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Abstract

A common way to circumvent the difficulties of applying QCD at low energies, in addressing the description and properties of baryons, consists in the use of models. The chiral quark-soliton model (CQSM) is one of such models, based mainly on the chiral symmetry of QCD and its spontaneous breaking. First of all, the model provides the framework to set up a consistent description of a baryon as a bound state of $N_c$ constituent quarks in the chiral meson field. These model baryons may then be used to compute observables in such states, which are in this work properties of the baryon octet, nucleon and hyperons, at low energies. The relevance of the physics going into the modeling of the baryon state, discounting effects of the approximations introduced, may be gauged by the agreement between the properties of the model states and the experimental properties observed for the actual baryon states.

In this work, the addressed properties concern the electromagnetic and weak interactions of this modeled baryon states. Results include electric and magnetic form factors and their associated static properties: radii and magnetic moments. These properties of the baryons are accessed in a flavor independent way, i.e. from the corresponding properties for each quark flavor current separately. Apart from interest in the form factors themselves, the calculation of the form factors for each flavor allows both the obvious study of the strange form factors of the nucleon and the phenomenological understanding of the influence of the strange sea quarks in the form factors of the nucleon. The results show a good agreement with the experimental data on form factors, including the strange form factors. There are also strong indications that point towards a sizeable influence of the strange sea quarks in the electric form factor of the neutron.

The model description of the electromagnetic current makes it also possible to study transition electromagnetic processes. As an exploratory study of the octet-to-decuplet electromagnetic transitions, results for the nucleon-$\Delta(1232)$ transition are presented. These results include the ratios $E_2/M_1$ and $C_2/M_1$ of the electric ($E_2$) and scalar ($C_2$) quadrupole amplitudes, respectively, to the magnetic dipole amplitude ($M_1$). Both ratios show a spatial deformation in the transition density in agreement with the experimental available data.

The axial current plays an important role in the physics of baryons since their weak interaction properties require both the knowledge of the matrix elements of the vector and axial currents. The axial form factors are obtained in this work from the axial currents in the the exact same scheme as the electromagnetic form factors were obtained from the vector currents. The results presented for the singlet and non singlet axial form factors agree with the experimental data. The singlet axial form factor in particular is related to the contribution of the quarks to the spin of the baryon they belong to. The results of the CQSM show a slight overestimation of the fraction of the proton spin carried by the quarks when compared to recent analysis of experimental results. On the contrary, the polarization of the strange quark compares favorably with such analysis.

The combined information from flavor decomposed electromagnetic and axial form factors allows to access the weak neutral properties of baryons. The properties of this current are experimentally accessible in parity-violating elastic scattering of polarized electrons on protons and light nuclei. Results for the form factors of the neutral weak current are presented, as well as for the P-odd electron-proton scattering asymmetry.
Abbreviations

CQSM - Chiral quark-soliton model
QCD - Quantum chromodynamics
CHPT - Chiral perturbation theory
SCSB - Spontaneous chiral symmetry breaking
PCAC - Partial conservation of the axial current
NJLM - Nambu–Jona-Lasinio Model
SM - Skyrme Model
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1 Introduction

The physical systems studied in this thesis are the octet baryons: Nucleons, i.e. protons and neutrons in their lowest state, and the lightest $J^P = 1/2^+$ hyperons ($\Sigma$'s, $\Lambda$ and $\Xi$'s). The properties of these systems to be addressed in this work are the form factors of the electromagnetic and weak currents in these baryonic states. The results are intended as a contribution to the yet unresolved problem of understanding the structure and interactions of these particles in terms of their inner constituents. The results presented in this thesis involved the reported theoretical calculations and the development of the corresponding computer codes.

The chapter begins with a review of the main experimental and theoretical steps leading from the first hints of compositness of the strong interacting particles until the present theory of strong interactions. One of these steps were the experimental data on elastic electron-proton scattering, which is best described in terms of form factors. Form factors are then highlighted in what regards their importance and meaning for the study of baryons. The form factors considered in this work and the main physics assumptions adopted are also summarized. The chapter ends with the organization of this monograph.

From phenomenology to QCD

The concept of an elementary particle has always played an important role in physics. In broad terms, this concept is understood in the sense of a structureless entity which, by its characteristics and interactions with other similar elementary particles, can explain the properties of larger assemblies of matter. Theoretically this is a working assumption, valid up to the distances that can be resolved at the energies and momentum transfers involved in the process under scrutiny. For instance, the electron is at present considered as an elementary (hence fundamental) particle, therefore structureless. This is so because this working description of the electron, when plugged into the formalism to obtain measurable quantities, leads to no contradiction with any known experiment, up to the highest energies presently attainable in the laboratory. One does not prove, thus, that a certain particle is elementary, one can but verify that its internal structure, if any, does not have experimental consequences up to the energies being considered.

Experimentally the physical objects that this notion encompassed have changed over time. Atoms were probably the single most important step in the search for the building blocks of matter. They were soon found to have an inner structure, though. Next came the discovery of the electrons, the atomic nucleus and, shortly after, the main constituents of the nucleus, the nucleons. Afterwards, the remaining octet members were discovered, together with many other alike particles, which were characterized by the fact that they interacted by the strong interaction. However, these so-called particles were not found to be elementary, at least in the same sense as, e.g. the electron. At sufficiently high energies of the probes they are studied with, the internal compound structure of these particles becomes clear. They are designated as particles for they are characterized by a definite set of measurable properties like mass, and magnetic moment and by a certain set of quantum numbers like charge, spin, isospin, strangeness/hypercharge, and parity$^1$.

Indeed, the strongly interacting particles are experimentally found to be bound states of the strong interaction. These are the only particles participating in strong interactions. They are collectively denoted by hadrons, which include mesons and baryons, with the octet as a small subgroup of these. The realization that hadrons were not structureless started to gain shape after

$^1$These quantum numbers are indicative of an isospin symmetry, which is however broken, as it is possible to infer from the mass differences within the octet.
the identification of the nucleons as the ordinary constituents of the atomic nuclei. This started with hints for the existence of the proton, around 1919 by E. Rutherford and co-workers, and ended with the discovery of the neutron, by J. Chadwick in 1932, the same year in which the deuteron was identified. It was then rather natural an assumption to regard the nucleus as made of protons and neutrons. These constituents could be considered as two states of the same particle\(^2\) the nucleon, and demanded a new type of interaction which could bind nucleons in the nucleus. The nuclear force\(^3\), the starting point for the strong interaction, was found to have characteristics which made it look neither simple nor fundamental.

The first progress in nuclear forces came with the work of H. Yukawa in 1935, who proposed a scalar meson exchange model for the interactions between nucleons. Development of these ideas and the experimental determination of the deuteron quadrupole moment in 1939 led Pauli to propose in 1946 a pseudoscalar isovector meson as mediator of the interaction between nucleons. Such particles were eventually identified with the pions, later discovered in 1947 and 1950 (neutral pion). The following years would lead to the development and refinement of these initial ideas. However, is was soon found that the nuclear force between two nucleons was repulsive at short range and attractive at longe range, and also that it depended on the spin and symmetry of the wave function of the system. This made it resemble more like the chemical force between atoms than the simpler electric forces within the atom. This resemblance suggested further that the nuclear force was a complex manifestation of more fundamental forces acting among internal constituents of the nucleons, very much in the same way the complexity of the chemical force is a manifestation of the internal structure of the atoms.

An early clue supporting this claim was the determination of the magnetic moment of the proton in 1933 by O. Stern, which was found to deviate much from the expected value in Dirac theory. This was also the case with the neutron in 1934. Both measurements were later confirmed by high-precision experiments by I. I. Rabi in 1939. Subsequent confirmations of this scenario were the discovery of the excited state of the nucleon, the \(\Delta(1232)\), the discovery of the “strange” particles and the results from elastic electron-proton scattering [1].

The \(\Delta\) particle was first hinted by H. Anderson and E. Fermi in 1952 and confirmed later by S. Lindenbaun and C. Yuan in 1953 and had a lifetime within the time scale of the strong interaction, \(10^{-23}\) s. The first “strange” particles were also discovered about the same time: The \(\Lambda\) and the charged \(\Sigma\) hyperons were confirmed in 1953, both by W. Fowler and co-workers. The \(\Xi^-\) was confirmed also in 1953 by H. Anderson and co-workers. In the late 50’s K. Nishijima and M. Gell-Mann independently predicted the \(\Sigma^0\) and \(\Xi^0\). These were later confirmed experimentally in 1957 and 1959. The results from elastic electron-proton scattering revealed in particular that the nucleons were extended objects with charge radii of the order of 0.8 fm. This size was experimentally in agreement with pion-nucleon collision results.

The “strange” particles brought a new additive quantum number to the hadron physics. This new quantum number was “strangeness”, \(S\), and was proposed independently by M. Gell-Mann and Nishijima. The grounds for this proposal were the fact that these particles appeared in pairs in pion-proton collisions and that some of these particles had unexpectedly large lifetimes when compared with the time scale of the strong interactions, in spite of the fact that their expected decays did not violate either charge nor baryon number conservation. In this scheme hadron were attributed an integer strangeness quantum number: \(S = 0\) (\(\pi, N, \Delta, \ldots\)), \(S = 1\) (\(K, \ldots\)), \(S = -1(\Lambda, \Sigma, \ldots)\), \(S = -2(\Xi, \ldots)\). The scheme also asserted\(^4\) that the strong and electromagnetic interactions would conserve \(S\).

The immediate expectation from having two additive quantum numbers, strangeness and isospin,\(^2\)First proposed by W. Heisenberg in 1932, was put into the isotopic spin formulation by B. Cassen and E. Condon in 1936.

\(^3\)The charge independence of the nuclear forces was proposed in 1936.

\(^4\)This explained the long lifetime of the \(\Sigma\). While it was produced in \(\pi^- p \rightarrow K^+\Sigma^-\), the \(\Sigma^-\) could only decay by the weak interaction \(\Sigma^- \rightarrow n\pi^-\), thus explaining the long lifetime.
was the attempt to enlarge the isospin symmetry of the nucleon to a larger group. Such new group would naturally fit the hadrons with similar properties into the the multiplet structure of the group. The mass differences between the hadrons made this a difficult task, but by 1961 the SU(3) group was proposed, which allowed to group the p, n, Σ⁺, Σ⁰, Σ⁻, Λ, Ξ⁰ and Ξ⁻ into the octet of Fig. 1.1. Although the large mass spread of almost 400 MeV makes it clear that the symmetry linking strange and nonstrange baryons is more approximate than isospin symmetry, it turned out a very good scheme to enumerate the hadron states. Defining the hypercharge \( Y = B + S \) (\( B \) is the baryon number), one obtains the Gell-Mann–Nishijima relation \( Q = T_3 + Y/2 \) (\( T_3 \) is the third component of isospin).

Figure 1.1: The baryon octet. The quantum numbers are hypercharge and third component of the isospin. Masses are in MeV/c².

However, pursuing further the above analogy with Chemistry, this SU(3) classifying scheme for the “elementary” particles would play the role of the Mandeleev’s table of chemical elements. As well as the Mandeleev’s table set the stage for the structure of atoms, the SU(3) group of isospin and strangeness set the stage for the proposal of quarks as the building blocks of hadrons. The role of the chemical elements was taken over by a growing experimental data on a growing number of strongly interacting particles acquired during the 1950’s. This still continues, e.g. with the recent discovery of the exotic pentaquark state \( Θ^+ \) [2], in accordance with predictions made in the model used in this work [3].

The first proposals for such constituents⁵ were made by Gell-Mann [4] and Zweig [5]. According to these proposals, the strongly interacting particles (hadrons) were built of a triplet⁶ of objects, named quarks (up \( u \), down \( d \), and strange \( s \)), and a corresponding triplet of antiquarks. These constituents were to have fractional electrical charge and spin 1/2. On general group theoretical grounds it was proposed that mesons were quark-antiquark states and (anti)baryons three (anti)quark states. Phenomenologically these proposals proved to be quite successful in particle classification and spectroscopy: masses, spin, and isospin [6]. It was nevertheless clear that apart from its heuristic value, the dynamics of the hadronic world was still unknown. At this point quarks were understood in a valence sense, i.e. the wave function of hadrons were simply a properly symmetrized combination of quark wave functions with definite spin and isospin quantum numbers making up for the hadron quantum numbers. In the context of the quark model, quarks are termed constituent to stress the fact that they are massive quarks, their masses being in the vicinity of one third of the baryon mass.

---

⁵Although since 1949 with E. Fermi there were models for pions as composite particles.

⁶The known hadrons at the time included only one additional quantum number besides isospin, strangeness.
The presence of constituents in the hadrons would gain further experimental support with the deep inelastic scattering (DIS) of electrons on protons [7, 8] and the discovery of scaling [9]. Although scaling is defined as a behaviour of the structure functions entering the cross sections, in simple words it means that, as the value of the squared four-momentum transfer increases enough, the virtual photon stops ‘seeing’ the proton as a whole and starts to resolve point like objects inside the proton, whose behaviour is almost that of noninteracting particles. These point like constituents were termed partons [10, 11].

After these indications about the main constituents of the hadrons, it was only a matter of finding the interaction between them. This interaction had to be entirely different from the other known interactions and demanded for a new charge for quarks, other then the electric charge and flavor charge (related to the weak interactions). The new charge supposedly carried by quarks was denominated color by M. Gell-Mann. The color quantum number of quarks was appealing for it solved one problem with certain baryon wave functions in the quark model. This difficulty was related to the fact that, without such additional quantum number, some three quark states seemed to violate the Pauli principle, e.g. the $\Delta^{++}$ would be, according to the quark model, made out of three up quarks in a spin $3/2$ state. However, the experimental results showed no evidence for neither asymptotic single quark states nor states with this new charge, i.e. with color [12, 13].

This difficulty was overcome by the twofold realization that the minimal number of colors necessary to overcome the difficulties with the Pauli principle was three and that a color $SU(3)$ group allowed for colorless quark-antiquark (meson) and three quark (baryon) states. It was however necessary to postulate a mechanism, termed confinement, responsible for the absence of detection of colored states. Color confinement means there are no states other then color singlets, i.e. only color singlets belong to the physical Hilbert space of strongly interacting particles. An understanding for this mechanism was not yet found.

After the proof of the renormalizability [14] of the Young-Mills theories [15], the theory of the strong interactions, Quantum Chromodynamics (QCD), was finally established. It is a non-Abelian Young-Mills quantum field theory, whose degrees of freedom are quarks, as the matter fields, interacting through the exchange of vector bosons denominated gluons, which carry color. This fact that the gluons carry themselves color allows for them to interact with each other, giving the theory a distinctive nonlinear behaviour, which accounts for the difficulties in working with the theory.

One of the first findings in QCD, which very much contributed to its establishment as the theory of the strong interactions, was the discovery of asymptotic freedom [16–19]. The proof of this property of the theory made the use of perturbation theory possible at sufficiently high energies or in special kinematic regimes, at the same time it justified many assumptions made in the study of DIS results. This was decisive in order to allow calculations with QCD, since strong interactions usually mean large coupling constants and the breakdown of perturbation theory. Although these calculations covered a small domain of the strong interaction phenomenology, their success was the key to the acceptance of the theory.

Today there is a wide belief that the microscopic theory of hadron structure and reactions is indeed QCD. This means that the synthesizing work from the strong interacting particles to their microscopic constituents, quarks and gluons, may be regarded as practically finished. However, the reverse path, that of going directly from the QCD degrees of freedom and dynamics to the many varied phenomenological aspects of hadrons and their interactions is still outside the domain of asymptotic freedom. In the remaining cases nonperturbative methods have to be employed to access the low energy regime\(^7\). These methods are in different extents inspired in QCD and belong, in most cases, to one of the following:

i- Lattice QCD;

---

\(^7\)Nuclear physics is placed in the very low energy regime and comprises methods of its own, although the frameworks in some models, as is the case studied here, are in large extents based in methods of nuclear physics.
ii- Chiral perturbation theory (CHPT);

iii- Models, inspired to different extents in QCD.

Lattice QCD, for a review see [20, 21], is the approach closer to QCD since it consists in implementing QCD in a discretized space-time lattice. The degrees of freedom are close to the QCD quarks and gluons and the lattice spacing provides a natural regularization. There are still unsettled issues, e.g. the restriction to the quenched approximation. The expectation is that these difficulties are surmountable and that lattice QCD is then limited by the available computing power only.

Chiral perturbation theory (CHPT) [22–26] is an effective field theory with hadronic degrees of freedom. It is constructed as the most general field theory compatible with the symmetries of QCD, especially the spontaneous (dynamical) breaking of chiral symmetry (The subject of Section B.3). Such an effective field theory is nonrenormalizable. However, it has been shown [27–29] that it nevertheless can form the basis of a consistent quantum field theory allowing a systematic low energy approximation in the nonperturbative regime. The parameter of the expansion is the momenta involved in the process and the expansion is thus limited to moderate momenta. The renormalization procedure also forces some phenomenological input into the theory in the form of the so-called low energy constants, an example of which are the electromagnetic radii. Although chiral perturbation is not suited for the study of observables depending on the quark content of hadrons, particularly in determining the low energy constants, it is possibly best suited to bridge the low energy regime of QCD with nuclear physics, namely the few nucleon systems [30–32].

Models are usually based on constituent (valence) quarks and effective interactions, which often replace any explicit gluonic degrees of freedom. Models do not show, in general, the same systematic character of CHPT and are inspired in QCD in very different ways, usually incorporating additional assumptions which render a simple classification of models very difficult. Some examples, closer related to the CQSM of this work, are left to Chapter 2 where they are briefly described in connection with the CQSM. Generally speaking, the insight into the nonperturbative regime of QCD that models are able to provide is most often hindered by the difficulty of disentangling model dependent from genuine QCD features.

Form factors

Although the main properties of the spectrum of hadrons, while strongly interacting particles, are expected to be computable from QCD, hadrons participate also in the electroweak interactions. On the one hand, this means that there will be corrections to the hadron properties due to these interactions. Quarks indeed are subject to electromagnetic and weak interactions as well as the strong interactions. On the other hand, the hadrons may be studied using probes which interact with them via the electromagnetic or weak interaction. The main point of this work consists in looking at the structure of the baryon octet in the light of the electromagnetic and weak interactions.

Even at low momentum transfers, experiment shows that hadrons have a finite spatial extent of the order of the fm ($10^{-15}$ m). The extended structure means compositeness and form factors are a very useful tool to study composite systems. In general terms, form factors are the ratios between a measured matrix element (or computed on the basis of an extended description for the particle) and the same matrix element considering a pointlike particle. They are functions of just the square of the transferred momentum from a probe to the hadron, i.e. the momentum carried out by the

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8 Basically consists in neglecting quark-antiquark pair creation from the vacuum

9 In Quantum Electrodynamics (QED) the electron current will also include form factors as the proton, but, while the proton current needs to be parameterized by form factors even at tree level, accounting for its extended structure, the electron form factors appear as a consequence of quantum effects, namely the renormalization of the vertex corrections. For the electron these quantum effects account for the tiny anomalous magnetic moment. In the case of the nucleon the anomalous magnetic moments are much larger, owing to their origin not in quantum radiative corrections, but on internal structure. This also explains why only the effects of internal structure are taken into account in this study of form factors.
current whose form factors are being determined\textsuperscript{10}. The form factors appear in the expressions for
the matrix elements of currents under general invariance principles of such matrix elements, as is
briefly explained in Appendix C. Cross sections are expressed in terms of these form factors.

The nature of the form factors differs according to the interaction between the probe and target
hadron. In this way one may consider electromagnetic, axial and strong form factors. In the case
of the electromagnetic form factors the probes are charged leptons and photons are the exchanged
particles. Axial form factors are usually measured with neutrinos as probes and the exchanged
particles the charged W bosons\textsuperscript{11}. In these two cases, considered in this work, the couplings
between probe and target are small, making it both possible to use the tree level approximation
and to expect that possible off-shell effects are negligible. Moreover, in both interactions, probe and
exchanged particles, are well understood up to energies much higher than the ones at which the form
factors are determined. For the strong form factors the probe particles are hadrons themselves and
the exchanged particles mesons. Since this involves just the strong interaction, these form factors
are difficult to define and also difficult to measure experimentally due to model dependence of the
data extraction.

As was already pointed out, the electromagnetic interaction had a key role in revealing the
composite structure of the nucleon. In 1937 the electron-neutron interaction (due to the neutron
anomalous magnetic moment) was detected [33]. By 1953 the electron scattering led to the identi-
fication of charge distributions inside the nucleus as well as to the study of the nuclear size [34,35].
In 1955 the charge distribution of the proton is investigated and values close to the today accepted
ones were obtained for the root mean square charge radius [1]. These first experimental studies
relied upon two assumptions: electrodynamics was valid for the leptonic probe and the one boson
exchange approximation was justified. The main consequence of the second assumption, namely
the one-photon exchange, was that the differential cross section for the elastic scattering of elec-
trons off an unpolarized target was expressed by the Rusenbluth formula [36], Appendix C, which
is based upon the smallness of the fine structure constant. By 1974 [37], both assumptions were
consistent with the experimental results, which already showed the main features they still possess
currently, as will be summarized in the beginning of Chapter 3.

