B_0 \neq 0\), they are a purely magnetohydrodynamical feature of a magnetized plasma and do not yield density and pressure disturbances. Obviously, the Alfvén dispersion relation (C.35) represents a magnetic disturbance moving parallel to the magnetic field with the Alfvén speed.
C.2.2 The Magnetosonic Wave Mode

The condition $D_2 = 0$ results in two solutions for the phase velocity, namely

$$\omega^2/k^2 = v_{ph}^2 = \frac{1}{2}(v_A^2 + c_s^2) \left[ 1 \pm \sqrt{1 - \frac{4c_s^2v_A^2 \cos^2 \theta}{(v_A^2 + c_s^2)^2}} \right]$$ (C.36)

Again, the phase velocity does not depend on the wave number, the solutions (C.36) represent non-dispersive dispersion relations. For the case of a vanishing Alfvén speed $v_A$, equation (C.36) yields $\omega = \pm c_s k$. This is the dispersion relation of the longitudinal sound wave. If a background magnetic field is present, the sound waves are modified. Equation (C.36) includes, besides the magnetic pressure represented by the Alfvén speed, an additional contribution resulting from the thermal pressure. Since (C.36) exhibits a dual character, the corresponding solutions are referred to as magnetosonic waves, where the plus and minus signs allow to distinguish between a fast and a slow wave mode, respectively. The expressions slow and fast are name tags of the mathematical solutions and do not reflect the underlying physics of the wave modes. In most astrophysical and heliospheric problems, it is often the case that the sound speed is less than the characteristic Alfvén speed, i.e. $c_s < v_A$, so that (C.36) can be reduced to

$$\omega^2 = (c_s^2 + v_A^2)k^2$$ \hspace{1cm} \text{for the fast wave mode} \hspace{1cm} (C.37)

$$\omega^2 = \frac{c_s^2v_A^2k_\parallel^2}{(c_s^2 + v_A^2)}$$ \hspace{1cm} \text{for the slow wave mode} \hspace{1cm} (C.38)

If $c_s$ vanishes completely, the shear Alfvén wave (C.35) and the fast magnetosonic mode (C.37) are the only non-vanishing wave modes in a cold MHD fluid. Since the fast magnetosonic wave behaves according to $\omega = v_A k$, which is quite similar to (C.35), it is also called the compressional Alfvén wave for $c_s = 0$. The states of polarization of both transverse waves can be derived via Faraday’s law combined with equation (C.29). One finds $\delta E \propto (0, 1, 0)$ and $\delta B \propto (1, 0, 0)$ for the shear Alfvén mode as well as $\delta E \propto (0, 1, 0)$ and $\delta B \propto (- \cos \theta, 0, \sin \theta)$ for the fast magnetosonic wave. The fast mode wave exhibits a non-vanishing contribution along the background magnetic field. This is the compressional component, which can provide for an effective energization of non-thermal particle populations.

In order to summarize the properties of the three MHD modes, it is common to illustrate the phase velocities (C.35) and (C.36) in a polar diagram, also called hodograph or Friedrich’s diagram. Figure C.1 shows such hodographs for arbitrary propagation angles with respect to the background magnetic field, given by $\theta$, and for four different values of the ratio of Alfvén speed to sound speed. Obviously, the phase velocities of the Alfvén wave (solid lines) and the slow magnetosonic wave (dashed curves) vanish at a perpendicular phase velocity. At parallel propagation, the slow mode can be a sound or an Alfvén wave, depending on the ratio of the Alfvén to the sound speed (see figure C.1(c) and C.1(a)). By comparing the slow modes in figures C.1(a) through C.1(d), it is clear that this mode vanishes for the cold plasma limit, so that the fast and the Alfvén modes are the only remaining waves. For parallel propagation, the fast magnetosonic wave (dotted curves) becomes purely transverse. With increasing inclination to the magnetic field, the fast mode is neither purely longitudinal nor transverse, it is a mixture of both. Approaching $\theta = \pi/2$, the polarization of the fast magnetosonic wave switches to a purely longitudinal state. In the case of parallel direction and $v_A \leq c_s$ or $v_A \geq c_s$, the fast mode is identical to the sound or the Alfvén wave, respectively.
Appendix C Theory of Plasma Waves

(a) The case $v_A/c_s = 1/\sqrt{2}$

(b) The case $v_A/c_s = 1$

(c) The case $v_A/c_s = \sqrt{2}$

(d) The case $v_A/c_s = 4$

Figure C.1: Polar diagrams of vector phase velocities given by the solutions (C.35) and (C.36) for four different ratios of the Alfvén and sound speed. Solid curves reflect the two solutions of the shear Alfvén wave modes (C.35), while dashed and dotted lines correspond to the slow and the fast magnetosonic waves given by the equation (C.36). Note that the phase velocities are normalized with respect to $\sqrt{v_A^2 + c_s^2}$. 
C.3 Kinetic Plasma Waves: Microscopic Approach

The MHD description of a plasma is valid only for the case that the frequencies of plasma waves are much smaller than collision frequencies represented by the scattering term on the right-hand side of equation (C.4). As it was shown, this fluid approach provides for low-frequency disturbances in the plasma. In contrast to the MHD approach, this section considers the case where wave frequencies exceed the collision rates. From this point of view, the individuality of plasma components becomes important, since the particles are allowed to interact with the waves. In order to include individual particle motions, one has to go back to Boltzmann’s equation (C.4) in which, from now on, the collision term is neglected. So, the following equation forms the basis for the microscopic approach:

$$\frac{\partial f_\alpha}{\partial t} + \dot{x} \cdot \nabla_x f_\alpha + \dot{p} \cdot \nabla_p f_\alpha = 0 \quad (C.39)$$

Equation (C.39) is, in combination with the equation of motion (C.5), referred to as Vlasov’s or collisionless Boltzmann equation. Together with Maxwell’s equations (C.8) through (C.11) and the charge and current densities (C.12) and (C.13), respectively, Vlasov’s equation (C.39) forms the appropriate fundamental basis of the kinetic point of view. As in the MHD approach, the equation system is strongly non-linear. A further treatment of the Vlasov-Maxwell equation set can be enabled if one assumes that small disturbances perturbate the initial equilibrium state slightly. The system of equations can then be linearized.