In the context of the aforementioned assumptions, the lepton scattering on a target $T$ measures
the matrix elements of the current $J_{\mu}$, electromagnetic or weak, between the target $T$ and a final
hadronic state $F$

$$
\langle F | J_{\mu}(0) | T \rangle,
$$

and the current undergoes a transfer of a space-like momentum\textsuperscript{12}. The elastic situation is charac-
terized by the final hadron state being equal to the target in constitution, but not in kinematics. In
this case, the matrix element is termed off-forward, a designation also valid when the final state has
a different constitution from the target. When the target and final states coincide in constitution
and kinematics one has a forward matrix element in the target state.

The results of this work concern the elastic form factors of the octet baryons in terms of which
the matrix element (1.1) can be expressed, as is explained in Appendix C. The exception is the
nucleon to delta transition studied in Chapter 4, where the final state is different from the target
state. The form factors are in this case referred to as transition form factors.

The main assumptions of the work reported in this thesis, explained with a little more detail in
Chapter 2, are: Spontaneous breaking of chiral symmetry, Section B.3, is the dominant dynamical
mechanism in QCD at the low energy, nonperturbative regime, where the form factors are computed
(\text{below 1 GeV}^2, i.e. the scale for chiral symmetry breaking); confinement effects are not significant
for the considered form factors; a picture of the hadron as a bound state of valence constituent

\textsuperscript{10}In the present work the form factors are for space-like squared momentum transfers only.

\textsuperscript{11}They also appear in the matrix elements of the neutral weak current.

\textsuperscript{12}The electron-positron annihilation into hadrons measures the matrix element of the electromagnetic current
between the vacuum and the hadronic final state $\langle F | J_{\mu}(0) | 0 \rangle$. In this case the current transfers a time-like momentum.
quarks interacting through the exchange of pseudo-Goldstone mesons is valid.

The original contributions of this work to the knowledge on form factors of the octet, to be reported and discussed along this thesis, include:

a) Electric, magnetic and axial form factors for individual flavor currents and associated static quantities;

b) Electromagnetic form factors of the octet baryons and axial form factors of the nucleon including kaon asymptotics in the strange form factor;

c) Electric, magnetic and axial radii, magnetic moments and axial constants;

d) Ratios of the electric quadrupole and scalar quadrupole amplitudes to the magnetic amplitude in nucleon to delta transitions;

e) Weak form factors of the neutral current for the nucleon;

f) Parity-violating asymmetry in elastic electron-proton scattering.

Contents

The general organization of this thesis was chosen such that the main body pursues the calculations actually carried out for the purpose of this work, leaving the general background and technicalities for the appendices. The monograph is then organized as follows: Chapter 2 provides mainly a short overall view of the theoretical framework under which the observables are computed, at the same time it serves as a guide to most of the appendices, where some topics are discussed in a somewhat more detailed way. The chapter introduces the chiral quark-soliton model (CQSM) together with the framework adopted to describe baryon properties. Emphasis is put into its theoretical foundation, particularly in chiral symmetry and spontaneous chiral symmetry breaking. The CQSM is then introduced and it’s origin from a model designed to explain chiral symmetry breaking in the QCD vacuum is briefly mentioned. The regularization used in this work and the way in which model parameters are fixed, in the vacuum and mesonic sectors, make the definition of the model complete. The chapter reviews the method to calculate observables within the model and ends with the approaches used in this work regarding the flavor SU(3) breaking and the symmetry conserving quantization.

The main results are reported in Chapters 3 to 6 and together with the most relevant experimental data. Chapter 3 contains the results for the electromagnetic form factors. They are obtained in a flavor independent way. This means that results include the flavor singlet and nonsinglet currents in terms of which the flavor currents can be expressed. The purpose of this flavor decomposition, already studied at exploratory level in [38], allows to address several questions regarding the form factors. First, obviously, it yields the strange current form factors, under experimental investigation by the SAMPLE (MIT-Bates), A4 (MAMI-Mainz), HAPPEX (TJNAL, Hall A, Newport News) and G0 (TJNAL, Hall C, Newport News) collaborations. The theoretical interest lies in the fact that the nucleon carries no net strangeness, but, according to the theory, strange quark-antiquark pairs may continuously form inside the nucleons. Second, the phenomenological study of the role played by the mass which characterizes the asymptotic behaviour of the meson fields is made possible due to this flavor decomposition. It is possible to obtain the different flavors with different asymptotic behaviors for the meson fields and then to see how these combination affects the form factors of the nucleon, namely the electric form factor of the neutron. Results concerning the behaviour of the form factors at higher momenta and the values for radii, magnetic moments are also presented.

In Chapter 4 the subject is the electromagnetic induced transition of the nucleon to the Δ isobar, either of the neutron to the Δ⁺ or of the proton to the Δ⁰⁺. The main point in studying this excitation of the nucleon lies in the very interesting experimental finding of non-vanishing scalar
and electric quadrupole amplitudes. This finding points towards a deformation of the density of
the nucleon from the spherical one. In the simple quark model this implies the introduction of a $D$
wave in the baryon wave function to supplement the natural prediction of this model of a complete
dominance of the spin-flip magnetic amplitude.

The subject of Chapter 5 consists in the evaluation the axial currents matrix elements and
determination of their associated form factors. The results single out the nucleon case both for
simplicity and for it is the most studied system experimentally. Results for the associated radii and
axial constants are given. The chapter includes the neutral weak form factors and the study of the
asymmetry in parity violating electron-proton scattering. From the point of view of the model, it is
a good general test for the form factors, as given by the model, since this asymmetry is a complex
function of the form factors, both electromagnetic and axial. From the experimental point of view,
the asymmetry is the quantity from which data on the strange form factors is deduced. In this way,
the CQSM is tested directly at the level of the strange form factors results and the asymmetry.

The overall summary of the conclusions drawn from the results and the outlook of this work
are the subject of Chapter 7.

The appendices are a repository of the main and auxiliary calculation aspects and are thus not
intended as an important part of the main reading. The main expressions are duplicated in the
text, with references to the corresponding appendix, in order to make direct need for the appendices
avoidable.

The notation used in this monograph and a list of most of the definitions used in the text is
to be found in Appendix A. Appendix B presents QCD with some more detail, with emphasis in
the role of chiral symmetry. The general aspects of form factors are explained in Appendix C.
The Appendix C shows briefly how to derive the expressions for the form factors, as used in the
text, from matrix elements of currents, using general principles requirements which such matrix
elements must comply with. A more detailed account of aspects related to the description of baryon
state is the subject of Appendix D. It summarizes the procedure to obtain the mean-field solution,
the soliton, as well as the procedure from which states with the appropriate baryon quantum
numbers can be constructed, particularly arriving at the quantization rules. Appendix E contains,
in spite of its apparent unreadableness, the main result of this work. It arrives to the general
expression for baryon observables in the framework adopted here, namely it applies to the matrix
element of an observable up to rotational and leading mass corrections in one-fermion (zero-meson)
loop approximation. The calculation of the so-called collective matrix elements is explained in
appendix F, which also lists the matrix elements used in this work. Appendix G collects some
numerical details of the calculation for completeness, including for example the calculations to fix
the model parameters. Finally, Appendix H collects the figures, for which there was no space in
the main text, illustrative of the main findings of the discussions, found in the main text, about
the behaviour of the model results with respect to its parameters.
2 The chiral quark-soliton model

This chapter contains a short account of the main features of the chiral quark-soliton model (CQSM) and the framework in which the model is used for the description of baryon states from which observables can be computed. It serves as a guide to the appendices where some aspects are presented with more detail, particularly those related closer to this work. Most of the topics discussed in this chapter are not new and have been studied by many authors in the past. They are repeated here for completeness. In this chapter and in what concerns the applications of the CQSM, we mainly follow the review [39], where additional references pertinent to this chapter may be found.

The definition and general properties of the theory of strong interactions, quantum chromodynamics (QCD), are left to Appendix B. This chapter starts by emphasizing, from Appendix B, the relevance of chiral symmetry and spontaneous chiral symmetry breaking (SCSB) to the description of hadrons. This is the main intersection between QCD and the CQSM. Clues as to the reason why this should be regarded as, perhaps, the most important mechanism in low energy QCD will be outlined. Other models of hadron structure, in particular those sharing with the CQSM the relevance given to SCSB and thus stand closer to the CQSM, will be referred to along the chapter.

The foundation of the CQSM in the instanton liquid model is briefly outlined, after which the model is defined. The definition include regularization and procedures to fix the model parameters. The next topic discussed is the model description of a baryon state with the appropriate quantum numbers. The chapter ends with the details of the calculation of form factors in the baryon states constructed within the model, which is the main contribution of this work.

2.1 From QCD to baryons

As already pointed out in Chapter 1, the baryon spectrum, especially the low energy part of it, fall into the domain where QCD is nonperturbative. The path from QCD to baryons, at least the ones made of the lightest quarks, as advocated in this work, relies on SCSB, which is connected with invariance of the QCD Lagrangian under global transformations, Section B.3. As a summary of Section B.3, the fate of the global transformations (B.7a) may be resumed in the following way: The vector symmetries appear to be realized in the manifest mode: $U(1)_V$ is an exact symmetry and represents baryon number conservation; $SU(N)_V$ is explicitly broken on a very small extent by the small current quark mass differences, which is small enough to allow these symmetries to be recognizable in the multiplet structure of the hadron spectrum. The axial symmetries share different fates. The axial $U(1)_A$ entails an anomaly [40, 41]. An anomaly is said to occur when a (classical) symmetry of the Lagrangian is absent at the quantum level\(^1\). The axial-vector symmetries are realized in a spontaneously broken way and are hidden from the hadron spectrum by the asymmetry of the vacuum. In the chiral limit, Section B.3, the vector charges are conserved and annihilate the vacuum

\[
Q^L |0\rangle = Q^a |0\rangle = 0,
\]

which is not the case for the axial-vector charges

\[
Q^5_5 |0\rangle \neq 0, \quad Q^a_5 |0\rangle \neq 0.
\]

In the chiral limit there is only just one quark bilinear which, and according to Goldstone’s theorem, can have a nonzero vacuum expectation value and produce the pattern of symmetry breaking of

\(^1\)Quantum corrections make that the divergence of the current in the chiral limit is $\partial^\mu J_{5\mu} = 3\alpha_s G^{a}_{\mu\nu} \tilde{G}^{\mu\nu}/8\pi$, with $\alpha_s$ the strong coupling constant.
The chiral quark-soliton model

(2.1) and (2.2). The quark bilinear is $\bar{q}q$ and its expectation value is the quark condensate $\langle \bar{q}q \rangle$, which is found to be nonvanishing. This condensate [42, 43] spontaneously breaks $U(N)_L \times U(N)_R$ down to $U(1)_V$, since $\bar{q}q$ is invariant under $U(1)_V$, but not under $U(1)_A$. The quark condensate is the order parameter of SCSB, since its nonvanishing value indicates a chiral broken phase.

To summarize, the evidence in favour of chiral symmetry breaking, one would start by the absence of mass degenerate parity doublets in the hadron spectrum. The mass difference of recognizable parity partners is quite large in the lower end of the hadron spectrum. The second piece of evidence is provided by the pseudoscalar being the lightest hadrons. They are thus easily interpreted as the (pseudo-)Goldstone bosons originating in the symmetry breaking. Lattice QCD contributes with the third piece of evidence by supporting a nonvanishing quark condensate. Theoretically the ’t Hooft’s anomaly matching conditions and confinement require SCSB for a number of flavors equal or higher than three [44, 45]. Also the Vafa-Witten theorem [46] asserts that the vector subgroup of the chiral group remains unbroken in vector-like gauge theories, as is the case of QCD with a vanishing vacuum angle ($\theta_{\text{QCD}} = 0$).

It is this scenario, regarding chiral symmetry breaking, which is at the basis of some of the most important approaches to low energy hadron physics. These are, namely, chiral perturbation theory (CHPT), current algebra based on partial conservation of the axial current (PCAC) and models displaying this pattern of symmetry breaking. However, one could ask why are these global symmetries more important than confinement at low energies. As a matter of fact, both the spontaneous breaking of chiral symmetry and confinement are not understood theoretically in terms of dynamical mechanisms of QCD.

Qualitatively one could say that a chiral asymmetric vacuum without confinement, first scenario, would not imply drastic consequences at low energies, while confinement without SCSB, second scenario, would. In the first scenario the lower part of the spectrum would still contain the light pions and heavy nucleons with differences appearing only in the excitation part of the spectrum. As pointed out, the pions being the lightest hadrons definitely give support to this scenario and are at the heart of the interest in chiral symmetry in the times before QCD. Without confinement there would appear massive constituent quarks in the spectrum. The expected role of confinement at these energies is just that of preventing this “ionization” of constituent quarks. In the second scenario the spectrum would look rather different from phenomenology, with heavy pions and light baryons (from the Goldberg-Treiman relation). This second scenario seems rather improbable at present. Confinement may nevertheless provide the inspiration for models of the baryon structure, as is the case of the MIT bag model [47].

2.2 The chiral quark-soliton model

The microscopic dynamical mechanism for SCSB in QCD is not yet fully understood. One of the mechanisms which was proposed to account for SCSB connects it with instanton-induced interactions among the light quarks. The instantons and their applications in the study of the QCD vacuum are manifold. Other than the SCSB, the instantons have been used in the context of, e.g. the aforementioned axial anomaly, the topological susceptibility of the QCD vacuum responsible for the $\eta'$ mass, the gluon condensate, and the vacuum energy density. For a review on instanton applications in QCD refer to [48–50].

For this work the most important fact is the possibility to deduce from this instanton picture of the vacuum an effective Lagrangian which is suited for the study of baryon physics. The deduction of such an effective Lagrangian, the chiral quark-soliton model (CQSM) is reviewed in [51]. The main features of the framework to compute baryon properties is also reviewed there. This framework is used here for the computation of form factors and as such will deserve a somewhat more detailed explanation below.

2 With explicit chiral symmetry breaking other vacuum expectation values acquire nonvanishing values, like $\langle \bar{q}\lambda^aq \rangle$, $a = 1, \cdots, N^2 - 1$, which break the vector symmetries and may imply, e.g. $\langle \bar{u}u \rangle \neq \langle \bar{d}d \rangle \neq \langle \bar{s}s \rangle$. 

Following [50, 51], observables can be computed from quarks in the instanton vacuum by averaging first over the instanton-antiinstanton ensemble in order to derive an effective four-fermion interaction. Under bosonization it becomes
\[ \mathcal{L} = \overline{\psi}(x) \left( i \partial_\mu - m - M(k) e^{i \gamma_5 \vec{\sigma} \cdot \vec{\tau}} \right) \psi(x). \]  
This effective Lagrangian may be used to compute observables. For such purpose it is not necessary to accept its origins in the instanton mechanism of SCSB.

The degrees of freedom of the CQSM (2.3) are constituent quarks and the pseudo-Goldstone bosons of chiral symmetry breaking (here at first sight appearing as auxiliary fields). Under the restriction to the chiral circle, these pseudo-Goldstone bosons are the pions in SU(2), and the pions and kaons in SU(3). The physics picture for the structure of baryons in the light of the model is then that of a bound state of constituent quarks in the chiral meson field, which is interpreted as the effective role of the quark Fermi see. Pions, and also kaons in flavor SU(3), are in turn understood in terms of collective excitations of sea quark-antiquark pairs. The restriction to the chiral circle means then that the degrees of freedom of equal or higher mass then that of the sigma are frozen and are not expected to be essential for the physics addressed with the model. This is closely related to the fact that the derivation of the low-energy effective Lagrangian from the instanton vacuum is carried out taking into account only those degrees of freedom whose masses are noticeable smaller than the inverse average instanton size, i.e. approximately 600 MeV. The degrees of freedom that fall in this regime are the constituent quarks, which are massive due to the breaking of the chiral symmetry, and the light mesons. The description of baryons carried out in this monograph rests solely upon these degrees of freedom.

The Lagrangian of the CQSM, as will be used in this work [50, 52–56], is obtained from (2.3) by taking the mass of the constituent quarks as a constant \( M = M(0) \), namely
\[ \mathcal{L}_{\text{CQSM}} = \overline{\psi}(x) \left( i \partial_\mu - m - M \gamma^\mu \right) \psi(x) \]
where the following notation is introduced:
\[ U^{\gamma^5} = e^{i \gamma^5 \vec{\sigma} \cdot \vec{\tau}} = \frac{1 + \gamma^5}{2} U + \frac{1 - \gamma^5}{2} U^\dagger, \quad U = e^{i \vec{x} \cdot \vec{\tau}}, \]
\[ M U^{\gamma^5} = \sigma + i \gamma^5 \vec{\pi} \cdot \vec{\tau}. \]
One may regard the fields \( \sigma \) and \( \pi \) as auxiliary fields for now. The main difference relative to the Lagrangian (2.3) is the fact that \( M \) is now a constant which represents the constituent quark mass. While in the context of the instanton liquid model (2.3) \( M(0) \simeq 350 \text{ MeV} \), in (2.4) \( M \) will be taken as a free parameter, the only free parameter in this work, as will be explained in Section (2.3). It has been shown in the instanton liquid model [57,58] that (2.4) is indeed able to describe low energy pion dynamics relevant for low-energy baryon physics. Leaving aside the current quark mass matrix for simplicity, the effective action is easy to derive from the partition function, in Euclidean space, since the integrand is quadratic in the quark fields,
\[ Z = \mathcal{N} \int [dU] \overline{[d\psi]} [d\psi] e^{-\int d^4 x \overline{\psi}(x) \left( i \partial^\mu + M \gamma^\mu \right) \psi(x)} \]
\[ = \mathcal{N}' \int [dU] \text{Det}^{N_c} D(U) = \int [dU] e^{-\text{Tr} \ln D(U) - \text{Tr} \ln D_0} \]
defining the Dirac operator in Euclidean space according to
\[ D(U) = i \partial^\mu + M \gamma^\mu. \]
The notation \( D_0 \) in (2.6) stands for the Dirac operator taken in the vacuum, i.e. \( U = 1 \) and its appearance is related to the normalization of the correlation function. Here we introduce the short notation \([dU]\) for the functional integration over the auxiliary fields \( \sigma \) and \( \pi \), which should be understood as a simple notational replacement of the functional integrations over \( \sigma \) and \( \pi \) together with a functional \( \delta \)-function imposing the restriction \( U^{\gamma_5} U^{\gamma_5} = 1 \). The action in terms of the
auxiliary fields is then, from (2.6),
\[
S_{\text{eff}}[\sigma, \vec{\pi}] = -N_c \left( \text{Tr} \ln D(\sigma, \vec{\pi}) - \text{Tr} \ln D_0 \right),
\]
\[
D(\sigma, \vec{\pi}) = i \partial + \sigma + i \gamma^5 \vec{\pi} \cdot \vec{\tau}, \quad D_0 \equiv D(U = 1) = i \partial + M.
\] (2.8a)

(2.8b)

While the integration over the fermion fields was exact, the remaining integration over the auxiliary fields is impossible to be carried out exactly, and requires thus some approximation. At this point, owing to the presence of the factor of $N_c$ in the action, i.e. the power of $N_c$ of the determinant, large $N_c$ may be invoked in order to allow the saddle point evaluation of the remaining functional integration. In this approximation the integral equals the integrand taken at the saddle point, which means that the partition function (2.6) is then determined by the action of the configuration of the fields $\sigma$ and $\vec{\pi}$, $\sigma_c$ and $\vec{\pi}_c$, respectively, which make the action stationary:
\[
\frac{\delta S}{\delta \sigma}_{|_{\sigma = \sigma_c}} = M, \quad \frac{\delta S}{\delta \vec{\pi}^a}_{|_{\vec{\pi}^a = \vec{\pi}_c}} = 0.
\] (2.9)

While the momentum dependent constituent quark mass $M(k)$ was a natural regulator for the Lagrangian (2.3), one has now to worry about the ultraviolet behavior of the effective action related to (2.4). At this point it is convenient to consider the real and the imaginary parts of the effective action separately. Writing $D = D(\sigma, \vec{\pi})$, the imaginary part of the effective action is found to vanish in SU(2). This is so in SU(2) because $D^* = VDV^\dagger$, making $\text{Det}(D/D^\dagger) = 1$ and, therefore, $\text{Im} \text{Tr} \log D \sim \text{Tr} \log(D/D^\dagger) = 0$. In SU(3) such relation between $D$ and $D^\dagger$ does no exist. Nevertheless, the imaginary part of the effective action is found to be finite. Regarding the real part of the effective action
\[
\text{Re} \text{Tr} \log D = \frac{1}{2} \text{Tr} \log \left(D^\dagger D\right),
\] (2.10)
a simple expansion in the derivatives of the fields $\sigma$ and $\vec{\pi}$ [59] shows that the real part of the effective action has divergent terms, which demand for some regularization procedure. The chosen regularization is an important aspect of the definition of the model, not only for computational purposes, but mainly because the model predictions for physical quantities should not depend on the regularization. Starting from (2.8a) and (2.8b),
\[
D^\dagger D = -\partial^2 + M^2 + i \partial \sigma - \partial \vec{\pi} \cdot \vec{\tau} \gamma^5,
\] (2.11)
the expansion in the derivatives of the meson fields leads, in lowest order, to
\[
\text{Re} S_{\text{eff}}[\sigma, \vec{\pi}] = -\frac{1}{2} N_c \left( \text{Tr} \log \left(D^\dagger D\right) - \text{Tr} \log \left(D_0^\dagger D_0\right) \right)
\]
\[
= -\frac{1}{2} N_c \text{Tr} \log \left(1 - \partial^2 + M^2\right)^{-1} \left(i \partial \sigma - \partial \vec{\pi} \cdot \vec{\tau} \gamma^5\right)
\]
\[
= -\frac{1}{2} N_c \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + M^2)} \int d^4x (\partial_\mu \sigma \partial_\mu \sigma + \partial_\mu \vec{\pi} \cdot \partial_\mu \vec{\pi}) + \cdots
\] (2.12)
This result not only indicates that the integral over $k$ is divergent, and therefore the effective action, but also leads to the identification of the kinetic term for the fields $\sigma$ and $\vec{\pi}$. It is then possible, in the context of the expansion in derivatives of the action, to associate the fields $\sigma$ and $\vec{\pi}$ in the Lagrangian with the physical sigma and pion meson fields. The integral over $k$ is proportional to the pion decay constant, under the chosen regularization.