C.3.1 Linearized Vlasov-Maxwell Equations

Based on the ansatz $f_\alpha(x, p, t) = f_\alpha^{(0)}(x, p, t) + \delta f_\alpha(x, p, t)$, where $f_\alpha^{(0)}$ and $\delta f_\alpha$ denote the stable and the perturbed phase space distribution contribution, respectively, the standard linearization method can be applied to the Vlasov-Maxwell equations. Neglecting all terms higher than the first-order in the small quantities, one obtains for the linearized version of Vlasov’s equation the form

$$\frac{\partial \delta f_\alpha}{\partial t} + v \cdot \nabla_x \delta f_\alpha + \dot{p} \cdot \nabla_p \delta f_\alpha = -q_\alpha \left( \delta E + \frac{1}{c} v \times \delta B \right) \cdot \nabla_p f_\alpha^{(0)} \quad (C.40)$$

where the time derivative of the particle momentum now reflects a trajectory in an unperturbed plasma, i.e. $\dot{p} = q_\alpha [v \times B_0(x, t)]/c$. The linearized Vlasov equation (C.40) couples the properties of the perturbative quantities to the equilibrium state. Since the left-hand side of equation (C.40) is integrable along the unperturbed trajectory, it can be shown via, Taylor’s expansion, that the associated formal solution for $\delta f_\alpha$ is given by a functional depending linearly in $\delta E(x, t)$ and $\delta B(x, t)$. Similarly, one can linearize Ampère’s and Faraday’s law, i.e. equations (C.10) and (C.11), respectively. This yields

$$\nabla \times \delta B = \frac{1}{c} \frac{\partial \delta E}{\partial t} + \frac{4\pi}{c} \delta J \quad (C.41)$$

$$\nabla \times \delta E = -\frac{1}{c} \frac{\partial \delta B}{\partial t} \quad (C.42)$$

where the fluctuating current density behaves according to

$$\delta J(x, t) = \sum_\alpha q_\alpha n_\alpha \int d^3p v_\alpha \delta f_\alpha(x, p, t) := \sigma \cdot \delta E \quad (C.43)$$

Equation (C.43) can be expressed, based on Faraday’s law and since $\delta f_\alpha$ is linear in $\delta E(x, t)$ and $\delta B(x, t)$, by the conductivity tensor $\sigma$ accompanied by the perturbative electric field $\delta E(x, t)$. The right-hand side of equation (C.43), which is introduced per definition, is referred to as the generalized Ohm’s law. It enables one to derive, via (C.41) and (C.42), a relation between the frequency and wavenumber of a plasma wave mode, the dispersion relation.
C.3.2 The Plasma Wave Dispersion Relation

The general dispersion relation and polarization properties of a plasma wave mode can be obtained by rearranging the set of Maxwell equations. In order to do so, a Fourier-Laplace representation for all fluctuating quantities, i.e., for $\delta\mathbf{E}(\mathbf{x}, t)$, $\delta\mathbf{B}(\mathbf{x}, t)$ and $\delta f_i(\mathbf{x}, t)$, is commonly used, requiring that the background plasma is considered to be homogeneous and stationary. The perturbative components are the only quantities varying locally and linearly in space and time. So, in order to get solutions for the linearized Maxwell equations, a Fourier-Laplace transformation is applied to equations (C.41) and (C.42). The Fourier transformation is defined by

$$\delta\mathbf{E}(\mathbf{x}, t) = \int d^3k \, \delta \hat{\mathbf{E}}(\mathbf{k}, t) e^{ik\mathbf{x}} \quad (C.44)$$

where $\delta \hat{\mathbf{E}}(\mathbf{k}, t)$ is given by the Laplace representation

$$\delta\mathbf{E}(\mathbf{k}, \omega) = \int_0^\infty dt \, \delta \hat{\mathbf{E}}(\mathbf{k}, t) e^{i\omega t} \quad (C.45)$$

where $\omega = \omega_R + i\omega_I$ is a complex plasma wave frequency for which the imaginary part $\omega_I = \Gamma$ is sufficiently large to provide for the convergence of the Laplace transformation. Equations (C.41) and (C.42) can then be manipulated to obtain

$$i\mathbf{c} \times \delta\mathbf{E}(\mathbf{k}, \omega) = i\omega \delta\mathbf{B}(\mathbf{k}, \omega) + \delta \hat{\mathbf{B}}(\mathbf{k}, t = 0) \quad (C.46)$$

$$i\mathbf{c} \times \delta\mathbf{B}(\mathbf{k}, \omega) = 4\pi \delta\mathbf{J}(\mathbf{k}, \omega) - i\omega \delta\mathbf{E}(\mathbf{k}, \omega) - \delta \hat{\mathbf{E}}(\mathbf{k}, t = 0) \quad (C.47)$$

Combining equation (C.46) with (C.47) and using the generalized Ohm’s law results in a single wave equation for $\delta\mathbf{E}(\mathbf{k}, \omega)$

$$\left(\frac{k^2 c^2}{\omega^2} - 1 - \frac{4\pi i}{\omega} \sigma \right) \delta\mathbf{E}(\mathbf{k}, \omega) - \frac{c^2}{\omega^2} (\mathbf{k} \cdot \delta\mathbf{E}(\mathbf{k}, \omega)) \mathbf{k} = \frac{i}{\omega} \mathbf{c} \times \delta \hat{\mathbf{B}}(\mathbf{k}, t = 0) - \frac{1}{\omega} \delta \hat{\mathbf{E}}(\mathbf{k}, t = 0) \quad (C.48)$$

Introducing the Maxwell operator

$$\Lambda_{rs} = N^2 \frac{k_r k_s}{k^2} + (1 - N^2) \delta_{rs} + \frac{4\pi i}{\omega} \sigma_{rs} \quad (C.49)$$

where $N^2 = k^2 c^2/\omega^2$ and $\delta_{rs}$ denote the refractive index and the Kronecker symbol, respectively, equation (C.48) reads

$$\Lambda \cdot \delta\mathbf{E}(\mathbf{k}, \omega) = \mathbf{A}(\mathbf{k}, \omega) \quad (C.50)$$

where

$$\mathbf{A}(\mathbf{k}, \omega) = \frac{i}{\omega} \delta \hat{\mathbf{E}}(\mathbf{k}, t = 0) - \frac{c}{\omega^2} \mathbf{k} \times \delta \hat{\mathbf{B}}(\mathbf{k}, t = 0) \quad (C.51)$$

is given by the initial conditions of the perturbation. Equation (C.50) can be solved to obtain

$$\delta\mathbf{E}(\mathbf{k}, \omega) = \Lambda^{-1} \cdot \mathbf{A}(\mathbf{k}, \omega) \quad (C.52)$$

where the inverse of the Maxwell operator is given by

$$\Lambda^{-1} = \frac{1}{\det[A]} \begin{pmatrix} A_{22} A_{33} - A_{23} A_{32} & A_{31} A_{23} - A_{21} A_{33} & A_{21} A_{32} - A_{31} A_{22} \\ A_{32} A_{13} - A_{12} A_{33} & A_{11} A_{33} - A_{31} A_{13} & A_{12} A_{31} - A_{32} A_{11} \\ A_{12} A_{23} - A_{13} A_{22} & A_{21} A_{13} - A_{11} A_{23} & A_{11} A_{22} - A_{12} A_{21} \end{pmatrix} \quad (C.53)$$
The inverse Laplace transformation results then, together with equation (C.52), in

$$\delta \hat{E}(k, t) = \frac{1}{2\pi} \int_C d\omega \Lambda^{-1} \cdot A(k, \omega) e^{-i\omega t}$$  (C.54)

where both the numerator and the denominator are entire functions of $\omega$. The contour $C$ describes an integration path passing above all singularities of $\delta E(k, \omega)$, for which $\Lambda^{-1}$ and $A$ are not analytical. The singularities are given, according to equation (C.52), by the condition

$$\det[\Lambda] = 0$$  (C.55)

This means that each singularity represents a plasma wave mode having the frequency $\omega = \omega_p$, with $p = 1, \ldots, N$. Here, $N$ denotes the number of possible wave modes having dispersion relations given by the condition (C.55). Waves obeying (C.55) are also called normal modes. It was Landau [1946] who first proposed to carry out the inverse Laplace transformation along a deformed integration path $C'$ instead of the originally specified contour $C$. Based on Cauchy’s integral and the residual theorem, the integral in equation (C.54) can be expressed by residues at the poles of $1/\Lambda$ at $\omega$, leading to

$$\delta \hat{E}(k, t) \propto \sum_{p=1}^{N} e^{-i\omega_{p} t + i\Gamma_{p}}$$  (C.56)

Here, the cases $\Gamma < 0$ and $\Gamma > 0$ describe sinusoidal fluctuations which amplitudes can decrease and increase with time, respectively. The first case, where the wave amplitude decreases in time, is credited to Landau, the so-called Landau damping. The damping is not associated with collisions, it results from a transfer of wave energy into the oscillation energy of resonant plasma particles. The case $\Gamma > 0$ represents instabilities triggered by several mechanisms, for example by particle beams.

As it is given by equation (C.55), the prescription for determining mode dispersion relations is to calculate $\det[\Lambda] = 0$, leading, in general, to a polynomial equation for $\omega$ of order $N$. Of course, the polynomial equation, and its corresponding solutions, depends on the structure of the Maxwell operator (C.49) determined by the so far unknown conductivity tensor $\sigma$. Following Schlickeiser [2002], the components of the conductivity tensor $\sigma$ can be calculated by integration of the linearized Vlasov equation (C.40) and, furthermore, by using the generalized Ohm’s law (C.43), which has to be subjected to the Fourier-Laplace transformation. Introducing cylindrical momentum coordinates and restricting the wave vector to the x-z-plane, i.e. $p_x = p_{\perp} \cos \phi$, $p_y = p_{\perp} \sin \phi$, $p_z = 0$ and $k = (k_\perp, 0, k_\parallel)$, respectively, one obtains for the conductivity tensor in a hot magnetized plasma the representation

$$\sigma_{rs}(k_\parallel, k_\perp, \omega) = \frac{1}{2t} \sum_\alpha \omega_{p,\alpha}^2 \int_{-\infty}^{+\infty} dp_{\parallel} \int_0^{+\infty} dp_{\perp} \sum_{n=-\infty}^{+\infty} \frac{Q_{rs}}{\omega - k_\parallel v_{||} - n\Omega_{\alpha}}$$  (C.57)

where $\omega_{p,\alpha}$ and $\Omega_{\alpha}$ denote the characteristic plasma frequency and the gyrofrequency of the plasma component $\alpha$. The tensor $Q_{rs}$ is split into two different terms,

$$Q_{rs} = T_{rs}^T \hat{U} f_\alpha^{(0)} + T_{rs}^L \hat{\nu} f_\alpha^{(0)}$$  (C.58)

where the tensor $T_{rs}^T$ is given by

$$T_{rs}^T = \begin{pmatrix}
    n^2 J_n^2(y) p_{\perp} / y^2 & m J_n(y) J_n^\prime(y) p_{\perp} / y & 0 \\
    -m J_n(y) J_n^\prime(y) p_{\perp} / y & [J_n(y)]^2 p_{\perp} & 0 \\
    n J_n^2(y) p_{\parallel} / y & 1 J_n(y) J_n^\prime(y) p_{\parallel} & 0
\end{pmatrix}$$  (C.59)
with \( J_n(y) \) and \( J'_n(y) \) being a Bessel function of the first kind and of the order \( n \) and its first derivative with respect to \( y = k_\perp v_\perp / \Omega_\alpha \), respectively. Note that the third column vanishes, pointing out that \( T_{rs}^L \) exhibits a transverse feature, whereas the tensor

\[
T_{rs}^L = \begin{pmatrix}
0 & 0 & n J_2^2(y) p_\perp / y \\
0 & 0 & -i J_n(y) J'_n(y) p_\perp \\
0 & 0 & J_2^2(y) p_\parallel
\end{pmatrix}
\]

in which the the third column is the only non-vanishing, is related to longitudinal contributions. The operators

\[
\hat{U} = \frac{\partial}{\partial p_\perp} + \frac{k_\parallel}{\gamma m_\alpha \omega} \left( p_\perp \frac{\partial}{\partial p_\parallel} - p_\parallel \frac{\partial}{\partial p_\perp} \right)
\]

and

\[
\hat{V} = \frac{\partial}{\partial p_\parallel} - \frac{n \Omega_\alpha}{p_\perp \omega} \left( p_\perp \frac{\partial}{\partial p_\parallel} - p_\parallel \frac{\partial}{\partial p_\perp} \right)
\]

entering the tensor \( Q_{rs} \) by the transverse and the longitudinal contribution, respectively, act on the equilibrium zeroth-order distribution \( f_\alpha^{(0)} \) depending only on \( p_\parallel \) and \( p_\perp \). It is noteworthy that the derivatives in the parentheses of equations (C.61) and (C.62) can be expressed by a derivative with respect to the particle pitch angle. Hence, the operators \( \hat{U} \) and \( \hat{V} \) include a measure of pitch angle anisotropy vanishing for isotropic plasmas.

Equation (C.55) provides, in combination with the Maxwell operator (C.49) and the conductivity tensor (C.57), for general dispersion relations of plasma waves propagating arbitrarily in a hot magnetized plasma, in which the particles are given by an arbitrary initial phase space distribution \( f_\alpha^{(0)} \). In general, plasma waves will be a mixture of both the transverse and the longitudinal contributions. Within the framework of the MHD approach considered above, this was already described for low-frequency fast magnetosonic waves, i.e. equation (C.36). Only for a parallel or perpendicular direction of propagation, the polarization of the fast mode wave is in a purely transverse or longitudinal state, respectively. In order to avoid the coupling of the two polarization states, the following considerations are restricted to plasma waves propagating only in parallel direction with respect to the uniform background magnetic field.