A last remark concerns the similarity of the Lagrangian (2.4) here employed with the modern version of one of the first models displaying SCSB, namely the Nambu–Jona-Lasinio model (NJLM) [60,61]. This model was revived after QCD, with the original nucleon fields replaced by quark fields, and (2.4) coincides with the nonlinear bosonized version of the NJLM. In fact most of the early applications of (2.4) to baryon observables was conducted within this model [39,62]. Reviews of the applications of the NJLM to mesons may be found in [63–66].
2.3 Regularization and parameters

As was discussed in the preceding section, the model Lagrangian (2.4), although easier to do calculations with, is nonrenormalizable. Many physical quantities calculated from it are divergent and hence some regularization must be introduced. Because this is an unavoidable choice, one should regard the regularization as part of the model. The regularization adopted in this work is the Schwinger proper-time regularization [67]. This choice is just based on convenience because it was found that the mean field solutions depend in a very small extent on the regularization [68].

The proper-time regularization is based on an integral representation for the logarithm. The difference of two logarithms can be represented in the form

\[ \ln B - \ln A = \int_{A}^{B} \frac{d\alpha}{\alpha} = \int_{0}^{\infty} du \int_{A}^{B} d\alpha \, e^{-\alpha u} = -\int_{0}^{\infty} \frac{du}{u} (e^{-uB} - e^{-uA}) \]

\[ = -\lim_{\Lambda^{-\infty}} \int_{1/\Lambda^2}^{\infty} \frac{du}{u} (e^{-uB} - e^{-uA}) \].

(2.13)

In calculations, however, in the last expression of (2.13), the proper-time regularization takes \( \Lambda \) as finite, which works as a cut-off. The proper-time regularization for the functional determinant is expressed, for positive definite operators \( A \) and \( B \) by

\[ \text{Tr} \left( \ln B - \ln A \right)_{\text{reg}} = -\int_{0}^{\infty} \frac{du}{u} \phi (u, \Lambda) \left( e^{-uB} - e^{-uA} \right), \]

(2.14)

where \( \phi (u, \Lambda) \) is a function that cuts the lower limit of integration and is otherwise arbitrary. In the original Schwinger proper-time regularization, as above,

\[ \phi (u, \Lambda) = \theta (u - 1/\Lambda^2) \].

(2.15)

In the context of this work, the difference of the logarithms will mean a vacuum subtraction, which is equivalent to take the fields in normal order.

The parameters of the model are the current quark masses, \( \overline{m} = m_u = m_d \), and the cut-off of the regularization functions, \( \Lambda \), in flavor SU(2), supplemented by \( m_s \) in the case of flavor SU(3). The strategy to fix these parameters in the vacuum sector consists in calculating the mass of the pseudo-Goldstone bosons, \( \pi \) (also the \( K \) mesons in the SU(3) case), and their decay constants, in terms of these parameters of the model. Using the experimental values of the masses and decay constants allows then to fix the parameters. The constituent quark mass \( M \) is always left as a free parameter.

In order to calculate the meson masses one needs the mesonic two-point functions. In the pure fermionic NJLM the evaluation of these two-point functions uses the Bethe-Salpeter equation in the ladder approximation [69,70]. In the present case, however, the effective action is bosonized and one can simply follow Refs. [39,71,72] with the advantage that the effective action is regularized from the beginning. The procedure for the calculation of the pion mass, in flavor SU(2), involves just writing the regularized effective action up to second order in the variations of the fields around their stationary vacuum configuration: \( \langle 0 | \sigma | 0 \rangle = \sigma'_0 = M' \), \( \langle 0 | \pi^a | 0 \rangle = \pi'_0 = 0 \) \( (M' = M + \overline{m}) \).

The coefficients of the second order terms in the field variations are identified with the inverse meson propagators. The meson propagators have a pole at the on-shell meson mass, which allows to obtain the meson mass. From Appendix G.1 one finds that

\[ m_{\pi}^2 = \frac{m_1}{M' - m_1} \frac{I_1 (\Lambda, M')}{-m_{\pi}^2, \Lambda, M'} \]

(2.16)

with the integrals defined by (G.8) and (G.12). This expression corresponds to isospin symmetry: \( m = m_1, m_1 = (m_u + m_d)/2 = \overline{m} \).

Equally important for fixing the parameters is the pion decay constant. To determine it one has just to find an expression for the axial current within the same procedure and then to compute the matrix element for the pion decay. Using the Noether theorem the axial current associated
to axial isovector rotations $e^{i\vec{a} \cdot \vec{r} \gamma_5}$ of $U\gamma^5$ is obtained from the first order variation of the action $\delta S_{\text{eff}} = \int d^4x A_{\mu a} (\partial_\mu \alpha_a)$. The matrix element of this current between the vacuum and one pion state yields the pion decay constant. The result from Appendix G.1 is

$$f_\pi^2 = 8N_c M^2 I_2 \left( q^2 = -m_0^2, \Lambda, M' \right).$$

(2.17)

Taking the expressions for (2.16) and (2.17) their product yields

$$m_\pi^2 f_\pi^2 = 8N_c M_1 (\Lambda, M') = -m_1 \langle \bar{\psi} \psi \rangle$$

(2.18)

where the quantity $\langle \bar{\psi} \psi \rangle = \langle \bar{u} u + \bar{d} d \rangle$ is the quark condensate.

$$\langle \bar{\psi} \psi \rangle = -8N_c M_1 (\Lambda, M').$$

(2.19)

In flavor SU(3) the procedure is quite similar, but lengthier. The main difference is now that $MU\gamma^5 = \sigma_\alpha \lambda_\alpha + i\gamma^5 \pi_\alpha \lambda_\alpha$. (2.20)

While in SU($N_f = 2$) the chiral symmetry was broken by nonvanishing vacuum expectation value of $\sigma$, in SU($N_f = 3$) there are two such non-vanishing values, that of $\sigma_0$ (taking over the role of $\sigma$) and $\sigma_8$, whose non-vanishing vacuum expectation value results from the strange quark mass, which breaks flavor SU(3). In this flavor case, it is possible to evaluate the pion and kaon masses and decay constants along the same lines as in flavor SU(2). For instance, the kaon mass is (here $m_0$ equals $m_1 = \overline{m}$ of the 2 flavor case [73–75]

$$m_K^2 = \left( m_0 I_1 (\Lambda, M) + m_3 I_1 (\Lambda, M) \right) 2I_2 \left( -m_K^2, \Lambda, M \right) + (m_1 - m_3)^2.$$ (2.21)

Using the experimental values $m_\pi =$ 140 MeV, $m_\pi =$ 93 MeV, $m_K = 495$ MeV, and choosing a value for the constituent quark mass $M$ (the only free parameter of the model), as the input for the respective equations yields for the nonstrange quark current moments $m_u = m_d \sim 7$ MeV, while the strange current quark mass comes around $m_s \sim 165$ MeV. These values are in good agreement with the phenomenological accepted ones, in particular $m_s / \overline{m} \simeq 24$. In the process the cutoff is determined around $\Lambda \sim 650$ MeV.

The effects of the choice of regularization in the vacuum sector were investigated in [76]. Earlier and more detailed work concerning the fixing of the parameters can be found in [68,76,77].

### 2.4 The model baryon state

A tool which can be used to study both the hadron spectrum and matrix elements of quark currents, the subject of the next chapter, is that of the correlation functions. The correlation function is defined by

$$C_{BB} (T) \equiv C_{BB} \left( T/2, x^i; -T/2, x \right) \equiv \langle 0 | J_B (T/2, x^i) J_B^\dagger (-T/2, x) | 0 \rangle$$ (2.23)

where the baryon current is defined by [39,78]

$$J_B (x) = \frac{1}{N_c} \Gamma_B^{(f_1 \ldots f_{N_c})} \varepsilon_{\alpha_1 \ldots \alpha_{N_c}} \bar{\psi}_{f_1 \alpha_1} (x) \cdots \psi_{f_{N_c} \alpha_{N_c}} (x).$$ (2.24)

This current carries no color due to the antisymmetric tensor in the color indices $\alpha_1, \ldots, \alpha_{N_c}$. It is no surprise to find $N_c$ here, since all of the framework relies on the large $N_c$ expansion. Only at the end does one replace $N_c$ by 3. The matrix $\Gamma_B^{(f)}$ combines the flavor quantum numbers of the $N_c$ quark fields $\{ f \} = \{ f_1, \ldots, f_{N_c} \}$ in order to construct the baryon quantum numbers, here summarized by $B$. As will be seen in the following, it does not have to be known, a difficult task since $N_c$ is not fixed, but it is enough that it exists.

One of the more evident characteristic properties is that, restricting it to a definite baryon, using a complete set of intermediate states $| b \rangle$ and Euclidean time, the correlation functions appears as

$$C_{BB} (T) \sim \sum_b | \langle 0 | J_B | b \rangle |^2 e^{-M_b T}$$

(2.22)

which allows to extract, at large $T$, the mass of the lowest lying state with the quantum numbers of $B$. 

---

[73–75]
The final result is given by
\[ C_{B'B}(T) \equiv \langle 0 | J_{B'}(x') J_{B}^†(x) | 0 \rangle^E \]
\[ = N \Gamma^{(H \ldots F_{Nc})} \Gamma^{(g_{1} \ldots g_{Nc})} \int [dU] \text{Det}^{N_{c}} [D^{E}(U)] \prod_{j=1}^{N_{c}} G^{E}_{j, \delta}(U; x', x) \quad (2.25) \]

In this expression the notation \([dU]\) is used, as in the previous section, to denote a functional integration over the \(\pi^{a} \) in \(U^{\gamma_{5}} = \exp i\gamma^{5} \pi^{a} \lambda^{a} \), including implicitly the nonlinear restriction due to \(U^{\gamma_{5}†}U^{\gamma_{5}} = 1 \). It is obvious that the remaining functional integration cannot be carried out exactly. The approximation to use is that of large \(N_{c} \), which is both the power of the determinant and the number factors in the product of propagators \(G(U)\).

The limit of large \(N_{c} \) allows to use the saddle point approximation to carry out the integral over \(U \). In this approximation the integral equals the integrand \(\exp \{ S_{\text{eff}} [U] \}\) at the configuration \(U_{c} \) which makes \(S_{\text{eff}} [U] \) stationary. One can obtain the expression for the effective action from Eq. (D.35), disregarding for this purpose the other terms it is deduced along with in Appendix (D). The effective action is, to lowest orders in \(N_{c} \) and \(\delta m\), given by
\[ S_{\text{eff}} [U] \equiv S_{\text{eff}}^{(G_{0}, \delta m_{0})} [U] = -T (M [U] - v.s.) , \quad (2.26a) \]
\[ \frac{1}{N_{c}} M [U] = \varepsilon_{\nu} [U] + \frac{1}{2} \sum_{n} \frac{1}{2\sqrt{\pi}} \int_{1/\Lambda^{2}}^{\infty} \frac{du}{u} e^{-u^{2} [U]} - v.s. , \quad (2.26b) \]
\[ \varepsilon_{n} [U] = \langle n | h (U) | n \rangle , \quad (2.26c) \]
where the last equation makes it clear that the spectrum of the operator \(h \) depends on \(U \) and \(h (U) | n \rangle = \varepsilon_{n} | n \rangle \), (v.s. stands for vacuum subtraction and represents simply the subtracted term with \(U = 1 \) and \(v \) is the valence level.) The operator \(h \) is the Dirac one-particle operator
\[ h (U) = -i\gamma^{0} \gamma^{i} \partial_{i} + \gamma^{0} MU^{\gamma_{5}} + \pi \gamma^{0} , \quad (2.27) \]
which depends on the fields \(U = e^{i\pi \cdot \vec{r}}\). A further approximation is used for these fields, namely that of the hedgehog, i.e.
\[ \pi \cdot \vec{r} = P (r) \hat{n} \cdot \vec{r} , \quad (2.28) \]
where \(P(r)\) is the so-called profile function of the meson fields and fulfills \(P(0) = -\pi\), which is related to the baryon number one. This configuration of the meson fields has been studied earlier and found to correspond to the stationary solution of (2.26), but in the case of the nonlinear sigma model \([79, 80]\). Its application in this model is favoured for the hedgehog is the most symmetric configuration as expected for the classical solution. It has recently been noticed that there may be, however, instabilities in SU(3) flavor CQSM due to the hedgehog \([81]\). The hedgehog configuration is not invariant under independent rotations in configuration and flavor spaces, but is invariant under the rotations generated by the grand-spin operator
\[ G^{a} = J^{a} + T^{a} , \quad T^{a} = \tau^{a}/2 , \quad a = 1, 2, 3 , \quad (2.29) \]
with \(J^{a} \) and \(T^{a} \) the angular momentum and isospin operators, respectively. The hedgehog is the most symmetric form for the meson field configuration and, therefore, makes it easier, in particular, to rewrite the stationary condition in terms of the profile function \(P (r)\). Using the hedgehog ansatz, the one-particle Hamiltonian becomes
\[ h (U) = -i\gamma^{0} \gamma^{i} \partial_{i} + \gamma^{0} M \left( \cos P (r) + i\gamma^{5} \hat{n} \cdot \vec{r} \sin P (r) \right) + \pi \gamma^{0} , \quad (2.30) \]

---

5The notation \(C_{B'B}(T)\) reminds that the correlation function is to be understood in the limit of large time separation without explicitly alluding to that in the expressions.

6Det\(^{N_{c}} [D^{E}(U)] = \text{exp} \{ N_{c} \text{Tr log} [D^{E}(U)] \}\).

\[ \tau e^{G^{a} \cdot n_{a}} \hat{n} \cdot \vec{r} e^{-G^{a} \cdot n_{a}} = \hat{n} \cdot \vec{r} . \]
The chiral quark-soliton model

and the condition to determine the classical profile $P_c(r)$ (soliton) is the stationary condition for the action (2.26a)

$$\frac{\delta S_{\text{eff}}[U]}{\delta U}|_{U=U_c} = 0.$$

(2.31)

The configuration $U_c$ is the one that makes the action stationary. Relation (2.31) corresponds, using (2.26a) and (2.30), to the following relation involving a scalar $S(r)$ and a pseudoscalar $P(r)$ densities:

$$S(r) \sin P_c(r) - P(r) \cos P_c(r) = 0,$$

(2.32)

$$\frac{1}{N_c M} S(r) = \int d\Omega \left( \overline{\phi \nabla} \phi - \sum_n \overline{\phi_n} (r) \phi_n (r) \mathcal{R}_1 (\varepsilon_n) - \text{v.s.} \right)$$

(2.33a)

$$\frac{1}{N_c M} P(r) = \int d\Omega \left( \overline{\phi \nabla} \phi \ gamma \tau n dot \overline{\phi \nabla} \phi + \sum_n \overline{\phi_n} (r) \ gamma \tau n dot \overline{\phi_n} (r) \mathcal{R}_1 (\varepsilon_n) - \text{v.s.} \right)$$

(2.33b)

with the regularization function given by

$$\mathcal{R}_1 (\varepsilon_n) = -\frac{\varepsilon_n}{2\sqrt{\pi}} \int_{1/\Lambda^2}^{\infty} du \frac{d \varepsilon}{u \sqrt{\varepsilon}}$$

(2.34)

and where the $\phi_n$ and $\varepsilon_n$ are the eigenfunctions and eigenenergies of $h (P(r))$

$$h (P(r)) \phi_n = \varepsilon_n \phi_n.$$

(2.35)

The profile for this classical solution, $P_c(r)$, is determined from (2.32) by a functional fixed point method (also called self-consistent), which consists in starting from a given profile, calculating the scalar (2.33a) and pseudoscalar (2.33b) densities and then use (2.32) to compute a new profile from these densities. The process is repeated until the difference between consecutive profiles is below a desired value. It yields thus the classical solution, $P_c$, first obtained with this fixed point method in [82,83]. This classical solution is classified as a nontopological soliton [84]. A nontopological soliton means that it is a localized stable bound state solution to the nonlinear problem, (2.31) and (2.26a), which is nondispersive, contrary to, e.g. wave packets. They are characterized by a finite stable shape in space. The nontopological character comes from the fact that their boundary condition at the infinity is the same as that of the vacuum, i.e.

$$P_c(r) \to 0.$$  

(2.36)

This implies a degenerate vacuum in contradistinction to the topological soliton which is associated with degenerate vacua.

The mean field solution described above corresponds to the Hartree-Fock approximation. It is a static solution with baryon number one. It may be described as a valence level (v) occupied by $N_c$ quarks bound by the chiral hedgehog field, whose energy is identified with the energy of the polarized quark sea. The spectrum of (2.35) is depicted in Fig. 2.1, and consists of a bound valence level and sea levels all occupied by $N_c$ quarks.

![Figure 2.1: Dirac one-particle spectrum.](image)
2.4 The model baryon state

The classical solution does not have, however, the proper quantum numbers of baryons. In order to obtain the states with proper baryon quantum numbers the semiclassical quantization of the zero modes is employed. Looking at the hedgehog configuration it is invariant neither with respect to spatial rotations nor to rotations in flavor space. Since it is localized it is also not translational invariant. In this way the soliton does not have definite momentum, angular momentum, isospin, and even center-of-mass position. This means that the rotational and translational symmetries in configuration space, and ‘rotational’ symmetry in flavor space are spontaneously broken, i.e. they are not observed due to the finite extension of baryons and to the mean field calculation. If the chiral field \( U_c(x) \) is the saddle point for the action so are the corresponding rotated fields in configuration space or in flavor space, due to isotropy of the functional of the chiral field (2.26) in configuration and flavor spaces\(^8\). The change of orientation of the soliton in configuration and flavor spaces requires no energy, hence the name zero modes. The mean field solution corresponds therefore to a family of degenerate solutions differing in orientation in such spaces. This fact allows for an easy introduction of collective coordinates closely connected with the orientation.

The consequence of this is that the changes of the soliton configuration compatible with the symmetries are unrestrained, i.e. not subjected to restoring forces. Fluctuations not compatible with the symmetries correspond to modes normal to the zero modes and are subject to restoring forces, like e.g. the vibrations of the soliton surface (which are related to meson loops). Since quantization means the establishment of proper commutation relations between coordinates and their conjugated momentum operators, it is necessary to have time dependent solutions, which make the action stationary, in order to be able to define momentum variables. From the above, it is clear that the most natural construction of such time-dependent solutions are solitons rotating and moving in configuration space with small time-dependent velocities. In fact, it will be the rotations in flavor SU(3) space, generated by the flavor generators \( \lambda^a, a = 1, \ldots, 8 \), which will be used below in order for the strange quark to also participate in the rotations and thus accessing the hyperons. In this picture the hedgehog soliton is pictured as a superposition of baryon states, with different quantum numbers, which become disentangled in the quantization procedure. In other words, the semiclassical quantization is a process to go beyond the mean field and to identify the low-lying excited rotational states generated by the rotating soliton with baryon states.