### C.3.3 The Limit of Parallel Wave Propagation

The advantage of restricting the considerations to parallel propagating waves is the following: the argument \( y \) vanishes and, subsequently, all Bessel functions can be reduced to a simple expression given by \( \delta_{n0} \). In other words, the Bessel function of zeroth-order is the only contributing function. For the further treatment, it is convenient to introduce for the unity vectors the helical representation \( \mathbf{e}_\pm = \mathbf{e}_z \pm i \mathbf{e}_y \). Equation (C.50) can then be rearranged. One arrives, after straightforward algebra, at

\[
\left[ 1 - N^2 + \pi \sum_\alpha \frac{\omega_\perp \omega_\parallel}{\omega} \int_{-\infty}^{\infty} dp_\perp \int_{0}^{\infty} dp_\parallel \frac{p_\perp}{\omega - k_\parallel v_\parallel - \Omega_\alpha} \hat{U} f_\alpha^{(0)} \right] \mathbf{e}_+ \cdot \delta \mathbf{E} = \mathbf{e}_+ \cdot \mathbf{A}
\]

\[
\left[ 1 - N^2 + \pi \sum_\alpha \frac{\omega_\perp \omega_\parallel}{\omega} \int_{-\infty}^{\infty} dp_\perp \int_{0}^{\infty} dp_\parallel \frac{p_\perp}{\omega - k_\parallel v_\parallel + \Omega_\alpha} \hat{U} f_\alpha^{(0)} \right] \mathbf{e}_- \cdot \delta \mathbf{E} = \mathbf{e}_- \cdot \mathbf{A}
\]

\[
\left[ 1 + 2 \pi \sum_\alpha \frac{\omega_\perp \omega_\parallel}{\omega} \int_{-\infty}^{\infty} dp_\perp \int_{0}^{\infty} dp_\parallel \frac{p_\perp}{\omega - k_\parallel v_\parallel} \frac{\partial f_\alpha^{(0)}}{\partial p_\parallel} \right] \mathbf{e}_z \cdot \delta \mathbf{E} = \mathbf{e}_z \cdot \mathbf{A}
\]
C.3 Kinetic Plasma Waves: Microscopic Approach

Apparently, the expressions in brackets appearing on the left-hand sides of equations (C.63), (C.64) and (C.65) represent, if set equal to zero, dispersion relations of waves having amplitudes aligned along e_+, e_- and e_z, respectively. The first two equations describe transverse plasma waves, while the last one represents the longitudinal mode. They are not coupled for the case of parallel propagation in a hot magnetized plasma. Since $T_{33}^L$ is non-vanishing for $n = 0$, it is obvious that the measure of particle pitch angle anisotropy does not occur in the dispersion relation for longitudinal modes, even for anisotropic distribution functions.

C.3.4 Transverse Fluctuations

Restricting the following considerations to transverse fluctuations, the expressions in brackets of equations (C.63) and (C.64) can be rearranged and summarized to yield

$$D(k, \omega_R, \omega_I) = 0 = \omega^2 - c^2 k^2_{\parallel} + \pi \sum_\alpha \omega^2_{p,\alpha} \int_{-\infty}^{\infty} dp_{\parallel} \int_{0}^{\infty} dp_{\perp} \frac{p_{\perp}^2}{\omega - k_{\parallel} v_{\parallel} \pm \Omega_\alpha} \left[ \omega \left( \frac{\partial f^{(0)}_{\alpha}}{\partial p_{\perp}} + \frac{k_{\parallel} v_{\parallel}}{p_{\parallel}} \left( \frac{\partial f^{(0)}_{\alpha}}{\partial p_{\perp}} - \frac{\partial f^{(0)}_{\perp}}{\partial p_{\perp}} \right) \right) \right] \tag{C.66}$$

Equation (C.66) describes the dispersion curve $\omega_R$ as well as the damping/growth rate $\omega_I = \Gamma(k_{\parallel})$ of transverse plasma waves in a hot plasma. For $|\Gamma| \ll |\omega_R|$, the so-called weak damping limit, one may approximate in equation (C.66) the parallel momentum integration by the Plemelj formula

$$\int_{-\infty}^{\infty} dx \frac{g(x)}{x - z} = \lim_{z \to 0} \left[ \int_{-\infty}^{x - z} + \int_{x + z}^{\infty} \right] dx \frac{g(x)}{x - z} + i \pi g(z) \bigg|_{x = z_R} \tag{C.67}$$

where $g(x)$ is an arbitrary function of the real variable $x$, while $z = z_R + i \omega_I$ denotes a complex parameter. The last term on the right-hand side selects the residue given by the resonance condition $x = z_R$. Using equation (C.67), the dispersion relation (C.66) can be separated into an imaginary part

$$\Im D = -\pi^2 \sum_\alpha \omega^2_{p,\alpha} \int_{0}^{\infty} dp_{\perp} p_{\perp}^3 \left[ \frac{\partial f^{(0)}_{\alpha}}{\partial p_{\perp}} + \frac{\Omega_\alpha}{k_{\parallel} v_{\parallel}} \frac{\partial f^{(0)}_{\perp}}{\partial p_{\perp}} \right] \bigg|_{p_{\parallel} = (\omega_R \pm \Omega_\alpha) m_{\alpha}/k_{\parallel}} \tag{C.68}$$

and a real part, $\Re D$, given by equation (C.66) for $\Gamma = 0$, i.e. $\omega = \omega_R$. Since $|\Gamma| \ll |\omega_R|$, which is not always satisfied, approximate solutions can be derived by using a Taylor expansion for the imaginary part around $\Gamma = 0$, yielding

$$\Im D \simeq \Im D + \Gamma \frac{\partial \Im D}{\partial \Gamma} \approx 0 \tag{C.69}$$

$D$ is an analytic function of the complex variable $\omega$ and, therefore, satisfies the Cauchy-Riemann relation

$$\frac{\partial \Im D}{\partial \Gamma} = \frac{\partial \Re D}{\partial \omega_R} \tag{C.70}$$

Substitution of (C.70) into equation (C.69) yields

$$\Gamma = -\frac{\Im D(k_{\parallel}, \omega_R, \Gamma = 0)}{\partial \Re D(k_{\parallel}, \omega_R, \Gamma = 0)/\partial \omega_R} \tag{C.71}$$

Equations (C.66), (C.68) and (C.71) are defined by the so far arbitrary initial distribution function. Therefore, a further treatment requires a $f^{(0)}_{\alpha}$, for which the integrals in (C.66) have to be evaluated. Since the zeroth-order functions must satisfy the zeroth-order Vlasov equation (C.39), they have
to be independent of the azimuthal coordinate $\phi$ in cylindrical momentum coordinates, implying that $f^{(0)}_\alpha$ can only depend on the perpendicular momentum as $p_\perp^2$. Furthermore, in order to simplify calculations, it is often assumed that the initial distribution function can be separated into partial momentum distributions defined with respect to the orientation of the uniform background magnetic field, i.e. $f^{(0)}_\alpha(p_\parallel, p_\perp) = f^{(0)}_\alpha(p_\parallel) f^{(0)}_\alpha(p_\perp)$. A standard function, satisfying the condition of separation, is the isotropic Maxwellian distribution, where particles have the same temperature in both directions. Under certain circumstances, however, several plasma constituents can exhibit substantial temperature anisotropies (see, e.g., Feldmann, 1979; Marsch et al., 1982; Toptygin, 1983). An often used distribution, allowing to include different particle temperatures in perpendicular and parallel direction, is the two-temperature bi-Maxwellian zeroth-order distribution

$$f^{(0)}_\alpha(p_\parallel, p_\perp) = \frac{1}{\pi^{3/2} a_\parallel,\alpha a_\perp,\alpha^2} \exp \left( -\frac{p_\parallel^2}{a_\parallel,\alpha^2} - \frac{p_\perp^2}{a_\perp,\alpha^2} \right)$$