The starting point for the procedure is the embedding of the flavor SU(2) hedgehog into a flavor SU(3) matrix

\[
U = e^{iP(r)} \sum_{a=1}^{8} n^a \lambda^a \rightarrow \begin{pmatrix} e^{iP(r)} \sum_{i=1}^{3} n^i \tau^i & 0 \\ 0 & 1 \end{pmatrix}.
\]

This embedding is the one that uses the fact that the SU(2) soliton leads to lowest energy solutions in most chiral models. The \( U \) matrix is invariant again with respect to rotations generated by the grand-spin operators and also, due to the form of the embedding, to rotations generated by \( \lambda^8 \). \( U_c \) will denote in flavor SU(3) the classical solution in the form of (2.37) with the flavor SU(2) soliton in the upper left corner.

Going back to (2.25), the functional integral over \( U \) is approximated by restricting the possible configurations of \( U \) to the ones corresponding to the zero modes. This is done in practical terms by replacing the functional integral according to

\[
U(x) \rightarrow A(t)U_c(x-z)A(t) \quad \text{and} \quad d[U] \rightarrow \int d^3 z \ d[A],
\]

where the SU(3) matrices \( A(t) \) (the explicit time dependence will be dropped in the following for simplicity), defining the finite rotation of the soliton, may be taken to represent the orientation of the soliton in flavor space, i.e. its collective coordinate in flavor space.

\(^8\)When considering both the rotations in configuration and flavor spaces, it can be shown that the first one can be absorbed in the second so that it is necessary to consider simply a rotation in flavor space.
Since the time-dependent fields will be derived from the static soliton, it is convenient to distinguish between a body-fixed coordinate system, attached to the static soliton, and a laboratory coordinate system, where the soliton rotates with small time-dependent angular velocity.

The spectral representation for the propagator in a static hedgehog field is

\[ \langle x^4 | G(U_c) | x^4 \rangle = \langle x^4 | D^{-1}(U_c) | x^4 \rangle = \theta(x^4 - x_{\text{min}}) \sum_{n \geq 0} e^{-\epsilon_n (x^4 - x_{\text{min}})} |n\rangle \langle n| - \theta(x^4 - x^4) \sum_{n < 0} e^{-\epsilon_n (x^4 - x_{\text{min}})} |n\rangle \langle n| \]  

(2.39)

where the states \(|n\rangle\) and energies are solutions to the one-particle Dirac Hamiltonian: \(h(U_c) |n\rangle = \epsilon_n |n\rangle\). The Dirac operator \(D(U)\), now \(U\) is given by the ansatz (2.38), becomes in this case

\[ D(U) = e^{iz \cdot P} A \left[ D(U_c) + A^\dagger \hat{A} + i \gamma_4 A^\dagger \delta m A \right] A^\dagger e^{-iz \cdot P}, \]  

(2.40a)

\[ D(U_c) = \partial_4 + h(U_c), \]  

(2.40b)

\[ \delta m = M_1 + M_2 \lambda^8, \]  

(2.40c)

and the quark propagator in the background of a slowly rotating hedgehog field

\[ G(U) = e^{iz \cdot P} A \left[ D(U_c) + A^\dagger \hat{A} + i \gamma_4 A^\dagger \delta m A \right]^{-1} A^\dagger e^{-iz \cdot P} \]

\[ = e^{iz \cdot P} A \left[ G(U_c) - G(U_c) \left( A^\dagger \hat{A} + i \gamma_4 A^\dagger \delta m A \right) G(U_c) \right. \]

\[ + G(U_c) \left( A^\dagger \hat{A} + i \gamma_4 A^\dagger \delta m A \right) G(U_c) \left( A^\dagger \hat{A} + i \gamma_4 A^\dagger \delta m A \right) G(U_c) + \cdots \left] A^\dagger e^{-iz \cdot P} \]  

(2.41)

where it is also shown how\(^9\) it is expanded\(^10\) in the quantity \(A^\dagger \hat{A}\) (angular velocity) and also in the strange-nonstrange quark mass difference \(\delta m\). For the fermionic determinant\(^12\) one finds

\[ \text{Tr} \log [D(U)] = \text{Tr} \log \left[ e^{iz \cdot P} A \left( D(U_c) + A^\dagger \hat{A} + i \gamma_4 A^\dagger \delta m A \right) A^\dagger e^{-iz \cdot P} \right] \]

\[ = \text{Tr} \log \left( D(U_c) + A^\dagger \hat{A} + i \gamma_4 A^\dagger \delta m A \right). \]  

(2.43)

Furthermore, introducing the definition of the angular velocities \(\Omega^\alpha_k(t)\)

\[ A^\dagger \hat{A}(t) = \frac{i \Omega^\alpha_k(t)}{2} \lambda^\alpha, \]  

(2.44)

and the definition of the matrices \(D^{(8)}_{\alpha \beta}(A)\) in the regular representation

\[ D^{(8)}_{\alpha \beta} = \frac{1}{2} \text{tr} \left( A^\dagger \lambda^\alpha A \lambda^\beta \right), \]  

(2.45)

which means that under rotations the Gell-Mann matrices transform according to

\[ A^\dagger \lambda^\alpha A = \left\{ \begin{array}{c} \lambda^0 , \chi = 0 \\ D^{(8)}_{\alpha \beta} \lambda^\beta , \chi \in \{1, \ldots, 8\} \end{array} \right\}, \]  

(2.46)

on can proceed by expanding (2.25) in powers of the angular velocities and \(\delta m\), both of which are considered as small. In Appendix D, eq. (D.34) one finds that (2.25) becomes

\[ \mathcal{C}_{BB'} \left( T, x' - x \right) = \langle 0 | J_{B'}(x') J_B(x) | 0 \rangle \]

\[ = \mathcal{N} \int dz \int dA \left( T/2 \right) \int dA \left( -T/2 \right) \Gamma_{B' B}^{(g_1 \ldots g_{N_c})} \Gamma_{B}^{(g_1 \ldots g_{N_c})} \]

\[ \times \left[ [dA] P \prod_{k} \left( A \phi_{V}(x' + z) \right)_{f_k} \left( \phi_{V}^\dagger(x + z) A^\dagger \right)_{g_k} P \exp \left( -S_{\text{eff}} [A] \right) \right]. \]  

(2.47)

\(^9\)The term associated with the translations \([39]\), of the form \(T^j \partial_x = -\partial_z \cdot \nabla\) is not written for simplicity.

\(^{10}\)\((A + B)^{-1} = A^{-1} - (A + B)^{-1} BA^{-1} = A^{-1} - A^{-1} BA^{-1} + A^{-1} BA^{-1} BA^{-1} + \cdots\)

\(^{11}\)The coordinate space matrix element of the propagator in the background of the chiral field is

\[ \langle x' | G(U) | x \rangle = \langle x' | A \left( D(U_c) + A^\dagger \hat{A} + \gamma_0 A^\dagger \delta m A (t) \right)^{-1} A^\dagger | x \rangle \]

(2.42)

\(^{12}\)\(\text{Det}_{N_c} [D^E(U)] = \exp \left( N_c \text{Tr} \log [D^E(U)] \right)\)
As is explained in Appendix D the product of valence functions comes from the product of propagators, since it is saturated at large Euclidean time by the valence functions. The effective action is

$$S_{\text{eff}} [A] = T L_E [A] = \int_{-T/2}^{T/2} d\tau L_E [A]$$

(2.48)

where $\tau$ is the Euclidean time and the Lagrangian\(^{13}\) is (D.48b)

$$L_E [A] = -M_c + \frac{1}{2} i I_1 i \Omega_E i \Omega_E + \frac{1}{2} i I_2 i \Omega_E i \Omega_E + \sigma + 2M_8 D_{8i}^{(8)} (A) K_1 i \Omega_E + 2M_8 D_{8a}^{(8)} (A) K_2 i \Omega_E - \frac{N_c}{2\sqrt{3}} i \Omega_E^3.$$

Changing variables in (2.47) according to $y = x + z, \ y' = x' + z$ the integration over $x, x'$ gives

$$\int d^3 x \ d^3 x' \ e^{i p \cdot x} e^{-i p' \cdot x'} \langle 0 | J_B' (x') J_B (x) | 0 \rangle =$$

$$\frac{1}{Z} \int d^3 z \int dA (T/2) \int dA (-T/2) \int [dA] \ e^{i p' \cdot z} \left[ \int d^3 y' \ e^{-i p' \cdot y'} \Gamma^{(f)}_{B'} \prod_{k=1}^{N_c} (A \phi_V (y'))_{f_k} \right]$$

$$\times \ e^{-i p \cdot z} \left[ \int d^3 y \ e^{i p \cdot y} \Gamma^{(g)}_{B} \prod_{k=1}^{N_c} \left( \phi_V (y) A \right)_{g_k} \right] e^{-S_{\text{eff}} (A U A')}$$

$$= \frac{1}{Z} \int d^3 z \ e^{i (p' \cdot p) \cdot z} \langle B' (T/2) | e^{-T H_{\text{coll}}} | B (-T/2) \rangle$$

(2.50)

where the definitions of the collective wave functions were introduced

$$\int d^3 y' \ e^{-i p' \cdot y'} \Gamma^{(f)}_{B'} \prod_{k=1}^{N_c} (A (T/2) \phi_V (y'))_{f_k} \rightarrow \langle B' | A (T/2) \rangle$$

(2.51a)

$$\int d^3 y \ e^{i p \cdot y} \Gamma^{(g)}_{B} \prod_{k=1}^{N_c} \left( \phi_V (y) A \right)_{g_k} \rightarrow \langle A (-T/2) | B \rangle.$$

(2.51b)

very much in order to comply with the general path integral formalism. In these expressions it is clear the large $N_c$ is not considered when choosing the baryon collective wave functions on the right-hand side of (2.51), which are for $N_c = 3$. In the path integral formalism, without stating exactly what the coordinates for the matrix $A$ are, the matrix element of the time evolution operator between coordinate states at different times equals the path integral of the effective action\(^{14}\), i.e.

$$\int [dA] \ e^{-S_{\text{eff}} (A U A')} = \int [dA] \ exp \left( - \int_{-T/2}^{T/2} d\tau L_E [A] \right) = \langle A (T/2) | e^{-T H_{\text{coll}}} | A (-T/2) \rangle.$$

(2.53)

Nonrelativistically, the final result fulfills

$$\langle B' (S'_t, p') | B (S_t, p) \rangle = (2\pi)^3 \delta (p' - p) \delta_{BB'}.$$

(2.54)

The baryon states are identified with the eigenstates of the collective Hamiltonian according to the previous discussion. From Appendix E, the collective Hamiltonian reads (D.64)

$$H [q, p] = M_c + H_{\text{sc}}^{\text{coll}} + H_{\text{ab}}^{\text{coll}},$$

(2.55a)

$$H_{\text{sc}}^{\text{coll}} = \frac{1}{2I_1} J_i J_i + \frac{1}{2I_2} J_a J_a,$$

(2.55b)

$$H_{\text{ab}}^{\text{coll}} = \sigma + 2M_8 K_1 D_{8i}^{(8)} J_i + 2M_8 K_2 D_{8a}^{(8)} J_a + O (M_8^2)$$

\(^{13}\) $\sigma = \Sigma [SU(2)] \left( M_1 + \frac{1}{2\sqrt{3}} M_8 D_{8}^{(8)} (A) \right) / m$ is the sigma term in $SU(2)$ (D.38).

\(^{14}\) Inserting complete states, $\Psi_B$ ($B$ stands for the quantum numbers of these eigenfunctions), of the collective Hamiltonian $H_{\text{coll}}$, one has further that

$$\langle A (T/2) | e^{-T H_{\text{coll}}} | A (-T/2) \rangle = \sum_B \langle \Psi_B (A (T/2)) \Psi_B (A (-T/2)) \rangle e^{-T E_{\text{coll}}}.$$
2.5 Computing form factors

It is composed of a classic part, the energy of the soliton, a component which does not break the flavor SU(3) symmetry, $H_{\text{coll}}^\text{sc}$, and one that does, $H_{\text{coll}}^{ab}$. The symmetric part coincides with the symmetries of the hedgehog shape of the chiral fields, in particular to its invariance under rotations generated by the grand-spin operators. It can be shown that the wave functions (A represents the direction in flavor space) for a representation of dimension $n$ are \[ \langle A | B^{\text{sc}} \rangle \equiv \langle A | B^{\text{sc}} (n; Y, T, T_3; Y', J, J_3) \rangle = \sqrt{\text{dim}(n)} \langle - \rangle^{Y/2+J_3} D_{(Y,T,T_3)(Y',J,J_3)}^{(n)} \rangle \] (2.56)

It is however necessary to take into account the symmetry breaking peace in the Hamiltonian (2.55c). This is done by means of perturbation theory. Treating the symmetry breaking piece linear in $M_8$ as a perturbation to the symmetric piece to first order in $M_8$ leads to the approximation of the octet baryon state used in this work \[ 74 \].

The expression for the collective Hamiltonian allows also to obtain the “quantization rules”. Although these follow from the commutation rules between collective coordinates and momenta, they are to be understood here as the prescription to replace the classical angular velocities once the quantum numbers are determined in \[ 39,74,75 \].

\[ |B, 8 \rangle = |B^{\text{sc}}, 8 \rangle + \sum_{n' \neq 8} \frac{\langle B, n' | H_{\text{coll}}^{ab} | B, 8 \rangle}{E_{\text{coll}}^{(n')} - E_{\text{coll}}^{(n)}} + \ldots \]

\[ \simeq |B^{\text{sc}}, 8 \rangle + c_{70}(Y, T) |B^{\text{sc}}, 70 \rangle + c_{27}(Y, T) |B^{\text{sc}}, 27 \rangle \]

where $B$, $B^{\text{sc}}$ stand for the quantum numbers $(Y, T, T_3; Y', J, J_3)$ and $n$ and $n'$ stand for the dimension of the representations. The only representations with which the octet is mixed by the perturbation are the representations $70$ and $27$. The coefficients $c_{70}$ and $c_{27}$ are linear in $M_8$ \( F.9 \) were determined in \[ 39,74,75 \].

The expression for the collective Hamiltonian allows also to obtain the “quantization rules”. Although these follow from the commutation rules between collective coordinates and momenta, they are to be understood here as the prescription to replace the classical angular velocities once the expansion in these velocities is made. From Appendix (D) and comparing with (2.55b), the quantization prescription consists in replacing in (2.44)

\[ \iota \Theta^E_{\alpha} = \left( \frac{J_{I_1} - 2 K_{I_1} M_8 D_{8i}^{(8)}}{I_1} \right) \delta_{\alpha 1} + \left( \frac{J_{I_2} - 2 K_{I_2} M_8 D_{8a}^{(8)}}{I_2} \right) \delta_{\alpha a} - \frac{N_c}{\sqrt{3}} \delta_{\alpha 8}. \]

Up to now the limits of small angular velocities and small strange-to-nonstrange quark mass difference, $\delta m$, were taken on the same footing. However, it is not certain that the two limits may be interchanged. It is clear from the previous considerations that the limit of small angular velocities is here taken before the limit of small $\delta m$.

2.5 Computing form factors

In order to compute observables, vector currents in the present case, the starting point is a correlation function which includes the appropriate vector current

\[ C_{\alpha \beta}^{\text{MN}}(T; x' - x) = \langle 0 | J_B(T/2, x') \psi^\dagger(y) O^{\alpha \beta} \psi(y) J_B^\dagger(-T/2, x) | 0 \rangle \]

\[ = \langle 0 | J_B(x') \psi^\dagger O^{\alpha \beta} \psi J_B^\dagger(x) | 0 \rangle_{V} + \langle 0 | J_B(x') \psi^\dagger O^{\alpha \beta} \psi J_B^\dagger(x) | 0 \rangle_{S}. \] (2.59)

The above correlation function splits into two different pieces, one of them is called valence contribution (v) and the other the sea contribution (s). As it is explained in Appendix E, the reason for this separate contributions lies in the fact that the external current associated with the vector current appears both in the product of propagators and in the fermionic determinant. The valence piece is connected to the product of propagators and the sea contribution to the determinant, as is clear from (E.8)

\[ \langle 0 | J_B(x') \psi^\dagger O^{\alpha \beta} \psi J_B^\dagger(x) | 0 \rangle_{V} = \frac{N_c}{2} \int \Gamma_B^{(f)} \Gamma_B^{(g)} \prod_{k=2}^{N_c} \langle T/2, x' | G_{f_k g_k}(U) | -T/2, x \rangle \]

\[ \Gamma_B^{(f)} \Gamma_B^{(g)} \]

\[ \prod_{k=2}^{N_c} \langle T/2, x' | G_{f_k g_k}(U) | -T/2, x \rangle \]
\[ \times \langle T/2, x' | G_{fd}(U) | 0, 0 \rangle \mathcal{O}^{d\overline{d}}_{dd} \langle 0, 0 | G_{d\overline{d}i}(U) | -T/2, x \rangle e^{N_c \text{Tr} \log D(U)}. \] (2.60)

for the valence part and from (E.9)

\[ \langle 0 | J_B(x') \psi \mathcal{O}^{\mu \nu \chi} \psi J_B^\dagger(x) | 0 \rangle_S = \frac{N_c}{Z} \frac{\delta}{\delta s^\mu(0, z)} \text{Tr} \{ \log D(U, s) \} |_{s^\mu = 0} e^{N_c \text{Tr} \log D(U)}. \] (2.61)

for the sea part, where \( s^\mu \) is the external current:

\[ D(U, s) = D(U) - s^\mu \mathcal{O}^{\mu \nu \chi}. \] (2.62)

Following the same kind of steps as in (2.50), the integration over \( x, x' \) gives (E.20)

\[ \int d^3x \, d^3x' \, e^{i(p - p') \cdot x} e^{-i(p' - p) \cdot x'} \langle 0 | J_B(x') \psi \mathcal{O}^{\mu \nu \chi} \psi J_B^\dagger(x) | 0 \rangle = \frac{1}{Z} \int d^3z \, e^{i(q - p) \cdot z} \int dA \, \Psi^*_B(A) \mathcal{F}^{\mu \nu \chi}_S(z) \Psi_B(A) \] (2.63)

from which follows that the valence part is

\[ \langle B(S_3', p') | \mathcal{O}^{\mu \nu \chi} \psi | B(S_3, p) \rangle = \int d^3z \, e^{iQ \cdot z} \langle B' | \mathcal{F}^{\mu \nu \chi}_S(z) | B \rangle \] (2.64)

with \( Q = p' - p \) the momentum transfer and the baryon states given by (2.57). The expression for \( \mathcal{F}^{\mu \nu \chi}_S(z) \) is given by (E.18). The sea contribution has the same form as (2.64) with \( \mathcal{F}^{\mu \nu \chi}_S(z) \) now given by (E.34).

The explicit forms for \( \mathcal{F}^{\mu \nu \chi}_V \) and \( \mathcal{F}^{\mu \nu \chi}_S \) are developed in (E.3), Section E.3.1 and Section E.3.2, respectively. The results are given in (E.52) for the valence and in (E.136) for the sea. These results are the outcome of an expansion up to terms linear in the angular velocities and \( \delta m \) (2.40c). One of the main features of this expansion is the time ordering of the collective operators. In this context the designation of collective operator applies both to the operators \( J_\theta \) in the quantization procedure (2.58), which may act upon the collective wave functions (2.57), and the matrices \( D_{\alpha \beta}(A) \) coming from the rotations like (2.46), which are similar to the baryon wave functions. These objects, \( J_\theta \) and \( D_{\alpha \beta}(A) \), do not commute and are inserted into the expressions for \( \mathcal{F}^{\mu \nu \chi}_V \) and \( \mathcal{F}^{\mu \nu \chi}_S \) with different time labels.

In the cases when both of these operators appear, it is necessary to order the time \( (t_f) \) of the angular velocity \( i\Omega^E_\theta(t_f) \) and the time \( (t_D) \) of the matrix \( D_{\alpha \beta}[A(t_D)] \), before replacing in the expansions the angular velocities \( i\Omega^E_\theta(t_f) \) by the operators \( J_\theta/I \) using (2.58). This time ordering is implemented, schematically, by

\[ \int dA \, (T/2) \int dA \, (-T/2) \int [da] \, e^{-S_{\text{eff}}(AUA^\dagger)} \langle B' | A(T/2) \rangle \, D_{\alpha \beta}[A(t_D)] \, i\Omega^E_\theta(t_f) \langle A(-T/2) | B \rangle. \] (2.65)

The expressions also take into account that the propagator is a direct sum of strange and nonstrange contributions, which can be separated using the appropriate projectors \( P_T \) and \( P_S \) (A.14)

\[ G(U_c) = G^T(U_c) \, P_T + G^S(U_c) \, P_S. \] (2.66)

While the spectral representation of \( G^T(U_c) \) is made from the eigenstates and eigenfunctions of the hedgehog Dirac Hamiltonian \( \mathcal{H}(U_c) \), the spectral representation of the strange part \( G^S(U_c) \) is made from solutions to the free Dirac equation. Writing the propagator according to (2.66) allows to clearly identify in the result the terms due to the strange quark.