where $a_\parallel,\alpha = \sqrt{2 m_\alpha k_B T_\parallel,\alpha}$ and $a_\perp,\alpha = \sqrt{2 m_\alpha k_B T_\perp,\alpha}$, with $T_\parallel,\alpha$ and $T_\perp,\alpha$ being the respective particle temperatures related to the parallel and perpendicular direction. However, equation (C.72) is a strongly idealized distribution function. Detailed analysis of data obtained by observations on spacecraft revealed non-Maxwellian high energy tails of electron and ion distributions in the Earth’s magnetosphere (Vasyliunas, 1968; Lui and Krimigis, 1983; Williams et al., 1988), in the solar wind (Abraham-Shrauner and Feldman, 1977; Gosling et al., 1981, Marsch et al., 1982) and near Jupiter (Leubner, 1982) as well as Saturn (Armstrong et al., 1983). The existence of such high energy tails can be described as follows: the particle mean free path is, in perpendicular direction, limited by the gyroradius of the particle, whereas in parallel direction along the ambient magnetic field no such restriction exists. Therefore, the efficiency for energization of particles along the magnetic field is higher than in the direction normal to the magnetic field, resulting in the formation of the tails. To take into account the non-Maxwellian feature of observed particle distributions at high energies, Thorne and Summers [1991] and Summers and Thorne [1992] extended and modified the classical approach for deriving dispersion relations on the basis of equation (C.72). Instead of using a bi-Maxwellian function, they used a so-called Kappa function. This distribution enables one to consider the influence of the high energy tails on plasma wave dispersion relations via a certain parameter, namely the index $\kappa$. This index represents the relative importance of the distribution tail. Performing numerical as well as analytical calculations, Summers and Thorne [1991, 1992] have shown that the influence of a different initial distribution function can be of importance for the properties of plasma waves, in particular for the damping and growth in a hot plasma. As it has been pointed out in several studies (Vasyliunas, 1968; Leubner, 1983; Summers and Thorne, 1992), the Kappa distribution contains the bi-Maxwellian distribution, i.e. the classical approach, as the asymptotic case $\kappa \to \infty$. More recently, Fichtner and Sreenivasan [1993] presented a more general approach than that by Thorne and Summers [1991] and Summers and Thorne [1992]. In order to study the parallel wave propagation in a hot non-Maxwellian plasma, they used polynomial distribution functions instead of Kappa distributions, allowing for a more general treatment of both equilibrium and non-equilibrium plasmas. However, since the solar wind plasma is observed to be Maxwellian-like to a good approximation, it is sufficient to use for the following treatment the bi-Maxwellian (C.72).

$^2$ Note that this restriction is only necessary if the plasma is magnetized, because it results from the Lorentz force term on the left-hand side of equation (C.39). In a magnetic field-free, uniform plasma, any function of $p_\parallel$, $p_\perp$ and $p_e$ satisfies the zeroth-order Vlasov equation. However, the independence of $f^{(0)}_\alpha$ with respect to $\phi$ was already used in the derivation of the conductivity tensor (C.57) where the integration over $\phi$ was already carried out. Hence, including different particle properties in three dimensions requires another calculation of the conductivity tensor (C.57).
Another interesting aspect, which can arise in space plasma physics, is the injection of a relatively tenuous beam of energetic particles into a thermal plasma of higher density. An example for this was already mentioned on page 153 of this chapter, namely cosmic rays exciting plasma waves in the interstellar, or interplanetary, plasma. This scenario is quite similar to the case where two components of a certain plasma constituent have comparable densities but different temperatures, so that one of the components drifts with respect to the other. This relative drift provides for a free energy pool from which instabilities can be produced. Such growing plasma wave modes are not considered here but, however, in order to generalize the approach to dispersion relations, the following drifting two-temperature bi-Maxwellian distribution is used for further considerations:

\[ f^{(0)}_{\alpha}(p||, p_\perp) = \frac{1}{\pi^{3/2}a_{\perp,\alpha}^2a_{\perp,\alpha}^2} \exp \left( -\frac{(p|| - P_{\parallel,\alpha})^2}{a_{\perp,\alpha}^2} - \frac{p_\perp^2}{a_{\perp,\alpha}^2} \right) \]  

(C.73)

Here, \( P_{\parallel,\alpha} \) denotes the relative drift momentum of the component \( \alpha \), which can be directed parallel or antiparallel with respect to the magnetic field. Substituting the distribution (C.73) into equation (C.66), making use of the integral

\[ \int_0^\infty dp_\perp p_\perp^n \exp(-p_\perp^2/a_{\perp,\alpha}^2) = \frac{\Gamma((n+1)/2)}{2a_{\perp,\alpha}^{n+1}} \]  

(C.74)

where \( \Gamma(x) \) denotes the Gamma function, and, furthermore, introducing for the integral with respect to the parallel contribution the standard plasma dispersion function (Fried and Conte, 1961)

\[ Z(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty dx \exp(-x^2) \]  

(C.75)

the real part of equation (C.66) can be expressed as follows:

\[ \Re D(k||, \omega_R) = \omega_R^2 - c^2 k^2 + \sum_\alpha \omega_{\perp,\alpha}^2 \left( \xi^{(0)}_\alpha Z(\xi^{(1)}_\alpha) - A_\alpha \left[ 1 + \xi^{(1)}_\alpha Z(\xi^{(1)}_\alpha) \right] \right) = 0 \]  

(C.76)

Here, the parameter \( A_\alpha = 1 - T_{\perp,\alpha}/T_{\parallel,\alpha} \) is a measure of the thermal anisotropy of the bulk of plasma particles. The complex quantities \( \xi^{(0)}_\alpha \) and \( \xi^{(1)}_\alpha \) are given by

\[ \xi^{(j)}_\alpha = \frac{(\omega_R - k|| V_{||,\alpha} \pm j \Omega_\alpha)}{k|| v_{th,||}^\alpha} \]  

(C.77)

where \( V_{||,\alpha} = P_{||,\alpha}/m_\alpha \) and \( v_{th,||}^\alpha = \sqrt{2k_B T_{||,\alpha}/m_\alpha} \) are the relative drift velocity and the parallel thermal speed of particles along the ambient magnetic field, respectively. The corresponding damping and growth rates of the modes, which are determined by (C.76), can be estimated by inserting the drifting bi-Maxwellian (C.73) into (C.71). One obtains, in agreement with Gary [1993],

\[ \Gamma = \text{sign}(k||) \sqrt{\pi} \left[ \frac{\partial \Re D(k||, \omega_R)}{\partial \omega_R} \right]^{-1} \sum_\alpha \frac{\omega_{\perp,\alpha}^2}{k|| v_{th,||}^\alpha} \exp \left[ -\frac{(\omega_R \pm \Omega_\alpha - k|| V_{||,\alpha})^2}{(k|| v_{th,||}^\alpha)^2} \right] \]

\[ \times \left[ \frac{T_{\perp,\alpha}}{T_{||,\alpha}} (k|| V_{||,\alpha} - \omega_R) \pm A_\alpha \Omega_\alpha \right] \]  

(C.78)

Note that the drift momentum is only included into the parallel contribution, since the initial distribution has to be independent of the azimuthal coordinate \( \phi \).