In order to understand what is involved in the calculation of observables one may write the results (E.52) or (E.136) appearing in (2.64) schematically as

\[ \langle B' | \mathcal{F}^{\mu \nu \chi}_S(z) | B \rangle = \langle B' \sum_\gamma f_\gamma^{\mu \nu \chi}(z) \Lambda^{\chi}_{\gamma} | B \rangle = \sum_\gamma f_\gamma^{\mu \nu \chi}(z) \langle B' \Lambda^{\chi}_{\gamma} | B \rangle. \] (2.67)

The matrix element of an observable may thus be written as a sum over the possible \( \gamma \) collective operators \( \Lambda^{\chi}_{\gamma} \, (D_{\alpha \beta}, J_\alpha, \text{or products of these}) \) associated with the observable in question. The
elements in the sum are products of the so-called densities \( f_{\mu \gamma}^{\chi}(z) \) with the matrix elements of the collective operators \( \langle B' | \Lambda_{\mu \gamma}^{\chi} | B \rangle \). The densities \( f_{\mu \gamma}^{\chi}(z) \) are mainly composed of matrix elements, or products of matrix elements, of the one-particle Dirac equation eigenfunctions, either in the chiral hedgehog field or in the vacuum (strange quarks), and functions of the eigenenergies, e.g. the regularization functions. For specific cases of the Lorentz index \( \mu \), i.e. depending on the hermiticity of the Dirac part of the observable, the densities may be simplified from the expressions in (E.52), (E.136) using the transformation properties of the matrix elements under the \( G_5 \) parity, Section G.2.1. The operators and their matrix elements \( \langle B' | \Lambda_{\mu \gamma}^{\chi} | B \rangle \) are described and computed in Appendix F. Expressions (E.52) and (E.136) may be easily generalized to other Lorentz structures, thus allowing to access a wealth of other possible observables, beyond the simple Lorentz-vector and axial-vector cases considered in this work.

2.6 Further aspects of the approach

Since this is a work in flavor SU(3), the “flavor symmetry”\(^{15}\) breaking is naturally the source of novel aspects with respect to the flavor SU(2). These comments summarize the approach used in this work for the symmetry breaking and also for the quantization difficulties brought about by flavor SU(3).

2.6.1 SU(3) symmetry breaking and zero modes

There are today a lot of indications that the strange sea quarks have sizeable influence upon the nucleon properties. In this way it is unavoidable to study the nucleon in flavor SU(3), while flavor SU(2) is inadequate for that purposes. The strange quark poses, however, some difficulties stemming from the fact that its mass is neither light nor heavy. In other words, while in the limit of isospin symmetry the mass of the lightest quarks \( u \) and \( d \) is almost equal and small, the mass of the strange quark is at least 20 times bigger, although small when compared to the charm mass which is higher that 1 GeV. Such scenario implies that SU(3) breaking effects must be taken into consideration with some care.

In the approach followed in this work, the symmetry breaking effects were dealt with in a perturbative approach, based on the assumption that the SU(3) symmetry breaking is not very large, so that the Hamiltonian may be divided into a symmetry conserving and a symmetry breaking parts (2.55a). In the CQSM this approach as led to correct mass predictions for spin 1/2 octet and spin 3/2 decuplet baryons [39,86]. There are nevertheless other approaches for including symmetry breaking, developed mainly in the Skyrme model, but applicable also in the CQSM. One is the bound state approach [87] which is based on the assumption that the symmetry breaking is large and that an hyperon is then a bound state of a SU(2) soliton and a kaon. There have been recently claims that the hedgehog may be unstable in the case of flavor SU(3) in the context of a simplified bound state approach for the strange degrees of freedom [81].

Another approach is that of [88] which corresponds to an intermediate strength, relative to the two previous approaches, of the symmetry breaking. The essential zero modes are taken to be those of the SU(2) case with the strange ones practically frozen.

Applications of these two methods to the CQSM have been restricted to bound state approach for the study of hyperons [89].

2.6.2 Asymptotics of the meson fields

In the perturbative approach to the SU(3) flavor symmetry breaking effects, used in this work, there is one other aspect to consider, namely that of the exact form of the perturbation. From the Dirac operator in the rotating hedgehog field (2.40a) one has that, inserting (2.40b) and (2.27),

\[
\gamma^0 D(U_c) + A^\dagger \delta m A = -i\alpha^i \partial_i + MU^{\gamma_5} + m + A^\dagger \delta m A
\]

\[
= -i\alpha^i \partial_i + MU^{\gamma_5} + A^\dagger m A. \tag{2.68}
\]

\(^{15}\)In spite of this usage, this is not a symmetry in the usual sense.
This expression shows that in (2.27) a diagonal mass term $m$ was included in the one particle Dirac Hamiltonian and at the same time it was subtracted from the perturbation term in $\delta m$. The importance of being able to chose $m$ is that its value determines the asymptotic behaviour of the meson fields. The reason why one should be interested in this asymptotics resides in the fact that this asymptotic behaviour is restricted to that of the pion if one would retain $m_1$ instead of $m$ in the Hamiltonian due to the trivial embedding (2.37). A careful choice of $m$ allows to obtain solitons with profiles corresponding to mesons having asymptotic tails governed by the kaon mass, at the expense of a slightly modified perturbation term. The difference between pion and kaon asymptotics is shown in Fig. 2.2. One of the consequences of the kaon asymptotics is a smaller spatial extension of the soliton.

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Illustration of the meson asymptotics corresponding to pion ($\mu = 140$ MeV) and kaon ($\mu = 490$ MeV) cases. $P(0) = -\pi$ in both cases.}
\end{figure}
\]

The correct description of the asymptotic behaviour of the pseudo-Goldstone fields is expected to play an important role in the quantities most sensitive to strangeness. This is but a semiphenomenological way of including such effects and therefore all the results show the two asymptotic behaviours, namely those of the pion, the most natural one, and of the kaon. To properly take into account these effects one should modify the formalism as was done in the Skyrme model in [90], although this approach was not carried out to quantities like form factors.

### 2.6.3 Symmetry conserving quantization

It was first realized in [91] that there is a problem with the quantization prescription (2.58). It does not mean there is a problem with the quantization rules themselves, but simply that the resulting formula for observables does not fulfill the expected relation between angular momentum and flavor operators from the quantization procedure. Concretely, in the quantization procedure, (2.58), it is found that right and left generators are related by (D.81)

\[
L_\alpha = D^{(n)}_{\alpha\beta}(A) R_\beta. \tag{2.69}
\]

It is then made plausible that left generators and right generators may be interpreted as generalized flavor, $T_\alpha$, and angular momentum, $J_\alpha$, operators, according to $L_\alpha = T_\alpha$ and $R_\beta = -J_\beta$ [73, 74]. Thus (2.69) relates generalized flavor and angular momentum operators according to

\[
T_\alpha = -D^{(n)}_{\alpha\beta}(A) J_\beta. \tag{2.70}
\]

It so happens that one may use the resulting expressions of this work to compute these observables. This is done in Section D.5 with the result (D.88)

\[
T_\alpha = -D^{(n)}_{\alpha\beta}(A) J_\beta - \sqrt{3} D^{(n)}_{\alpha\beta}(A) \frac{P_\alpha}{I_2} \tag{2.71}
\]
where $I_2$ is the moment of inertia of the soliton (3.29) and $I'_2$ is a similar term to this moment of inertia but in an antisymmetric form (D.86).

One of the consequences of (2.71) is that it breaks the Gell-Mann–Nishijima relation (3.31) due to the presence of the last term in (2.71). Also (2.69) reflects the symmetry of the hedgehog, while (2.71) breaks this symmetry. Hence the name “symmetry conserving quantization” given to the prescription for disregarding the terms related to $I'_2$ in the expressions for observables, (E.52) and (E.136). The rationale for this prescription in [91] was the agreement of the results for observables in the CQSM with the corresponding ones in the quark model in the limit of small soliton size. In this work the offending terms are in agreement with [91] and are identified with the terms of (E.52) and (E.136) which have the commutators $[J_a, D_{\alpha\beta}]$. The aforementioned terms are apparently independent of the regularization and may originate from both real and imaginary part of the effective action. In this work one adopts the symmetry conserving approach of [91].

2.6.4 Related models

As a last remark, apart from the NJL model, there are other models to which the CQSM is more or less related. These models include the Skyrme model (SM) [92], for a review see [93], and the quark model as two extreme cases of the CQSM. Looking at Fig. 2.1 it is compelling to relate the CQSM soliton at large sizes, in which case the valence level plunges into the lower Dirac continuum, to the Skyrme soliton. At small soliton sizes, the sea contribution to observables vanishes, and the CQSM is expected, with only the valence level, to reduce to the quark model [94]. One other model, with quarks and pions, also based in chiral symmetry and SCSB is the sigma model with quarks of Gell-Mann and Levy [95]. This model is also a revived form of the original model with nucleons. Contrary to the NJL model, however, this model is renormalizable and includes pions explicitly.
3 Electromagnetic form factors

Since the quarks are charged particles they interact electromagnetically. Hadrons, while objects made of quarks, have an extended electromagnetic structure, which may be described by form factors. Form factors may in turn be used to characterize the charge and magnetization distributions of the extended object, allowing to describe the charge distribution in terms of length parameters, like radii, which give an idea about the spatial dimension of the hadron. Form factors are specific to a given interaction so that the characterization of the spatial extension of the hadrons in terms of radii is different according to the interaction.

The electromagnetic form factors have been the subject of extensive experimental and theoretical work due to the fact that the electromagnetic interaction is very well understood. This eliminates difficulties with the probe, placing all the emphasis in the target hadron. Such situation is not valid in the case of strong form factors. The experimental setup of an experiment is also easier in the case of electromagnetic interaction. However, the information one is able to extract from the electromagnetic interaction corresponds to an incomplete picture of the hadron, for the gluon distributions inside the hadron do not appear in it.

The general notion of a form factor is schematically given by

\[ \frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{point}} |F(q)|^2 \]  

(3.1)

where \( q \) is the four-momentum transferred from probe to target and \( F(q) \) is a form factor. In the special case of a static target the form factor is just the Fourier transform of the charge distribution,

\[ F(q) = \int d^3x \rho(x) e^{i\mathbf{q} \cdot \mathbf{x}}. \]  

(3.2)

The size of the charge distribution may be quantified in terms of its radius, understood in this work as the r.m.s radius, defined by

\[ \langle r^2 \rangle = \int r^2 dr \rho(r) r^2. \]  

(3.3)

The simplicity of using the electromagnetic interaction to study the hadrons explains why it was the first interaction for which form factors were experimentally determined [1]. It does not mean, however, that a very detailed set of data was meanwhile gathered. In most cases the short lifetime of the hadrons puts large restrictions on the use of electron scattering to study them. Even the neutron, which decays due to the weak interaction, can be studied with the electromagnetic interaction only while bounded within a nucleus, most of the times by studding the deuteron or \(^3\text{He}\). In this way the extraction of the neutron form factors imply a detailed knowledge of the few-nucleon systems, which requires a theoretical analysis and some model dependence at times. This explains why the data on the neutron form factors is worse than the data on the proton, which is a stable particle. Even in the case of the proton not all the form factors are known with the same accuracy for all values of \( Q^2 \) (\( = -q^2 \)) due to the method of extracting the form factors from the electron scattering cross sections.

In the case of the proton and in the space-like case for the exchanged four-momentum one uses electron-proton scattering to determine the form factors. As is developed in Section C.3, the amplitude for electron-proton scattering factorizes, on account of the one-photon exchange approximation, into

\[ M = \frac{e^2}{q} \bar{u}(k')\gamma^\mu(k) \left( p' \right) J^{\text{EM}}_\mu \left( p \right), \quad q = k - k' = p' - p \]  

(3.4)
where $\bar{u}(k')$, $u(k)$ are Dirac spinors for the incident and outgoing electron (or muon). The cross section for elastic electron-proton scattering (C.58) is given by the Rosenbluth formula [36] which may be written in the form

$$\frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} \left[ A(Q^2) + \tan^2(\theta/2)B(Q^2) \right], \quad Q^2 = -q^2$$

(3.5)

where $d\sigma_{\text{Mott}}$ is the Mott cross section (C.59), which represents the cross section for the scattering of an electron off a pointlike spin 1/2 object, and $\theta$ is the electron scattering angle. From (3.5), the ratio between the cross section and the Mott cross section is, at a fixed $Q^2$ ($E$ and $\theta$ may vary), a linear function of the variable $\tan^2 \theta/2$. The intersection of this straight line (Rosenbluth plot) with the ordinate axis and the slope of the line determine $A$ and $B$, respectively, from which the form factors are determined. This process of determining the form factors is known as the Rosenbluth separation.

The experimental situation for the nucleon form factors is summarized in the following. For recent reviews see [96–98]. The Rosenbluth separation method has been used in elastic electron-proton scattering to determine the Sachs [99] form factors of the proton, $G^p_E$ and $G^p_M$, up to 9 GeV$^2$. At higher values of $Q^2$, up to 30 GeV$^2$, because the magnetic form factor is multiplied by $\tau = Q^2/4M^2$ in (3.5) making the magnetic form factor dominate $A$, the magnetic form factor is obtained directly from the cross section while the electric form factor is affected by large uncertainties. The measurement of the components of the recoil polarization in double polarization experiments on electron-deuteron scattering [100] yields higher accuracy for the electric form factor. The overall situation may be summarized by the so-called dipole $G_D$ behaviour

$$G^p_E(Q^2) \approx G_D(Q^2) = \frac{1}{(1 + Q^2/0.71)^2} \quad (3.6a)$$

$$G^p_M(Q^2) \approx \mu_p G_D(Q^2) \quad (3.6b)$$

where $\mu_p$ is the magnetic moment of the proton. Using recoil polarization experiments, it has been found [101,102] that $\mu_p G^p_E/G^p_M$ decreases with $Q^2$ up to $Q^2 = 5.6$ GeV$^2$. It was also observed form factor scaling, $\mu_p G^p_E/G^p_M = 1$, at low $Q^2$ [103,104].

Since there are no neutron targets (a situation shared by the hyperons), the neutron form factors were initially accessed through scattering of slow neutrons off atomic electrons (this only determines the slope of the electric form factor at $Q^2 = 0$), elastic electron-deuteron scattering and deuteron electrodesintegration. The closest situation to electron-neutron scattering is quasi-free scattering of electrons on deuterium and $^3\text{He}$. The difficulties with the Rosenbluth separation are even larger here due to the small electric form factor $G^n_E$, limiting the application of the method to $Q^2 = 4$ GeV$^2$ [105]. At low $Q^2$ the extraction of the neutron form factors is made easier due to the advances in understanding the nuclear effects and final state interactions [106]. This is achieved for $G^n_M$ in inclusive quasi-elastic of polarized electrons on polarized $^3\text{He}$ [108,109]. The neutron magnetic form factor is also found to have an approximate dipole shape at low $Q^2$

$$G^n_M(Q^2) \approx \mu_n G_D(Q^2). \quad (3.7)$$

For the electric form factor $G^n_E$, the situation is the less favourable of all the four form factors of the nucleon. The situation is gradually improving (see legend of Fig. 3.12 for the latest experimental results), although the more extensive set of results is still that obtained in elastic electron-deuteron scattering [110,111].

In this chapter, the main task is the calculation of the form factors of the flavor singlet baryon $V^B_\mu = V^B_\mu/N_c$ current, the isospin current $V^I_\mu = V^I_\mu$ and the hypercharge current $V^Y_\mu = V^Y_\mu/\sqrt{3}$. All the flavor currents are simply combinations of these three currents. The corresponding Sachs form factors follow the same relations as the currents. To determine the Sachs form factors of the octet baryons it would suffice to know the isospin and hypercharge currents. The flavor singlet current is essential to obtain a complete flavor separation, thus allowing to compute the strange form factors. The results discussed in the chapter concern the implications for various aspects of form factors as
3.1 Definition of the vector currents

The definitions for the hadron currents are as follows: Baryon current

\[ V_\mu^B = \frac{1}{N_c} \bar{V}_\mu^0 = \frac{1}{N_c} \bar{V}_\mu^0 \gamma_\mu \psi, \]

(where the superscript 0 refers to the identity matrix), hypercharge current

\[ V_\mu^Y = \frac{1}{\sqrt{3}} \bar{V}_\mu^8 = \frac{1}{\sqrt{3}} \bar{V}_\mu^8 \gamma_\mu \lambda^8 \psi, \]

and the isospin current

\[ V_\mu^T = V_\mu^3 = \bar{V}_\mu^3 \gamma_\mu \lambda^8 \psi. \]

These definitions of the currents normalize their electric form factor (\( \mu = 0 \)) to 1 at \( Q^2 = 0 \), except when they vanish at this point.

The hadron currents (3.8,3.9, 3.10) can be written in terms of flavor currents:

\[ V_\mu^0 = \bar{u} \gamma_\mu u + \bar{d} \gamma_\mu d + \bar{s} \gamma_\mu s, \]

\[ V_\mu^3 = \bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d, \]

\[ V_\mu^8 = \frac{1}{\sqrt{3}} (\bar{u} \gamma_\mu u + \bar{d} \gamma_\mu d - 2\bar{s} \gamma_\mu s). \]

Inverting these relations, the flavor currents are given by

\[ \bar{u} \gamma_\mu u = \frac{1}{\sqrt{3}} V_\mu^0 + \frac{i}{2} V_\mu^3 + \frac{1}{2 \sqrt{3}} V_\mu^8 = V_\mu^B + \frac{1}{2} V_\mu^T + \frac{1}{2} V_\mu^Y, \]

\[ \bar{d} \gamma_\mu d = \frac{1}{\sqrt{3}} V_\mu^0 - \frac{i}{2} V_\mu^3 + \frac{1}{2 \sqrt{3}} V_\mu^8 = V_\mu^B - \frac{1}{2} V_\mu^T + \frac{1}{2} V_\mu^Y, \]

\[ \bar{s} \gamma_\mu s = \frac{1}{\sqrt{3}} V_\mu^0 - \frac{i}{\sqrt{3}} V_\mu^3 = V_\mu^B - V_\mu^Y. \]

The form factors for the currents \( V_\mu^0, V_\mu^3, \) and \( V_\mu^8 \) are denoted, respectively, by \( G^{(0)}, G^{(3)}, \) and \( G^{(8)} \).

The definition of the Sachs form factors can be extended to any of these flavor currents, defining the respective flavor (f) form factors in a baryon state \( B \). Using the definition of the Sachs form factors (C.45,C.47a) the flavor form factors are given by the following matrix elements:

\[ \langle B(S_3, p') | J_f^\mu(0) | B(S_3, p) \rangle = \delta_{S_3 S_3'} G_f^f(q^2), \]

\[ \langle B(S_3, p') | J_f^\mu(0) | B(S_3, p) \rangle = \frac{i}{2 m_N} \epsilon^{klm} \langle S_3' | \sigma^l \sigma^m | S_3 \rangle G_M^f(q^2). \]

where \( q = p' - p \) is the four-momentum transferred between the photon and the current.

Looking at the charge matrix

\[ Q_{SU(3)} = \text{diag} \left( \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right) = \frac{1}{2} \left( \lambda^3 + \frac{1}{\sqrt{3}} \lambda^8 \right) = Q_u + Q_d + Q_s \]

one has

\[ Q_u = \frac{2}{3} \text{diag} (1, 0, 0) = \frac{2}{9} \lambda_3 + \frac{1}{3} \lambda^3 + \frac{1}{3 \sqrt{3}} \lambda^8, \]

\[ Q_d = -\frac{1}{3} \text{diag} (0, 1, 0) = -\frac{1}{9} \lambda_3 + \frac{1}{6} \lambda^3 - \frac{1}{3 \sqrt{3}} \lambda^8, \]

\[ Q_s = -\frac{1}{3} \text{diag} (0, 0, 1) = -\frac{1}{9} \lambda_3 + \frac{1}{3 \sqrt{3}} \lambda^8. \]

The flavor form factors in the baryon state \( B \) are then given by the linear combinations:

\[ C^{u,B}_{E,M}(q^2) = \frac{2}{9} c^{(0)B}_{E,M}(q^2) + \frac{1}{3} c^{(3)B}_{E,M}(q^2) + \frac{1}{3 \sqrt{3}} c^{(8)B}_{E,M}(q^2), \]

\[ C^{d,B}_{E,M}(q^2) = -\frac{1}{9} c^{(0)B}_{E,M}(q^2) + \frac{1}{6} c^{(3)B}_{E,M}(q^2) - \frac{1}{6 \sqrt{3}} c^{(8)B}_{E,M}(q^2), \]

\[ C^{s,B}_{E,M}(q^2) = -\frac{1}{9} c^{(0)B}_{E,M}(q^2) + \frac{1}{6} c^{(3)B}_{E,M}(q^2) - \frac{1}{6 \sqrt{3}} c^{(8)B}_{E,M}(q^2), \]
\[ G_{E,M}^{s,B}(q^2) = \frac{1}{9} G_{E,M}^{(0)B}(q^2) + \frac{1}{3\sqrt{3}} G_{E,M}^{(8)B}(q^2). \]  

(3.16c)

The form factor for the baryon state \( B \) becomes thus a sum of flavor components\(^1\)

\[ G_{E,M}^{B}(q^2) = \sum_{f=u,d,s} G_{E,M}^{f,B}(q^2). \]  

(3.17)

According to (3.16) the electric form factor for a given flavor current is normalized to the number of valence quarks of that flavor in the baryon considered times the electric charge of the flavor. This can be seen from the normalizations at the photon point of the form factors of the singlet, triplet and nonsinglet currents:

\[ G_{E}^{(0)B}(0) = 3; \quad G_{E}^{(3)B}(0) = 2 \langle T^{3} \rangle_{B}; \quad G_{E}^{(8)B}(0) = \sqrt{3} \langle Y \rangle_{B}. \]  

(3.18)

with \( Y \) the hypercharge and \( T^{3} \) the third component of the isospin of the baryon state to each the form factors refer. These lead to

\[ G_{E}^{a,B}(0) = \frac{2}{9} G_{E}^{(0)B}(0) + \frac{1}{3} G_{E}^{(3)B}(0) + \frac{1}{3\sqrt{3}} G_{E}^{(8)B}(0) = \frac{2 + 2 \langle T^{3} \rangle_{B} + \langle Y \rangle_{B}}{3}, \]  

(3.19a)

\[ G_{E}^{d,B}(0) = -\frac{1}{9} G_{E}^{(0)B}(0) + \frac{1}{6} G_{E}^{(3)B}(0) - \frac{1}{6\sqrt{3}} G_{E}^{(8)B}(0) = -\frac{2 + 2 \langle T^{3} \rangle_{B} - \langle Y \rangle_{B}}{6}, \]  

(3.19b)

\[ G_{E}^{s,B}(0) = -\frac{1}{9} G_{E}^{(0)B}(0) + \frac{1}{3\sqrt{3}} G_{E}^{(8)B}(0) = \frac{1}{3} + \frac{1}{3} \langle Y \rangle_{B}, \]  

(3.19c)

which show, e.g. for the proton \( \bar{\Sigma} \),

\[ G_{E}^{sp}(0) = \frac{4}{3}; \quad G_{E}^{dp}(0) = -\frac{1}{3}; \quad G_{E}^{sp}(0) = 0. \]  

(3.20)

In order to make the comparison between flavors in baryons with different number of valence quarks of that flavor, it is useful to introduce form factors \( \tilde{G} \) which correspond to the normalization of each flavor form factor to the product of number of valence quarks \( n_{f} \) with the charge of the flavor \( q_{f} \):

\[ \tilde{G}_{f,B} = \frac{G_{f,B}}{n_{f}^{2} q_{f}}. \]  

(3.21)

This definition is used in the following for both electric and magnetic flavor form factors. In particular, all the electric form factors \( \tilde{G}^{f} \) are 1 or 0 at the photon point \( Q^{2} = 0 \). The definition for \( \tilde{G}^{f} \) conforms with the definition used for the strange form factors and allows to easily relate different flavor form factors in a certain baryon. Indeed, \( V_{\mu}^{Y} \) is the same for the baryons which have the same hypercharge, and \( V_{\mu}^{T} \) is proportional to \( T_{3} \). This leads to the following relations, valid (in the absence of isospin breaking) for electric and magnetic form factors:

\[ \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}, \quad \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}, \quad \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}. \]  

(3.22a)

\[ \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}, \quad \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}, \quad \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}. \]  

(3.22b)

\[ \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}, \quad \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}, \quad \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}. \]  

(3.22c)

\[ \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}, \quad \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}, \quad \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}. \]  

(3.22d)

\[ \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}, \quad \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}, \quad \tilde{G}_{u} = \tilde{G}_{d} = \tilde{G}_{s}. \]  

(3.22e)

Finally, to obtain the baryon currents one must multiply the individual flavor currents by the flavor charge:

\[ V_{\mu} = \frac{2}{3} \bar{u} \gamma_{\mu} u - \frac{1}{3} \bar{d} \gamma_{\mu} d - \frac{1}{3} \bar{s} \gamma_{\mu} s = \frac{1}{2} \bar{u} \gamma_{\mu} \lambda^{3} u + \frac{1}{2\sqrt{3}} \bar{d} \gamma_{\mu} \lambda^{8} s = \frac{1}{2} \left( V_{\mu}^{T} + V_{\mu}^{Y} \right). \]  

(3.23)

As was natural from the definition of the currents, the quantities which are calculated in the CQSM are thus \( G_{E,M}^{0}(q^2) \), \( G_{E,M}^{3}(q^2) \), and \( G_{E,M}^{8}(q^2) \). All results regarding form factors are obtained

\(^{1}\text{In SU(2)} \quad Q_{SU(2)}^{SU(2)} = \text{diag} \left( \frac{2}{3}, -\frac{1}{3} \right) = \frac{1}{6} + \frac{1}{2} \delta^{3} = Q_{SU(2)}^{S_{U(2)}} + Q_{SU(2)}^{S_{U(2)}} \)

with \( Q_{SU(2)}^{S_{U(2)}} = \frac{2}{3} \text{diag} (1, 0) = \frac{1}{3} \bar{1}_{2} + \frac{1}{3} \bar{r}_{3} \), \( Q_{SU(2)}^{S_{U(2)}} = -\frac{1}{3} \text{diag} (0, 1) = -\frac{1}{6} \bar{1}_{2} + \frac{1}{6} \bar{r}_{3} \).
from these. This is the main subject of the next section.

3.2 Expressions for the electromagnetic form factors in the CQSM

3.2.1 Electric form factors

Starting from its definition (3.13a) the electric form factors for a specific baryon state \( B \),

\[
G_E^{(1)}(q^2) = \langle B(S_3, p') | j^{0\lambda}(0) | B(S_3, p) \rangle = \int d^3z \ e^{i \mathbf{q} \cdot \mathbf{z}} \langle B(S_3) | \mathcal{F}^{0\lambda}_E(z) | B(S_3) \rangle ,
\]

one obtains, after an average over the orientations of the momentum transfer,

\[
G_E^{(1)}(q^2) = \frac{1}{4\pi} \int d^3z \int d^3q \ e^{i \mathbf{q} \cdot \mathbf{z}} \langle B(S_3, p') | \mathcal{F}^{0\lambda}_E(z) | B(S_3, p) \rangle
\]

\[
= \int d^3z \ j_0(|q| |z|) \langle B(S_3) | \mathcal{F}^{0\lambda}_E(z) | B(S_3) \rangle
\]

for \( \chi = 0, 3, 8 \). The form factor can be written as \((\chi = 3, 8)\)

\[
G_E^{(1)}(q^2) = \int d^3z \ j_0(|q| |z|) G_E^{(1)}(z)
\]

with the charge distribution or density given from (E.137) by

\[
G_E^{(1)}(z) = \frac{1}{\sqrt{3}} \left\langle D^{(8)}_\chi(z) \right| B(z) - \frac{2}{I_1} \left\langle D^{(8)}_\chi J_1 \right| B(z) - \frac{2}{I_2} \left\langle D^{(8)}_\chi J_2 \right| B(z)
\]

\[
- \frac{4M_8}{I_1} \left\langle D^{(8)}_8 D^{(8)}_8 \right| B \left( I_1 K_1(z) - K_1 I_1(z) \right)
\]

\[
- \frac{4M_8}{I_2} \left\langle D^{(8)}_8 D^{(8)}_8 \right| B \left( I_2 K_2(z) - K_2 I_2(z) \right)
\]

\[
- 2 \left( \frac{M_1}{\sqrt{3}} \left\langle D^{(8)}_8 \right| B + \frac{M_8}{3} \left\langle D^{(8)}_8 D^{(8)}_8 \right| B \right) C(z)
\]

The functions of the spatial coordinate entering this expression are written in terms of the one-particle wave functions \( \langle z | n \rangle = \phi_n(z) \), eigenfunctions of the one-particle Dirac Hamiltonian:

\[
\frac{1}{N_c} B(z) = \langle v | z | v \rangle - \frac{1}{2} \sum_{n} \text{sgn}(\varepsilon_n) \langle z | n \rangle \langle n | z \rangle,
\]

\[
\frac{1}{N_c} C(z) = \sum_{n \neq 0} \frac{1}{\varepsilon_n - \varepsilon_0} \langle v | \tau | z | n \rangle \langle n | \gamma^0 | v \rangle + \frac{1}{2} \sum_{n,m} R_3(\varepsilon_n, \varepsilon_m) \langle m | z \rangle \langle n | \gamma^0 | m \rangle,
\]

\[
\frac{6}{N_c} I_1(z) = \sum_{n \neq 0} \frac{1}{\varepsilon_n - \varepsilon_0} \langle v | \tau | \tau | z | n \rangle \langle n | \gamma^0 | \tau | v \rangle + \frac{1}{2} \sum_{n,m} R_3(\varepsilon_n, \varepsilon_m) \langle m | z \rangle \langle n | \gamma^0 | m \rangle,
\]

\[
\frac{4}{N_c} I_2(z) = \sum_{n \neq 0} \frac{1}{\varepsilon_n - \varepsilon_0} \langle v | \gamma^0 | z | n \rangle \langle n | v \rangle + \sum_{n,m} R_3(\varepsilon_n, \varepsilon_m) \langle m | z \rangle \langle n | m \rangle,
\]

\[
\frac{6}{N_c} K_1(z) = \sum_{n \neq 0} \frac{1}{\varepsilon_n - \varepsilon_0} \langle v | \gamma^0 | z | n \rangle \langle n | v \rangle + \sum_{n,m} R_3(\varepsilon_n, \varepsilon_m) \langle m | z | n \rangle \langle n | \gamma^0 | m \rangle.
\]

The integrals of the last four densities correspond to the moments of inertia \( I_1, I_2, K_1, \) and \( K_2 \):

\[
\int d^3z \ I_i(z) = I_i, \quad \int d^3z \ K_i(z) = K_i, \quad i = 1, 2.
\]

The case \( \chi = 0 \) can be extracted from the result for \( \chi = 3, 8 \), as explained in Section E.2.1:

\[
G_E^{(0)}(q^2) = \int d^3z \ j_0(|q| |z|) G_E^{(0)}(z),
\]

\[
G_E^{(0)}(z) = \mathcal{B}(z) + 2 \left( M_1 + \frac{1}{\sqrt{3}} M_8 \left\langle D^{(8)}_8 \right| B \right) C(z).
\]
The effects due to $G$ and taking into consideration that effects due to upon the model parameters, it is easy to conclude from Fig. H.3 that these electric form factors are effectively those of the baryon, isospin and hypercharge currents. The differences at the level at the slope at the origin are not that large. Regarding the dependence of these form factors upon the model parameters, it is easy to conclude from Fig. H.3 that these electric form factors show a very small dependence on the strange quark mass. This shows that symmetry breaking effects due to $m_s$ is small for $m_s$ in the range $0 - 180$ MeV. The same conclusion applies to the main model parameter, which is the constituent quark mass $M$, as may be concluded from Fig. H.4. The effects due to $M$ in the range $400 - 450$ MeV are of the order of those due to $m_s$.

\[ \int Q^2 [\text{GeV}^2] \]

\[ G_E^0 / \sqrt{3}, G_E^3, G_E^8 / \sqrt{3} \]


Figure 3.1: The singlet, triplet and octet electric form factors (normalized as to represent the baryon, isospin and hypercharge currents) for the proton quantum numbers ($Y = 1$, $T_3 = 1/2$) as a function of $Q^2$. The constituent quark mass is $M = 420$ MeV and the strange quark mass is $m_s = 180$ MeV.

**Normalization and the Gell-Mann-Nishijima relation**

It is rather important to know the form factors at the photon point, $Q^2 = 0$. In particular for the electric case, one expects to obtain the charge of the baryon at $Q^2 = 0$. In SU(3) one thus expects to obtain the Gell-Mann-Nishijima relation

\[ G_E^B(Q^2 = 0) = \langle T_3 \rangle_B + \frac{1}{2} \langle Y \rangle_B . \]  

(3.31)

The singlet form factor is straightforwardly related to the baryon number:

\[ G_E^{(0)}(q^2 = 0) = \int d^3z \left[ B(z) + 2 \left( M_1 + \frac{1}{\sqrt{3}} M_S \left< D_{18}^{(8)} \right>_B \right) C(z) \right] = \int d^3z B(z) = N_c = 3 \]  

(3.32)

As to the triplet and nonsinglet form factors $G_E^{(3)}B(0)$, $G_E^{(8)}B(0)$, one starts by noticing that by the definition of the moments of inertia

\[ \int d^3z \left( I_1 K_1(z) - K_1 I_1(z) \right) = \int d^3z \left( I_2 K_2(z) - K_2 I_2(z) \right) = 0. \]  

(3.33)

Remembering, from Section D.4, that the collective operator $J_S$ is the constraint

\[ J_S = - \frac{N_c}{2\sqrt{3}} = - \frac{\sqrt{3}}{2} \]  

(3.34)

and taking into consideration that $\int d^3z C(z) = 0$, then

\[ G_E^{(3)}B(0) = \frac{N_c}{\sqrt{3}} \left< D_{18}^{(8)} \right>_B - 2 \left< D_{31}^{(8)} \right>_B - 2 \left< D_{3a}^{(8)} J_a \right>_B = 2 \sum_{a=1}^{8} \left< \sum_{\alpha} D_{3a}^{(8)} R_{\alpha} \right>_B = 2 \langle T_3 \rangle_B \]  

(3.35)
where $R_\alpha = -J_\alpha$ are the right generators of SU(3). Similarly, in the nonsinglet case ($\chi = 8$)
\[
G_E^{(8)B}(0) = \frac{1}{\sqrt{3}} \left\langle D_{88}^{(8)} \right| B \right\rangle - 2 \left\langle D_{88}^{(8)} J_1 \right| B \right\rangle - 2 \left\langle D_{88}^{(8)} J_3 \right| B \right\rangle = 2 \sum_{\alpha=1}^{8} D_{88}^{(8) R_\alpha} \right\rangle = \sqrt{3} \left\langle Y \right\rangle_B \tag{3.36}
\]
In this way the Gell-Mann-Nishijima formula is obtained:
\[
G_E^B(0) = \frac{1}{2} G_E^{(3)B}(0) + \frac{1}{2 \sqrt{3}} G_E^{(8)B}(0) = \frac{1}{2} (T_3)_B + \frac{1}{2 \sqrt{3}} \left\langle Y \right\rangle_B = (T_3)_B + \frac{1}{2} \left\langle Y \right\rangle_B . \tag{3.37}
\]

### 3.2.2 Magnetic form factors

The magnetic form factors are obtained in the same way as the electric ones. Starting from the definition (3.13b), one has for a baryon state $B$
\[
\left\langle B(S_3, p') \left| j_3^{(X)}(0) \right| B(S_3, p) \right\rangle = \int d^3z e^{iqz} \left\langle B(S_3, p') \right| \bar{\mathcal{F}}_M^{k_B}(z) \right\rangle B(S_3, p) = \frac{i}{2M_N} \varepsilon^{kml} \left\langle S_3' \right| \sigma^l \left| S_3 \right\rangle q^m G_M^{k_B}(q^2) . \tag{3.38}
\]

Since the current is conserved, which is easy to see from the product of the matrix element with $q^k$, the previous relation can be used in order to express the magnetic form factor in terms of the magnetic density. Defining $G^{k_B}(q^2)$, it is possible to average in the orientation $\hat{q}$ of the moment in the Breit-frame
\[
\varepsilon^{kml} \int d^3z e^{iqz} \left\langle B(S_3) \right| \bar{\mathcal{F}}_M^{k_B}(z) \left| B(S_3) \right\rangle = \frac{i}{2M_N} \varepsilon^{kml} \int d^3z \left| q \right| \hat{z} \cdot j_1(\mid q \mid z) \left\langle B(S_3) \right| \bar{\mathcal{F}}_M^{k_B}(z) \left| B(S_3) \right\rangle = \frac{i}{2M_N} \int d^3q \left( q^2 - (q^3)^2 \right) G^{k_B}(q^2) = \frac{i4\pi}{3\delta_{MN}} q^2 G^{k_B}(q^2) , \tag{3.39}
\]

which allows to extract the form factor in terms of the of the magnetic density. Defining $\bar{\mathcal{F}}_M = \bar{\mathcal{F}}_M^{k_B} e^k$.

The magnetic form factor is given by
\[
G^{(8) B}(q^2) = 3M_N \int d^3z \frac{1}{\left| q \right| z} \left\langle B(S_3) \right| \left\{ z \times \bar{\mathcal{F}}_M(z) \right\}^3 \left| B(S_3) \right\rangle . \tag{3.42}
\]

Taking into account that
\[
\gamma_0^k = \gamma_5^k , \quad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \tag{3.43}
\]
and that the collective matrix elements fulfill
\[
\left\langle N(S_3) \left| J_1 \right| N(S_3) \right\rangle = \delta_{33} \left\langle N(S_3) \right| J_3 \left| N(S_3) \right\rangle , \tag{3.44a}
\]
\[
\left\langle N(S_3) \left| D_{88}^{(8)} \right| N(S_3) \right\rangle = \delta_{33} \left\langle N(S_3) \right| D_{88}^{(8)} \left| N(S_3) \right\rangle , \tag{3.44b}
\]
the magnetic form factor is ($\chi = 3, 8$), from (E.142),
\[
G^{(8) B}(q^2) = \frac{M_N}{\left| q \right|} \int d^3z \frac{j_1(\mid q \mid z)}{\mid z \mid} G_M^{(8) B}(z) . \tag{3.45}
\]

The magnetic density is written in terms of the one-particle wave functions according to
\[
G^{(X) B}(z) = \left\langle D_{X^3}^{(8)} \right| B \left( \mathcal{Q}_0(z) + \frac{1}{I_1} \mathcal{Q}_1(z) \right) \right\rangle .
where\(^2\) the densities have the following definitions:

\[
\frac{1}{N_c} Q_0(z) = \langle \psi | J_0 | \psi \rangle + \sum_{m} \langle \psi | \bar{n} \rangle \langle n | J_0 | n \rangle + \langle n | \bar{\epsilon} | m \rangle \langle m | n \rangle \langle \bar{\epsilon} | \bar{\epsilon} | 0 \rangle | n \rangle | \epsilon | | m \rangle,
\]

\[
\frac{1}{N_c} \chi_1(z) = \sum_{n \neq 0} \frac{1}{\sqrt{n^2 - \epsilon^2}} \langle \psi | M_1 | \psi \rangle \langle n | \bar{\epsilon} | n \rangle - \langle n | \bar{\epsilon} | m \rangle \langle m | n \rangle | \epsilon | | m \rangle,
\]

\[
\frac{1}{N_c} \chi_2(z) = \sum_{n \neq 0} \frac{1}{\sqrt{n^2 - \epsilon^2}} \langle \psi | M_2 | \psi \rangle \langle n | \bar{\epsilon} | n \rangle - \langle n | \bar{\epsilon} | m \rangle \langle m | n \rangle | \epsilon | | m \rangle,
\]

\[
\frac{2}{N_c} Q_1(z) = \sum_{n \neq 0} \frac{1}{\sqrt{n^2 - \epsilon^2}} \langle \psi | M_1 | \psi \rangle \langle n | \bar{\epsilon} | n \rangle - \langle n | \bar{\epsilon} | m \rangle \langle m | n \rangle | \epsilon | | m \rangle,
\]

\[
\frac{3}{N_c} M_0(z) = \sum_{n \neq 0} \frac{1}{\sqrt{n^2 - \epsilon^2}} \langle \psi | M_0 | \psi \rangle \langle n | \bar{\epsilon} | n \rangle - \langle n | \bar{\epsilon} | m \rangle \langle m | n \rangle | \epsilon | | m \rangle,
\]

\[
\frac{3}{N_c} M_1(z) = \sum_{n \neq 0} \frac{1}{\sqrt{n^2 - \epsilon^2}} \langle \psi | M_1 | \psi \rangle \langle n | \bar{\epsilon} | n \rangle - \langle n | \bar{\epsilon} | m \rangle \langle m | n \rangle | \epsilon | | m \rangle,
\]

\[
\frac{3}{N_c} M_2(z) = \sum_{n \neq 0} \frac{1}{\sqrt{n^2 - \epsilon^2}} \langle \psi | M_2 | \psi \rangle \langle n | \bar{\epsilon} | n \rangle - \langle n | \bar{\epsilon} | m \rangle \langle m | n \rangle | \epsilon | | m \rangle.
\]

For the singlet case, which can be deduced from the case $\chi = 3, 8$, Section E.2.1,

\[
G_M^{(0)}(q^2) = \frac{M_N}{g^2} \int d^4 z \frac{j_1(|q||z|)}{|z|} G_M^{(0)}(z)
\]

with the singlet magnetic density given by

\[
G_M^{(0)}(z) = - \langle J_3 | B \rangle B_1 \chi_1(z) - 2 M_0 \langle D_8^{(8)} | B \rangle \left( - \frac{K_1}{I_1} \chi_1(z) + 3 M_1(z) \right).
\]

The results for the singlet, triplet and magnetic form factors for the proton are presented in Fig. 3.2 using the same normalization as in the electric case, Fig. 3.1. The comparison is here

\[\text{\footnotesize \textsuperscript{2}The expression (3.46) differs from the corresponding one in [112] in the absence of the term } + \frac{1}{2} Q_2(z), \text{ with}\]

\[\frac{2}{N_c} Q_2(z) = \sum_{n \neq 0} \frac{1}{\sqrt{n^2 - \epsilon^2}} \langle \psi | M_2 | \psi \rangle \langle n | \bar{\epsilon} | n \rangle - \langle n | \bar{\epsilon} | m \rangle \langle m | n \rangle | \epsilon | | m \rangle.
\]

coming from (E.142e) due to the symmetry conserving quantization.
effects due to the model parameters are the same as in the electric case. Namely, from Fig. H.3 and type of result as was observed for the electric form factors, Fig. 3.2. The conclusions regarding the effects due to the model parameters are the same as in the electric case. Namely, from Fig. H.3 and Fig. H.4, one finds that the effects of changing $m_s$ and $M$ are small in the ranges of, respectively, $0 – 180$ MeV and $400 – 450$ MeV.

\[
0 - 180 \text{ MeV and } 400 - 450 \text{ MeV.}
\]

Figure 3.2: The singlet, triplet and octet magnetic form factors (in units of n.m.) for the proton as functions of $Q^2$. Conventions and model parameters as in Fig. 3.1.