Remember that the classical plasma dispersion function is based on the Maxwellian distribution (C.73). For the generalization of the dispersion function to a modified plasma dispersion function, which can be derived from the Kappa distribution, see Thorne and Summers [1991] and Summers and Thorne [1992].
Equation (C.78) provides for the damping or growth rates of parallel propagating transverse fluctuations in a plasma, consisting of particles with an initial phase space behavior given by (C.73). From this expression it is clear that the sign of \( \Gamma \) will change for certain magnitudes of the drift velocity \( V_{\parallel,\alpha} \) and/or for different ratios of perpendicular to parallel particle temperatures. The derivative of the real part with respect to \( \omega_R \) has to be carried out by using (C.76). Since equation (C.76) includes the complex plasma dispersion function, a variety of numerical solutions were presented in the past (see, e.g., Kennel and Scarf, 1968; Gary et al., 1976, 1984, 1985; Gary and Feldman, 1978; Gary, 1993). However, using series representations of the plasma dispersion function, one can derive simple approximations of the general dispersion relation (C.76), which then represents transverse modes in certain frequency regimes. It is often assumed, for simplicity, that the sound speed is small enough, so that the argument of the plasma dispersion function satisfies \(|\xi| > 1\). This approximation requires non-resonant particles, i.e. \( \omega_R - k_\parallel V_{\parallel,\alpha} \pm j\Omega_\alpha \neq 0 \). Using the series expansion

\[
Z(\xi) \simeq i\sqrt{\pi}\sigma \exp(-\xi^2) - \frac{1}{\xi} \left[ 1 + \frac{1}{2\xi^2} + \frac{3}{4\xi^4} + \cdots \right]
\]  

where \( \sigma = 0, 1 \) and 2 corresponds to the cases \( \Re\xi > 0 \), \( \Re\xi = 0 \) and \( \Re\xi < 0 \), respectively, and dropping the subscript \( R \), equation (C.76) leads to the following algebraic expression:

\[
\omega^2 \simeq c^2 k_\parallel^2 + \sum_{\alpha} \omega_{p,\alpha}^2 \left[ \frac{\omega - k_\parallel V_{\parallel,\alpha}}{\omega - k_\parallel V_{\parallel,\alpha} \pm \Omega_\alpha} - \frac{A_{\alpha}}{2} \frac{k_{\parallel,\alpha}^2}{(\omega - k_\parallel V_{\parallel,\alpha} \pm \Omega_\alpha)^2} \right]
\]  

The first term in brackets represents the cold plasma limit, while the second contribution describes a thermal correction term and includes the measure of temperature anisotropy. Equation (C.80) can be described as a seed equation for a variety of different wave dispersion relations. The relative drift velocity \( V_{\parallel,\alpha} \) and the temperature anisotropy \( A_\alpha \) allow to consider instabilities, i.e. \( \Gamma > 0 \), evolving out of stable waves. The corresponding growth rates of the instabilities can be estimated by using the expression (C.78). Gary [1993] has investigated temperature anisotropy and drift instabilities in detail by solving equations (C.76) and (C.78) numerically for many scenarios, e.g. for an electron-ion plasma where the ions are represented by a single Maxwellian distribution and the electrons are given by a bi-Maxwellian, and vice versa. Since instabilities are not considered here in detail, the interested reader is referred to his monograph on a variety of microinstabilities.

A further analytical simplification of (C.80) can only be achieved, if additional assumptions are made: first, it is assumed that the plasma consists of only electrons and protons. Furthermore, in order to avoid drift instabilities, relative drift velocities are neglected, i.e. \( V_{\parallel,p} = V_{\parallel,e} = 0 \). However, to keep the calculations more general, it is assumed that the electrons have different temperatures in parallel and perpendicular direction, i.e. \( A_e \neq 0 \), while protons are isotropic, \( A_p = 0 \). Restricting further considerations to left-handed circular polarized waves by taking the lower sign in front of \( \Omega_\alpha \) (right-handed circular polarized waves are then defined for negative frequencies), equation (C.80) reads

\[
\omega^2 = c^2 k_\parallel^2 \left[ 1 - A_e \frac{\beta_e}{2} \frac{\Omega_e^2}{(\omega - \Omega_e)^2} \right] + \omega \sum_{\alpha=p,e} \frac{\omega_{p,\alpha}^2}{\omega - \Omega_\alpha}
\]  

where \( \beta_e = 8\pi n_e k_B T_{\parallel,e}/B_0^2 \) denotes the plasma beta of the electrons. Note that the electron cyclotron resonance appears at negative frequencies, since \( \Omega_e = -|\Omega_e| \). A further treatment of (C.81) depends on several frequency regimes, for which the dispersion relation can be simplified. Considering first low-frequency waves with \(|\omega| \ll \Omega_e, \Omega_p, \Omega_c \), the dispersion relation (C.81) can be well approximated by the following expression:

\[
\omega^2 = \frac{v_A^2 k_\parallel^2}{1 + v_A^2/c^2} (1 - A_e \beta_e/2)
\]
For the case \( A_e = 0 \), one recovers the dispersion relation for the shear Alfvén wave already derived by using the MHD approach. In contrast to equation (C.35), the Alfvén dispersion relation (C.82) reveals an additional term stemming from the displacement current neglected in MHD. If \( A_e \neq 0 \), the dispersion relation is modified by the electrons having different temperatures in normal and parallel direction. Equation (C.82) predicts instability of Alfvén waves in an anisotropic plasma if the parallel particle pressure \( P_\parallel \) (which has not to be confused with \( p_\parallel \) and \( P_\parallel \)) exceeds the magnetic pressure \( P_M \), resulting from the ambient magnetic field \( B_0 \), and the perpendicular particle pressure \( P_\perp \). The instability condition is then given by \( P_\parallel > P_\perp + P_M \). This is the firehose instability, a result of pressure anisotropy, which also occurs for anisotropic proton distributions and, furthermore, for non-vanishing drift velocities \( V_{\parallel,a} \).