3.3 The electromagnetic form factors in SU(2)

The baryon octet form factors may be found directly from (3.23) and the electric form factors (3.26,3.30,3.46) and (3.50). The electric and magnetic form factors in SU(2) are easily obtained form the same quantities, the difference coming solely from the collective matrix elements.

In the electric case, for a nucleon state $B$ (here just the nucleon), it is, with $G_E^{(0)SU(2)}$ from (3.26) and $G_E^{(3)SU(2)}$ from (3.30), given by

\[
G_E^{SU(2),B}(q^2) = \frac{1}{6} G_E^{B(0)SU(2)}(q^2) + \frac{1}{2} G_E^{B(3)SU(2)}(q^2)
\]

\[
= \int d^3 z \, j_0(|q| \, |z|) \left[ \frac{1}{6} B(q^2) + \frac{1}{2} \left( -\frac{1}{I_1} \, D_3^{(3)J_3} \right) \right] B(q^2) \quad (3.51)
\]

where

\[
\langle D_3^{(3)J_3} \rangle_B = \sum_i \sum_{j_3, j_3'} \langle T_3, J_3 | D_3^{(8)} | T_3', J_3' \rangle \langle T_3', J_3' | J_3 | T_3, J_3 \rangle = -\delta_{j_3 j_3'} T_3^B. \quad (3.52)
\]

In the magnetic case there are again only two contributions: $G_M^{(0)SU(2)}$ and $G_M^{(3)SU(2)}$. One then has from (3.50)

\[
G_M^{(0)SU(2)}(q^2) = -\frac{M_N}{|q|} \, \langle J_3 \rangle_B \, \frac{1}{I_1} \int d^3 z \, j_1(|q| \, |z|) \, \lambda_1(z) \quad (3.53)
\]

and from (3.46)

\[
G_M^{(3)SU(2)}(q^2) = \frac{M_N}{|q|} \, \langle D_3^{(T=J)} \rangle_B \, \frac{1}{I_1} \int d^3 z \, j_1(|q| \, |z|) \, \left( Q_0(z) + \frac{1}{I_1} Q_1(z) \right) \quad (3.54)
\]

with the collective matrix element $\langle D_3^{(T=J)} \rangle_B = -2T_3^B / 3$. For the complete form factor in the
nucleon $B$

$$G_M^{B,\text{SU}(2)}(Q^2) = \frac{1}{6} G_M^{B(0),\text{SU}(2)}(Q^2) + \frac{1}{2} G_M^{B(3),\text{SU}(2)}(Q^2). \quad (3.55)$$

The results for these form factors are presented and discussed in Section 3.7. They coincide with the previous results in the CQSM of [113].

### 3.4 Flavor form factors for the octet

The independent flavors in the baryon octet\(^3\), as given by (3.22), are presented in Fig. 3.3 for valence quarks and in Fig. 3.4 for sea quarks. The elements of these form factors, magnetic moments and radii, are collected in Table 3.1.

The normalization used for the valence flavor form factors is that of (3.21). In this way, all of the valence electric form factors fulfill $G_E^{B}(0) = 1$ and are thus easier to compare among each other. It is clear from Fig. 3.3 that they are not much different from one other, both for the $u$ and $s$ valence quarks. (The form factors for the $d$ quark are obtained from the ones of the $u$ quark using (3.22).) In the case of the sea quarks, Fig. 3.4, the mass of the sea quark has an observable effect. Returning to Fig. 3.3, the magnetic form factors seem to have a much differentiated behaviour. For the $u$ quark, the sign of the magnetic form factors is clearly opposite between the neutron and $\Xi^0$, which have both a single $u$ quark, and the proton and $\Sigma^+$, which have two $u$ quarks. It is interesting to notice the different behaviour of $\Lambda$ and $\Sigma^0$, one $u$ quark: while the $\Sigma^0$ follows the proton and $\Sigma^+$, the $\Lambda$ has a particular behaviour, quite distinct from the others, characterized by a very small form factor and a large magnetization radius for this flavor.

This is also the case for the $s$ quark in the $\Xi^0$. The behaviour of the magnetic form factors for $d$ as sea quark in $\Sigma^+$ and $\Xi^0$ is also quite interesting, Fig. 3.4, and they clearly are smaller than the strange magnetic form factor in the nucleon.

Notice that the large magnetization radius of $u$ in $\Lambda$ and $d$ quark in the $\Xi^0$ is a consequence of the small magnetic moment, which normalize the radius according to (C.90). In Table 3.1 one should remember that the expression for the radii is always the same, namely (C.90). In this way, in order to construct the tables of the radii, Tab. 3.2, and magnetic moments, Tab. 3.3, for the baryon octet from Tab. 3.1, taking into account (3.21) and (C.90) one has

$$\langle r_B^2 \rangle_E = \frac{2}{3} n_u \langle r_u^2 \rangle_E - \frac{1}{3} n_d \langle r_d^2 \rangle_E - \frac{1}{3} n_s \langle r_s^2 \rangle_E, \quad (3.56a)$$

$$\mu_B = \frac{2}{3} n_u \mu_u - \frac{1}{3} n_d \mu_d - \frac{1}{3} n_s \mu_s, \quad (3.56b)$$

$$\langle r_B^2 \rangle_M = \frac{1}{\mu_B} \left( \frac{2}{3} n_u \langle r_u^2 \rangle_E - \frac{1}{3} n_d \langle r_d^2 \rangle_E - \frac{1}{3} n_s \langle r_s^2 \rangle_E \right). \quad (3.56c)$$

Appendix H contains more figures with flavor form factors which try to illustrate some aspects of these form factors. Fig. H.6 studies the differences on the $u$ quark form factors between two baryon states which transform into each other by changing a $d$ quark into a $s$ quark or vice-versa. In the sea, the form factors of $u$ decrease when a $d$ quark of the $\Sigma^-$ transforms into a $s$ of the $\Xi^-$. The form factors of the $u$ quark do not change noticeably when the two $d$ quarks of the neutron turn into the two $s$ quarks of the $\Xi^0$. However, changing a $d$ into a $s$ in the neutron, or a $s$ into a $d$ in the $\Xi^0$, which leads the form factors of the spectator $u$. The $u$ magnetic

\[^3\text{The figure shows the valence quark structure of the baryon octet, recalling that the distinction between the } \Sigma^0 \text{ and } \Lambda \text{ comes from the } u, d \text{ quarks being in isotriplet and isosinglet combinations, respectively.}\]
3.4 Flavor form factors for the octet

Figure 3.3: Independent electric and magnetic flavor form factors for the valence quarks of the baryon octet. The constituent quark mass is 420 MeV, the strange quark mass 180 MeV and the profile is characterized by the pion for $u$ and $d$ quarks and by the kaon mass for $s$ quarks.

<table>
<thead>
<tr>
<th>$S$</th>
<th>$T_3$</th>
<th>Flavor ($f$)</th>
<th>Baryon</th>
<th>$(r_f^2)_E$</th>
<th>$(r_f^2)_M$</th>
<th>$\mu_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$u$</td>
<td>$p$</td>
<td>0.633</td>
<td>0.634</td>
<td>1.211</td>
</tr>
<tr>
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<td>$p$</td>
<td>0.440</td>
<td>0.778</td>
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<td>$p$</td>
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<td>0.115</td>
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<td>$\Lambda$</td>
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<td>0.012</td>
</tr>
<tr>
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<td>0</td>
<td>$s$</td>
<td>$\Lambda$</td>
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<td>0.414</td>
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</tr>
<tr>
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<td>0</td>
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<td>$\Sigma^0$</td>
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<td>0.619</td>
<td>1.238</td>
</tr>
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<td>0</td>
<td>$s$</td>
<td>$\Sigma^0$</td>
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<td>-0.248</td>
</tr>
<tr>
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<td>$+1$</td>
<td>$u$</td>
<td>$\Sigma^+$</td>
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<td>0.633</td>
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</tr>
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<td>$d$</td>
<td>$\Sigma^+$</td>
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<td>0.042</td>
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<td>$\Xi^0$</td>
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</tr>
</tbody>
</table>

Table 3.1: Flavor radii (in fm$^2$) and magnetic moments $\mu$ in (n.m.). Model parameters and conventions as in Fig. (3.3).
form factors changes sign relative to neutron and $\Xi^0$: stays small in the $\Lambda$ but is large in the $\Sigma^0$, Fig. H.6 and Fig. H.10. The $u$ form factors of a two valence $u$ quarks do not depend on whether the spectator is a valence $d$ (proton) or $s$ quark ($\Sigma^+$), Fig. H.6.

Changing a $u$ in the proton into a $d$ of the neutron has large implications for the form factors of the $u$, (3.3). These differences are not so large when a $u$ in the $\Sigma^+$ is turned into the $d$ of the $\Sigma^0$, Fig. H.7, but are large if the $d$ is of the $\Lambda$, Fig. H.10. The form factors of the $s$ quark also strongly depend on whether the valence quark is one, as in $\Lambda$ and $\Sigma$’s, or two as in the $\Xi$’s, which may be seen from Fig. H.8. A direct comparison between the flavor form factors of the $\Sigma^0$ and $\Lambda$ are depicted in Fig. H.10. At the level of the quark model the difference between $\Sigma^0$ and $\Lambda$ is the isor triplet and isosinglet character, respectively, of the $u, d$ part of the wave function. In principle, the model predictions should approach the quark model results in the limit of small soliton size. Therefore, the consequences of the symmetries of the wave functions of the quark model should also characterize the CQSM results in this limit.

As a concluding remark, one may comment on the approximate dipole behaviour of the flavor form factors. It is tempting to interpret this behaviour as a consequence of the composite nature of the constituent quarks. The similarities between baryon form factors and their flavor form factors seem to indicate that the internal structure of constituent quarks in terms of current QCD quarks and gluons is expected to be resolved at higher enough values of $Q^2$, close to the values where the same occurs for the baryon themselves. The grounds for this observation rest upon the local character of the electromagnetic current being studied. When this is not the case and the operators are nonlocal, on the light cone, then one obtains parton distributions [114,115], which refer directly to current QCD quarks. Some of these distributions, the generalized ones, have the form factors as the local limit. Generalized parton distributions, being connected to important issues like the contribution of quarks to the spin of the baryons, do seem to encompass the form factors and provide new insight into the structure of hadrons.

### 3.5 Octet electromagnetic form factors

The form factors for the baryon octet, shown in Fig. 1.1 together with their quantum numbers, masses and valence flavor content, have been studied in the CSQM in [112], without the symmetry conserving quantization. Incidentally, the problem which would lead to the symmetry conserving quantization was avoided in the case of the electric form factors by omitting the offending term.

Figure 3.4: Independent electric and magnetic form factors for the sea quarks of the baryon octet. Model parameters and conventions as in Fig. (3.3).
This was not the case with the magnetic form factors, where the analogous problematic term with respect to the quantization was kept. The symmetry conserving quantization consistently accounts in this work for the omission of the offending terms. At the same time it guarantees the fulfillment of the Gell-Mann–Nishijima relations, it also leads to a larger underestimation of the magnetic moments, as compared with those of [112]. Differences in the electric form factors may also be caused by the inclusion in this work of the C-term (3.28b) omitted in [112]. However, the main differences between the results of this work and those of [112] originate in the fact that in the present case one may reconstruct the the octet form factors from the flavor form factors, as is explained in the following, instead of using the triplet and octet components of the octet vector currents as in (3.23).

The form factors for the octet baryons are most easily obtainable from the flavor form factors using relations (3.17) and (3.21), i.e.

\[
G_{E,M}^B(Q^2) = G_{E,M}^{uuB}(Q^2) + G_{E,M}^{ddB}(Q^2) + G_{E,M}^{ssB}(Q^2)
\]

(3.57)

The main point of this chapter is to replace \(G^{(0)}, G^{(3)}\) and \(G^{(8)}\) in the role of an independent set of form factors by the set of flavor form factors \(G^{uu}, G^{dd}\) and \(G^{ss}\). In so doing we sidestep for the moment the fact that the flavor form factors are in the first place calculated in terms of the first set of form factors. The next modification is to take an asymptotic behaviour for the form factors of flavors \(u\) and \(d\) dominated by the pion mass and an asymptotic behaviour dominated by the kaon mass for the strange quark. These modifications lead to the mixed asymptotic description of the form factors obtained from (3.57) in the form

\[
G_{E,M}^B(Q^2) = q_u n_u \tilde{G}_{E,M}^{uB}(Q^2) + q_d n_d \tilde{G}_{E,M}^{dB}(Q^2) + q_s n_s \tilde{G}_{E,M}^{sB}(Q^2).
\]

(3.58)

where \((\pi)\) represents a pion tail quantity and \((K)\) a kaon tail one. To judge the implications of ansatz (3.58) for the form factors of the baryon octet, one may construct the set \(G^{(0)}, G^{(3)}\) and \(G^{(8)}\) of form factors as obtained from the flavor ones in (3.58) and compare with the initial set of \(G^{(0)}, G^{(3)}\) and \(G^{(8)}\). This is done in Fig. H.5 for the case of the proton. It clearly shows that the differences of (3.58) to the original set with pion tail are small and affect solely the singlet and the octet currents. It is thus to expect that the effects of this mixed asymptotic description of form factors will be most visible in the case of differences of these currents, as will indeed be the case with the neutron in the next section. The results for the form factors are presented in Fig. 3.5, for the charged octet baryons, and in Fig. 3.6, for the neutral ones.

**Electromagnetic radii**

The results of this work for the radii of the octet baryons are listed in Tab. 3.2. The experimental value for the electric radius of the proton in Tab. 3.2 is taken as \(\langle r_E^2 \rangle_p^E = 0.729 \pm 0.024\) fm², which corresponds to the mean value of \(\langle r_E^2 \rangle_p^E = 0.862 \pm 0.012\) fm from [116] and \(\langle r_E^2 \rangle_p^E = 0.847 \pm 0.009\) fm from [117]. It is found to be well reproduced in the present work. For the magnetic radius the value quoted as experimental, \(\langle r_M^2 \rangle_p^E = 0.836 \pm 0.009\) fm, is from [117].

In the case of the neutron the experimental value \(\langle r_n^2 \rangle_n^E = -0.113 \pm 0.007\) [118] for the charge radius is slightly underestimated. For the experimental value of the neutron magnetic radius the experimental value quoted corresponds to the mean value of [117] \(\langle r_n^2 \rangle_n^M = 0.898 \pm 0.009\) fm and [119] \(\langle r_n^2 \rangle_n^M = 0.873 \pm 0.011\) fm, i.e. \(\langle r_n^2 \rangle_n^M = 0.776 \pm 0.020\) fm².

The experimental results for the charge distribution of the \(\Sigma^-\) are from the SELLEX \(\langle r_{\Sigma^-}^2 \rangle_E = 0.61 \pm 0.21\) fm² [120] and \(\langle r_{\Sigma^-}^2 \rangle_E = 0.91 \pm 0.72\) fm² [121] both using the scattering of an elastic beam of \(\Sigma^-\) off atomic electrons.

Comparing these results, computed according to

\[
\langle r^2 \rangle_{E,M} = -\frac{6}{G_{E,M}(Q^2 = 0)} \frac{d}{dQ^2} G_{E,M}(Q^2)|_{Q^2 = 0},
\]

(3.59)
with the CQSM model results [112] without the symmetry conserving quantization one finds significant differences in the charge radii of the neutral octet baryons. Taking into account that the electric form factors of the neutral hyperons are very small, the C-term (3.28b), which is in the present work properly included, and the effects of the combined pion and kaon asymptotics in the different flavor components of the form factors are enough to change the sign of the form factors of $\Lambda$ and $\Xi^0$ as compared to the respective form factors in [112]. The charge radii change sign accordingly. This corresponds to a larger symmetry breaking effect in this calculation than the one found in [112].

The relation between charged radii for the baryon octet found in the present work follows the relation
$$\langle r_{\Xi^+}^2 \rangle_E > \langle r_{\Xi^0}^2 \rangle_E > \langle r_{\Xi^-}^2 \rangle_E$$
and for the neutral baryons
$$\langle r_{\Xi^0}^2 \rangle_E > \langle r_{\Xi^0}^2 \rangle_E > \langle r_{\Lambda}^2 \rangle_E.$$  
These relations are in agreement with two recent analysis carried out in the perturbative chiral
3.5 Octet electromagnetic form factors

Baryon & \langle r^2 \rangle_E & \text{Exp.} & \langle r^2 \rangle_M & \text{Exp.} \\
\hline
p & 0.728 & 0.729 \pm 0.024 & 0.649 & 0.699 \pm 0.018 \\
n & -0.097 & -0.113 \pm 0.007 & 0.677 & 0.776 \pm 0.020 \\
\Lambda & 0.039 & - & 0.457 & - \\
\Sigma^- & 0.662 & \{0.61 \pm 0.21 \} & 0.718 & - \\
\Sigma^0 & 0.075 & - & 0.550 & - \\
\Sigma^+ & 0.811 & - & 0.619 & - \\
\Xi^- & 0.546 & - & 0.318 & - \\
\Xi^0 & 0.102 & - & 0.535 & - \\
\hline

Table 3.2: Electric and magnetic radii and magnetic moments $\mu$. The constituent quark mass is 420 MeV, the strange quark mass 180 MeV and the flavors combined are characterized by the pion mass asymptotic behavior for the $u$ and $d$ quarks and by the kaon mass asymptotic behavior for the $s$ quarks. Units are n.m. and fm$^2$. For references on the experimental data see text.

Concerning the magnetization radii, the values in Tab. 3.2 differ much from the quoted ones in [112]. The origin for these differences consists in the use of the symmetry conserving quantization, which is also responsible for the change in the magnetic moments, and the mixed asymptotic description of the meson cloud of (3.58). In particular, agreeing with [122, 123] the radius for the $\Xi^-$ is found to be very small in comparison to the other octet baryons.

**Magnetic moments**

The results for magnetic moments are listed in Tab. 3.3 and are also presented in Fig. 3.7. The experimental data for the magnetic moments is from the Particle Data Group compilation [124]. At a broad level, from Tab. 3.3 and the corresponding data in Fig. 3.7 denoted by (⋄) for the model results and by (⋆) for the experimental values, it is possible to see that the pattern of SU(3) flavor ‘symmetry’ breaking is well reproduced. Also at broad level, one may notice a systematic underestimation of the magnetic moments.