Considering now subluminal waves, i.e. \( \omega / k \ll c \), in the limit \( \omega \ll \Omega_e \), equation (C.81) yields

\[
\omega_{f,b} = \frac{v_A^2 k^2}{2 \Omega_p} \left( 1 - A_e \beta_e / 2 \right) \left[ \sqrt{1 + \frac{4 \Omega_p^2}{v_A^2 k^2 (1 - A_e \beta_e / 2)}} - 1 \right] \tag{C.83}
\]

where \( j = +1 \) and \( j = -1 \) refer to forward (f) and backward (b) propagating waves, respectively. For small wavenumbers, \( k \ll \Omega_p / v_A \), equation (C.83) reduces to

\[
\omega \simeq j v_A k \sqrt{1 - A_e \beta_e / 2} \tag{C.84}
\]

which describes, without displacement current correction term, non-dispersive forward \( (v_{ph} > 0) \) and backward \( (v_{ph} < 0) \) propagating Alfvén waves which are either left-handed \( (\omega > 0) \) or right-handed \( (\omega < 0) \) circular polarized. For \( k > \Omega_p / v_A \), equation (C.83) leads to

\[
\omega_{f,b} \simeq j \Omega_p + (j - 1) \frac{v_A^2 k^2}{2 \Omega_p} (1 - A_e \beta_e / 2) \simeq \begin{cases} \Omega_p & \text{for } j = +1 \\ - \frac{v_A^2 k^2}{\Omega_p} (1 - A_e \beta_e / 2) & \text{for } j = -1 \end{cases} \tag{C.85}
\]

The left-handed Alfvén branch develops into the forward and backward propagating left-handed proton cyclotron wave given by the upper case, whereas the right-handed Alfvén wave develops, for \( A_e = 0 \) into the dispersive right-handed whistler wave, which moves both forward and backward. It is worthy to consider the whistler dispersion relation for \( A_e \neq 0 \) in more detail, i.e.

\[
\omega \simeq - \frac{v_A^2 k^2}{\Omega_p} \left( 1 - \frac{\beta_e}{2} \left[ 1 - \frac{T_{\perp,e}}{T_{\parallel,e}} \right] \right) \tag{C.86}
\]

This is the so-called right-hand polarized whistler anisotropy instability evolving out of the whistler mode for \( T_{\perp,e} > T_{\parallel,e} \) (see Gary [1993] and references therein). In the opposite case, \( T_{\perp,e} < T_{\parallel,e} \), the electron firehose instability evolves out of the right-hand whistler wave (see Gary [1993] and references therein). In contrast to the whistler anisotropy instability, the electron firehose instability undergoes a significant shift in \( \omega \) with decreasing \( T_{\perp,e}/T_{\parallel,e} \). Although evolving out of a right-hand circularly polarized mode, the instability becomes left-hand polarized within the unstable wavenumber range. The switch in the state of polarization can easily be explained by considering the last term of the damping/growth rate (C.78), because it provides for the instability condition

\[
- \frac{T_{\perp,e}}{T_{\parallel,e}} \pm \left( 1 - \frac{T_{\perp,e}}{T_{\parallel,e}} \right) \frac{\Omega_e}{\omega} > 0 \tag{C.87}
\]

Making use of \( \Omega_e = -|\Omega_e| \) and, furthermore, since the polarization of the modes can be written as \( G = \pm \omega/|\omega| \), where \( G = +1 \) and \( G = -1 \) correspond to the right-handed and left-handed state of
polarization, respectively, the instability condition can be manipulated to obtain
\[
\left( \frac{T_{\perp,e}}{T_{\parallel,e}} - 1 \right) \left( G \frac{\Omega_e}{|\omega|} - 1 \right) > 1
\]  
(C.88)

Since \(|\omega| < |\Omega_e|\), it is clear that the condition of instability can be satisfied only for certain ratios of \(T_{\perp,e}/T_{\parallel,e} > 1\) and \(T_{\perp,e}/T_{\parallel,e} < 1\) if \(G = +1\) (right-handed) and \(G = -1\) (left-handed), respectively. The latter case points out that the wave is left-handed circularly polarized in the unstable regime \(\Gamma > 0\). For \(T_{\perp,e} = T_{\parallel,e}\), it is obvious that the condition is not fulfilled and no instabilities occur. Considering the derivation of equation (C.88) in more detail and replacing the electrons by protons, it is instantaneously clear that protons with \(T_{\perp,p}/T_{\parallel,p} > 1\) have the potential to drive left-handed instabilities, i.e. \(G = -1\), whereas a proton species with \(T_{\perp,p}/T_{\parallel,p} < 1\) can produce right-handed unstable modes, i.e. \(G = +1\). The switch in polarization by changing the particle species results from the different charge states of protons and electrons.

With \(A_e = 0\) and at higher frequencies, i.e. \(|\omega| \gg \Omega_p\), the dispersion relation (C.81) can be approximated by
\[
\omega \simeq -|\Omega_e| \frac{k^2c^2 + \omega^2_{p,e}}{k^2c^2 + \omega^2_{p,e}} \left\{ \begin{array}{ll}
-|\Omega_e| & \text{for } k \gg \omega_{p,e}/c \\
\frac{\nu^2A^2}{\Omega_p} & \text{for } k \ll \omega_{p,e}/c
\end{array} \right.
\]  
(C.89)

Equation (C.89) indicates that the right-handed whistler branch develops into the right-handed electron cyclotron wave regime for sufficient large wavenumbers. For the case \(A_e = 0\) and at very high frequencies, i.e. \(|\omega| \gg \Omega_e \gg \Omega_p\), equation (C.81) reduces to
\[
\omega^2 = k^2c^2 + \omega^2_{p,p} + \omega^2_{p,e} \simeq k^2c^2 + \omega^2_{p,e}
\]  
(C.90)

This corresponds to the dispersion relation of an unmagnetized plasma and describes transverse ordinary left- and right-hand polarized electromagnetic waves. Obviously, such waves can only propagate if the frequency exceeds the sum of the characteristic plasma frequencies. The sum is usually determined by the plasma component having the smallest mass, i.e. electrons. For vanishing plasma frequencies, i.e. the waves propagate as ordinary electromagnetic radiation through space, the phase velocity, \(v_{ph} = \omega/k_{||}\), and group velocity, \(v_g = \partial \omega/\partial k\), are the same, i.e. \(v_{ph} = v_g = \pm c\).

Considering equation (C.90), it is clear that \(k = \pm \sqrt{\omega^2 - \omega^2_{p,e}/c}\) is a real quantity, provided that \(\omega^2 > \omega^2_{p,e}\), so that the waves can propagate through the plasma. At frequencies below the cutoff at \(\omega^2 = \omega^2_{p,e}\), the wave number becomes imaginary, so that the waves can grow or decay exponentially in space. Considering, for instance, slowly varying number densities of the plasma components in space, the plasma prevents the propagation of the waves, they penetrate only to a distance given by an electromagnetic skin depth \(\Lambda_e\). Hence, the waves cannot propagate beyond \(\Lambda_e\), but they are reflected at that point. This is an important tool for studies concerning Earth’s upper atmosphere, i.e. the ionosphere. The distances to layers of different plasma frequencies can be measured based on the reflection of the waves back to the ground level.

For different frequency regimes and \(A_p = A_e = 0\), figure C.2 summarizes the properties of the solution (C.83) and its corresponding sub- and superluminal asymptotic shapes given by equations (C.84), (C.85), (C.89) and (C.90). Curves with \(\omega > 0\) (\(\omega < 0\)) and \(k > 0\) (\(k < 0\)) are referred to as forward (f) propagating waves, while modes with \(\omega > 0\) (\(\omega < 0\)) and \(k < 0\) (\(k > 0\)) are backward (b) moving. Modes with positive frequencies are left-handed, whereas waves having negative frequencies are right-hand polarized. The dotted lines represent the light cone distinguishing between the sub- and superluminal range.
Figure C.2: Illustration of the dispersion relation (C.81) describing for different frequency regimes left- and right-hand circularly polarized parallel propagating waves in a cold electron-proton plasma.
Appendix D

Alternative Derivation of the Drift Coefficient for Slab Turbulence

In this appendix, a more enlighting derivation of the transport parameter for particle drift, $\kappa_T$, in a slab turbulence is presented. The fundamental assumption is that the components of the particle’s velocity normal to the background magnetic field are given by the drift velocity of the well-known $\mathbf{E} \times \mathbf{B}$ drift. The appropriate components of this drift velocity are used to define a second-order correlation function. For this, the transport parameter (3.23b) and the definition of the FPCs, equation (3.14), are first merged into one expression, so that

$$\kappa_T = \frac{1}{2} \Re \int d\mu \int_{-1}^{1} ds < \bar{g}_x(t)\bar{g}_y^*(t + s) > $$ (D.1)

The second-order correlation function appearing in the integrand is so far unknown. In order to specify this function, the particle drift velocity for $\mathbf{E} \times \mathbf{B}$ drift is used, i.e.

$$\mathbf{V}_d = \frac{c(\mathbf{E} \times \mathbf{B})}{B^2}.$$ 

The magnetic field $\mathbf{B}$ may be expressed as a superposition of the large-scale background magnetic field and the small-scale magnetic fluctuations, so $\mathbf{B} = B_0 \mathbf{e}_z + \delta \mathbf{B}$. The electric field $\mathbf{E}$ is given by the small-scale electric field of the turbulence, i.e. $\mathbf{E} = \delta \mathbf{E}$. Since transverse waves are assumed, it becomes clear that $\delta E_z$ and $\delta B_z$ vanish for slab geometry. The perpendicular contributions of the drift velocity $\mathbf{V}_d$ can then be cast into the following forms:

$$V_{d,x} = \frac{cB_0}{B^2} \delta E_y$$ and $$V_{d,y} = -\frac{cB_0}{B^2} \delta E_x$$ (D.2)

On making use of the Fourier transformation (A.6), one obtains

$$V_{d,x} = \frac{cB_0}{B^2} \int d^3k e^{i\mathbf{k} \cdot \mathbf{x}(t)} E_y(k||)$$ and $$V_{d,y} = -\frac{cB_0}{B^2} \int d^3k e^{i\mathbf{k} \cdot \mathbf{x}(t)} E_x(k||)$$ (D.3)

where $E_y$ and $E_x$ now denote the Fourier transformed small-scale electric fluctuations. For the further treatment, it is assumed that the particle motion along the background magnetic field may be described by the unperturbed particle orbit, i.e. $z(t) = v|| t$, used in appendix A. Assuming that $V_{d,x}$ and $V_{d,y}$ may be identified with $\bar{g}_x$ and $\bar{g}_y$, respectively, one obtains for the corresponding second-order correlation function the form

$$< \bar{g}_x \bar{g}_y^* > = -\frac{c^2 B_0^2}{B^4} \int d^3k e^{-ik||v||s} R_{yx}(k||, s)$$ (D.4)

where the electric correlation function

$$R_{yx}(k||, s) = < E_y(k||, t) E_x^*(k||, t + s) > = \sum_{j=\pm 1} e^{+j\omega_j s} < E_y(k||) \hat{j} E_x^j*(k||) >$$ (D.5)
is introduced. Here, it was assumed that the Fourier components are not correlated at different wave vectors. For the evaluation of \( R_{xy} \), it is instructive to recall section 4.1.1 of chapter 4. There, Faraday’s law is used to express the plasma wave electric field by its magnetic counterpart. Making use of equation (4.3), taking into account that \( \mathbf{k} \cdot \mathbf{x} = k_x \|z(t) = k_y \|v_y t \), and performing in Faraday’s law the derivatives with respect to \( z \) and \( t \), the electric field components \( E_y \) and \( E_x \) can be expressed as

\[
E_y^j = \frac{(\omega_j - k \|v_y)}{ck} B_y^j \quad \text{and} \quad E_x^j = -\frac{(\omega_j - k \|v_y)}{ck} B_x^j
\]

with \( \omega_j = \omega_{R,j} + i \Gamma_j \) being the plasma wave dispersion relation. This yields

\[
R_{yx} (k || s) = -\frac{1}{c^2 k^2} \sum_{j=\pm 1} |\omega_j - k \|v_y|^2 e^{+i \omega_j s} P_{xy}^j (k ||)
\]

where \( P_{xy}^j = \langle B_x (k ||) B_y^*(k ||) \rangle \) is the purely magnetic contribution of each wave mode. The correlation function (4.4) can then be written as

\[
\langle \bar{y}_x \bar{y}_y \rangle = \frac{B_0^2}{B^4} \sum_{j=\pm 1} \int d^3 k e^{-i (k || v_y - \omega_j) s} k^{-2} |\omega_j - k \|v_y|^2 P_{xy}^j (k ||) \]

so that

\[
\kappa_T = \frac{B_0^2}{2B^4} \sum_{j=\pm 1} \Re \int d\mu \int d^3 k \mathcal{R}_j k^{-2} |\omega_j - k \|v_y|^2 P_{xy}^j (k ||)
\]

where \( \mathcal{R}_j \) is the resonance function (4.8) for \( n = 0 \). As it was shown in section (4.3.2), the slab model leads to \( P_{xy}^j = \omega_j A^j (k ||) \) with \( \sigma^j \) and \( A^j \) being the magnetic helicity and the wave power spectrum. Upon using \( \Re (i \mathcal{R}_j) = -\Im \mathcal{R}_j \) the power spectrum (4.59) for slab turbulence and the resonance function (4.8), one finds

\[
\kappa_T = \frac{\pi B_0^2}{B^4} \sum_{j=\pm 1} \sigma^j g_0^j \int d\mu \int_{k_{\min}}^{k_{\max}} dk || k^{-2} (\omega_{R,j} - k \|v_y) |\omega_j - k \|v_y|^2 \]

Considering now undamped Alfvén waves, i.e. \( \omega_{R,j} = \omega_j^* = \omega_j = j \sigma v_A k || \), one arrives at

\[
\kappa_T = -\frac{\pi B_0^2}{B^4} \sum_{j=\pm 1} \sigma^j g_0^j \int d\mu \int_{k_{\min}}^{k_{\max}} dk || k^{-1} (j \sigma v_A - v_y)
\]

The pitch angle integration of the second term in parenthesis yields zero. Performing the integration with respect to \( k \), making use of the normalization (4.60) and expressing the minimum and maximum wavenumbers by the corresponding outer and inner scale lengths, respectively, one finally obtains

\[
\kappa_T = -l_{\max} v_A \left( \frac{q - 1}{2\pi q} \right) \left[ 1 - (l_{\min}/l_{\max})^q \right] \sum_{j=\pm 1} \left( \frac{\delta B^j}{B_0} \right)^2 j \sigma^j
\]

where it was made use of \( B^2 = B_0^2 + (\delta B)^2 \simeq B_0^2 \). Equation (4.12) is in excellent agreement with (6.38) derived by using the general Foeldker-Planck approach.
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