<table>
<thead>
<tr>
<th>Baryon</th>
<th>$\mu$ (⋄)</th>
<th>$\mu$ (⋆)</th>
<th>Exp. (⋆)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>1.759</td>
<td>2.400</td>
<td>2.793</td>
</tr>
<tr>
<td>$n$</td>
<td>-1.210</td>
<td>-1.651</td>
<td>-1.913</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>-0.478</td>
<td>-0.652</td>
<td>-0.613</td>
</tr>
<tr>
<td>$\Sigma^-$ &amp; -0.702</td>
<td>-0.958</td>
<td>-1.16</td>
<td></td>
</tr>
<tr>
<td>$\Sigma^0$</td>
<td>0.495</td>
<td>0.675</td>
<td>-</td>
</tr>
<tr>
<td>$\Sigma^+$</td>
<td>1.692</td>
<td>2.309</td>
<td>2.458</td>
</tr>
<tr>
<td>$\Xi^-$</td>
<td>-0.444</td>
<td>-0.606</td>
<td>-0.651</td>
</tr>
<tr>
<td>$\Xi^0$</td>
<td>-1.030</td>
<td>-1.405</td>
<td>-1.250</td>
</tr>
</tbody>
</table>

Table 3.3: Magnetic moments of the octet baryons. Model Parameters and conventions as in Tab. 3.2. Units are n.m. (⋆, ⋄) and “soliton magneton” (⋄) (The symbols ⋄ and ⋆ appear in Fig. 3.7). The soliton magneton equals the n.m. multiplied by 1.364, which is the ratio between the soliton mass and the physical proton mass. For references on the experimental data see text.

In order to find some possible origin to the discrepancies between the model and phenomenology, one may take several combinations of the magnetic moments. The easiest consists in taking the sum of all the magnetic moments. In order to obtain the magnetic moment of the $\Sigma^0$ one may use the isospin symmetry related expression

$$
\mu_{\Sigma^0} = \frac{1}{2}(\mu_{\Sigma^+} + \mu_{\Sigma^-}).
$$

(3.62)
From the experimental data one finds from (3.62) \( \mu_{\Sigma_0} = 0.649 \) using the aforementioned isospin relation. For the sums one finds
\[
\sum_{B \in \text{octet}} \mu_{\text{CQSM}}^B = 0.082, \quad \sum_{B \in \text{octet}} \mu_{\text{exp}}^B = 0.313. \tag{3.63}
\]
Although there is still a significant difference, the situation, owing probably in a large extent to the symmetry conserving quantization, has qualitatively improved in comparison with past analysis in the context of the CQSM: the sum was \(-0.17\) in [125], \(-0.21\) in [112, 126]. Comparing with the Skyrme Model (SM), it was found for the sum of the octet magnetic moments the value \(-0.94\) [127].

It is also worthwhile to look at the magnetic moments from the point of view of flavor symmetry breaking. To this end one may consider the Coleman-Glashow relations [128], which would
\[
\mu_{\Sigma^+} \mu_{\Sigma^-} = 0.
\]
In the presence of symmetry breaking, one may look at the Caldi-Pagels relations [129] between the magnetic moments obtained in chiral perturbation theory, which are independent of leading-order corrections. For the first relation (3.64a), already encountered above (3.62), which is based on isospin symmetry, one finds
\[
\frac{1}{2}(\mu_{\Sigma^+} + \mu_{\Sigma^-}) = \mu_{\Sigma^0} = -\mu_\Lambda. \tag{3.64a}
\]

Figure 3.7: Octet magnetic moments. The experimental values [124] are denoted by (⋆). The values obtained with the model using the physical nuclear magneton (n.m.) are indicated by (⋆). The values with (⋆) are the model values for the magnetic moments in terms of the “soliton magneton”, i.e. the soliton defined with respect to the mass of the soliton. Model parameters as in (3.5).
0.649  --  0.613  exp., \hfill (3.64b)
0.495  0.495  0.478  CQSM, \hfill (3.64c)
0.675  0.675  0.652  scaled. \hfill (3.64d)

This relation is satisfied in the model almost exactly and the scaled version comes very close to the experimental values. The second Caldi-Pagels relation is \((3.65a)\)

\[
\mu_{\Xi^-} + \mu_{\Xi^0} + \mu_p + \mu_n = 2\mu_\Lambda, \quad (3.65a)
\]

\[
-1.021  -1.226  \exp., \quad (3.65b)
\]

\[
-0.925  -0.956  \text{ CQSM}, \quad (3.65c)
\]

\[
-1.262  -1.304  \text{ scaled}. \quad (3.65d)
\]

This relation is again well satisfied in the model.

An analysis of higher order terms in the strange quark mass and/or in the angular velocity, not included in this calculation, [126], namely terms of the orders \(O(m_s/N_c)\) and \(O(m_s^2)\), are found to be much less important than the terms \(O(1/N_c)\) (rotational corrections) and \(O(m_s)\) considered here. The neglect of such terms puts an upper limit of about 15% on the accuracy to which magnetic moments can be computed in the framework used in this study. It is fairly reasonable to conclude that these model results have reached such an accuracy and that higher order terms and, eventually, effects from the meson-loops have to be included in order to improve this description of the magnetic moments.

The dependence of the magnetic moments with the strange quark mass \(m_s\) in the range \(0 - 180\) MeV was studied in [130] using a model independent approach.

A comparison with the results in the NJLM [85] reveals that there are large differences at the level of charge radii, magnetic radii are systematically larger and magnetic moments are closer to experiment than the ones obtained here. In the framework of CHPT [131,132], the one significant difference to the present results is the opposite sign of the charge radius of the \(\Sigma^0\). For the magnetic radii CHPT predicts, as is the case here, the radius of the \(\Sigma^-\) to be the largest, but the value of the radius of the \(\Xi^-\) is not particularly small as happens here. Two recent calculations in the quark model, the perturbative chiral quark model [122] and a relativistic quark model in the a Bethe-Salpeter approach [123], obtain results quite similar to the ones presented.

### 3.6 Strange form factors of the nucleon

Before discussing the strange electric and magnetic Sachs form factors presented in Fig. 3.8, it may be interesting to review the indications pointing towards a strange quark contributions to the nucleon observables. This a question with important phenomenological implications. In fact, while the quark model excludes strange quarks from the nucleon wave function, since the net strangeness of the nucleon is zero and its wave function contains only valence quarks contributions. On the contrary, QCD allows strange-antistrange quark pairs to continuously form from the vacuum and return to it. This picture due to QCD should have its imprints upon the properties of the nucleon.

One of the original clues as to this influence of strangeness came from the analysis of the \(\pi - N\) sigma term, \(\Sigma_{\pi N}\) [133]. In [133] an improved extrapolation to the value of \(\pi - N\) sigma term at the Cheng-Dashen point \((Q^2 = -q^2 = 2m_N^2)\) allowed \((M\) is the nucleon mass\) to obtain \(\sigma\)

\[
\sigma = \frac{1}{2M} \left\langle p \left| \overline{m}(\bar{u}u + \bar{d}d) \right| p \right\rangle, \quad \overline{m} = \frac{1}{2} (m_u + m_d). \quad (3.66)
\]

Then, approximate SU(3) symmetry connected the \(\sigma\)-term matrix element with a similar matrix element including strangeness

\[
\sigma = \frac{\sigma_0}{1 - y}, \quad \sigma_0 = \frac{1}{2M} \left\langle p \left| \overline{m}(\bar{u}u + \bar{d}d - 2\bar{s}s) \right| p \right\rangle \quad (3.67)
\]

From the analysis [133] it was concluded that \(\sigma_0 = 35 \pm 5\) MeV and then

\[
y = 2 \frac{\left\langle p \left| \bar{s}s \right| p \right\rangle}{\left\langle p \left| (\bar{u}u + \bar{d}d) \right| p \right\rangle} \sim 0.2 \quad (3.68)
\]
which indicates that the scalar matrix element $\langle p | \bar{s}s | p \rangle$ is non-zero. The implication is a strange quark contribution of approximately 130 MeV/c² to the nucleon mass.

The other important clue as to the role of strangeness came from DIS of polarized electrons (SLAC) and muons (EMC and SMC) on polarized targets. For a review see [134]. The observables are the cross section asymmetries. The asymmetries are used to extract the polarized structure function $g_1(x)$. The first moment of $g_1(x)$ may be expressed in terms of the constants (5.33), less logarithmic $Q^2$ corrections. From [134] it was concluded that

$$\Delta u + \Delta \bar{u} = 0.78 \pm 0.03, \quad \Delta d + \Delta \bar{d} = -0.48 \pm 0.03, \quad \Delta s + \Delta \bar{s} = -0.14 \pm 0.03. \quad (3.69)$$

The first outcome of these results is that the sum of these quantities is just $0.16 \pm 0.08$. Since this sum is interpreted as the contribution to the nucleon spin, one sees that this contribution amounts only to 16%, in clear contradiction to the quark model. The second outcome is the markedly different from zero strange contribution to spin.

The currently more promising tool to search for the strangeness effects is related to the vector current $\langle p | \bar{s} \gamma^\mu s | p \rangle$ and its form factors ($g_1$ is related to a axial vector current $\langle p | \bar{s} \gamma^5 \gamma^\mu s | p \rangle$). The initial proposals pointed the neutral weak current as a means to measure the strange vector current $\langle p | \bar{s} \gamma^\mu s | p \rangle$ of the proton [135], which started [136,137] an extensive experimental program in the study of asymmetries in parity-violating electron-proton scattering (see Chapter 6) which provide information on the strange electromagnetic form factors.

The strange vector form factors of the nucleon were the subject of previous studies in the CQSM. The first work [138] had one arithmetic mistake in the strange magnetic form factor, which by that calculation, should have been very small. It also did not take into account the symmetry conserving quantization. This explains the large difference from the result in [138,139] to the present result [140,141] for the magnetic form factor.

The results [140] for the strange Sachs electric and magnetic form factors $G_E^s$, $G_M^s$ are shown in Fig. 3.8 while the Dirac and Pauli strange form factors are shown in Fig. 3.9. Results for radii, defined as

$$\langle r^2 \rangle_E^s = -6 \frac{dG_E^s(Q^2)}{dQ^2} \bigg|_{Q^2 = 0}, \quad \langle r^2 \rangle_M^s = -6 \frac{dG_M^s(Q^2)}{dQ^2} \bigg|_{Q^2 = 0}, \quad (3.70)$$

and magnetic moments are presented in Tab. 3.6. for $M = 420$ MeV and both the pion ($\pi$) and kaon (K) asymptotics.

![Graph](image_url)

**Figure 3.8:** The strange electric and magnetic form factors $G_E^s$, $G_M^s$ as a function of $Q^2$. The constituent quark mass is $M = 420$ MeV and the strange quark mass $m_s = 180$ MeV.

Fig. 3.8 shows the Sachs form factors with explicit dependence on the asymptotic behaviour

---

$x$ is the Bjorken variable, $x = Q^2/2M \nu$, $Q^2$ and $\nu$ are the four-momentum and the energy transferred, respectively.
of the meson field. It contains results for both the pion asymptotics ($\mu = 140$ MeV) and ‘kaon asymptotics’ ($\mu = 490$ MeV) in the spirit of Section 2.6.2. One finds that the values of the form factors depend differently on the mass governing the asymptotics of the meson field: the electric form factor decreases with this mass while the magnetic form factor is enhanced. The data point in Fig. 3.8 for the magnetic form factor comes from a measurement by the SAMPLE collaboration [142]. It was found ($Q^2$ in (GeV/c)$^2$) that 
\begin{equation}
G_M^s(Q^2 = 0.1) = +0.14 \pm 0.29 \pm 0.31 \text{ (n.m.)}
\end{equation}
A new determination of the magnetic form factor by the same collaboration gave [143] 
\begin{equation}
G_M^s(Q^2 = 0.1) = +0.37 \pm 0.20 \pm 0.26 \pm 0.07 \text{ (n.m.)}
\end{equation}
This last experimental result, due to the large error bars, only seems to give further support to the first one in favour of a positive strange magnetic moment.

From the comparison between model and the SAMPLE value (3.71) one finds that the CQSM is not contradicted by the data. It is not confirmed either, due to the large error bars. One should note here that there is some support for the need to take into account the proper asymptotic behaviour of the meson fields, as is revealed by the phenomenological agreement obtained by the kaon mass description of the meson asymptotics. The effect should, nevertheless, be put into more solid theoretical ground. In the spirit, to be discussed below together with the octet magnetic moments, of using a “soliton magneton” instead of the nuclear magneton, one would find a smaller difference between the CQSM result and the SAMPLE value. This would lead again to no conclusions on the account of the large error bar.

Support to the finding of a positive strange magnetic moment is provided by the model independent analysis of [130], which finds $\mu_s = +0.41 \pm 0.18$ n.m. on the basis of the SU(3) model.
algebra with input from the hyperon magnetic moments. This model independent approach consisted in keeping the collective matrix elements in (2.67) and replacing the values of the densities at $Q^2 = 0$ by constants which are then fitted to the experimental values of the magnetic moments. A reasoning similar to this one has been followed in the SU(3) versions of other models. Taking from the treatment of SU(3) in the Skyrme model, the MIT bag [47] and the chiral bag model [144] just the SU(3) structure and using phenomenological information like magnetic moments and decay constants to determine the inertia parameters, it has been found [145, 146] support for positive strange magnetic moment, similarly to [130].

The dependence of the results on the model parameters are shown in Appendix H Fig.s H.1 and H.2. In Fig. H.1 it is clear that both form factors grow with growing strange quark mass. In Fig. H.2 one observes that the strange electric form factor decreases with growing constituent quark mass, the strange magnetic form factor has the opposite behaviour.

Due to their importance in understanding the structure of the nucleon, strange form factors attracted, and still attract, a lot of theoretical interest. In the following, the quoted pair of numbers will represent the predictions ($\mu_s, \langle r^2 \rangle_s$) for the strange magnetic moment and electric Sachs (r.m.s) radius, with the first in n.m. and the second in fm$^2$.

In the NJLM, [147] a study of the strange form factors has been made with similar methods of those used here. The quoted results ($\mu_s = -0.05 \rightarrow 0.25, \langle r^2 \rangle = -0.2 \rightarrow -0.1$) agree qualitatively with the ones presented here and the ones in the SM ($-0.13, -0.11$) [148] and ($-0.05, 0.05$) [127]. The approach of [147] is, however, different since in [147] the imaginary part of the action is regularized, rotational corrections are not included and the soliton is not treated self-consistently.

One of the methods to study the strange form factors, mainly their associated radius and the strange magnetic moment, consists in making a dipole fit analysis of the isoscalar form factors of the nucleon using dispersion relations. Early work in this direction gave ($-0.31 \pm 0.09, 0.14 \pm 0.07$) [149] while an update based on an improved dispersion-theoretical fit of the nucleon electromagnetic form factors yielded ($-0.24 \pm 0.03, 0.21 \pm 0.03$) [150]. As in most models, the conclusions point towards a negative magnetic moment. Contrary to most models, the sign of the strange charge radius is positive. For a review see [151].

Another common approach to the strange form factors is based on the consideration that the strange content of the nucleon manifests itself as fluctuations of the bare nonstrange nucleon into a kaon-hyperon pair, e.g. $K\Lambda$ and $K\Sigma$ components. These types of approach are described as based on kaon loops, although they may incorporate other aspects of the nucleon structure. Early work including kaon loops gave ($-2.6, -0.97 \times 10^{-2}$) [152], later improved to ($-0.31 \rightarrow -0.40, -2.71 \rightarrow -3.23 \times 10^{-2}$) [153] by including seagull terms. The peculiar aspect of this approach was the small and negative strange radius, as compared to the pole-fit analysis. In order to bring these two approaches to a common ground it has been proposed to include vector meson dominance (VMD) and $\omega - \phi$ mixing into the kaon loop approach. The results included ($-0.24 \rightarrow -0.32, -3.99 \rightarrow -4.51 \times 10^{-2}$) [154], ($-1.69 \times 10^{-2}$) [155]. More recent work included the $K^*$ mesons [156]. The $Q^2$ dependence of form factors was obtained in [157] for several assumptions concerning the $K^*$ loops. In the same line of work [158], giving emphasis to the OZI allowed coupling of the $\phi$ meson to the nucleon and including kaon loops and hyperon excitations, a small but positive strange magnetic moment is obtained $\mu_s = 0.003$ (n.m.). In the chiral quark model [159, 160] it is found that the kaon loops have a small contribution to the strangeness of the nucleon with a magnetic moment of $-0.05$ (n.m.).

The strange form factors also attracted the attention of heavy baryon CHPT (HBCHPT). In [161] the strange magnetic form factor was analyzed at one-loop order with the result $\mu_s = 0.03 \rightarrow 0.18$ (n.m.). The strange chiral nuclear form factors are reported in [162] using HBCHPT to third order in the chiral expansion. The findings of [162] for the strange radii are $\langle r^2 \rangle_E = 0.05 \pm 0.09$ fm$^2$, which still allows for a small negative value as obtained in this work, and $\langle r^2 \rangle_M = -0.14$ fm$^2$ which has the opposite sign relative to the present work. The result for the strange magnetic moment is found positive 0.18 n.m. and compares well with the one obtained here. The strange magnetic
radius is further analyzed in HBCHPT in [163].

In the perturbative quark model [164] the electric form factor is qualitatively like the one presented in Fig. 3.8, albeit smaller. The strangeness radius is thus smaller $-0.011 \pm 0.003$ fm$^2$. The strange magnetic form factor differs in sign: the magnetic moment is $-0.048 \pm 0.012$ n.m. and the magnetic radius is positive, according to our normalization, and higher.

In lattice QCD, an analysis based on chiral symmetry gives for the magnetic moment $\mu_s = -0.16 \pm 0.18$ n.m., which does not exclude positive values for $\mu_s$. The strange form factors obtained in quenched lattice QCD and quenched CHPT of ref. [165] are in quite good agreement with the results of the CQSM. For extensive reviews on the strange form factors of the nucleon, both theoretical and experimental, see [166, 167].

3.6.1 The SAMPLE, HAPPEX, A4, and G0 experiments

The results of the CQSM may be used to make predictions for ongoing and future experiments, summarized in the title of this section [140, 141, 168, 169].

Even though SAMPLE has been the only collaboration to present results for a single strange vector form factor, namely the magnetic form factor (3.71, 3.72), other collaborations have succeeded in extracting data on these form factors of the nucleon, but only as a linear combination of electric and magnetic form factors. The HAPPEX collaboration announced such kind of results [170].

From the parity-violating polarized electron scattering asymmetry $A_{th}$ they arrive after various approximations at the singlet form factor relation

$$\frac{G_E^{(0)} + 0.392G_M^{(0)}}{G_M^{p\gamma}/\mu_p} (Q^2 = 0.477) = 1.527 \pm 0.048 \pm 0.027 \pm 0.011. \quad (3.73)$$

Using the available data on the nucleon electromagnetic form factors and the relation between singlet and strange form factors

$$G_{s,M} = G_{E,M}^{(0)} - G_{p\gamma}^{p\gamma} - G_{E,M}^{p\gamma} \quad (3.74)$$

leads to a result for the strange vector form factors

$$(G_E^s + 0.392G_M^s)(Q^2 = 0.477) = 0.025 \pm 0.020 \pm 0.014. \quad (3.75)$$

In Tab. 3.5 we present the results for the combination (3.73) of $G_E^s$ and $G_M^s$ and for the combination of $G_E^s$ and $G_M^s$ in (3.75) as given by HAPPEX [170]. Tab. 3.5 further distinguishes between the cases with the HAPPEX collaboration phenomenology based value $\beta = 0.392$ and the model calculation of the quantity $\beta = \frac{\tau(Q^2)G_M^{p\gamma}(Q^2)}{\epsilon(Q^2)G_E^{p\gamma}(Q^2)}$, as given by

$$\beta(Q^2, \theta) = \frac{\tau(Q^2)G_M^{p\gamma}(Q^2)}{\epsilon(Q^2)G_E^{p\gamma}(Q^2)}, \quad (3.76a)$$

$$\tau = Q^2/(4M_N^2), \quad \epsilon = [1 + 2(1 + \tau)\tan^2(\theta/2)]^{-1}, \quad (3.76b)$$

using the model electromagnetic form factors, calculated with the symmetry conserving quantization, as they are obtained in the model, without any scaling. For the combination of $G_E^{(0)}$ and $G_M^{(0)}$ and for both cases of $\beta$ the HAPPEX result (3.73) falls within the interval of model results between the pion ($\pi$) and kaon (K) asymptotics. The combination of $G_E^s$ and $G_M^s$ seems to overestimate the HAPPEX result (3.75) for both cases of $\beta$. One notices that the kaon asymptotics leads to results closer to the experiment. One has to take into account, however, that the HAPPEX value (3.75) could be higher, by as much as 0.020, if a different input for the electromagnetic form factors is considered [170].

Although the results in Tab. 3.5 depend on $M$ it is found that, for physically reasonable values of $M$ between 400 and 450 MeV the dependence on $M$ is much weaker than the effect of the meson asymptotics.

Predictions for HAPPEX and A4 experiments are summarized in Tab. 3.6. The combination $G_E^s(Q^2) + \beta(Q^2, \theta)G_M^s(Q^2)$ is depicted in Fig. 3.10 for the experimental setup ($\theta$) of the G0 (left) and A4 (right) experiments. The function $\beta(Q^2, \theta)$ in Tab. 3.6 and in Fig. 3.10 is calculated in the model without any scaling for the magnetic form factor. These model calculations indicate that the experiment A4 at backward angles is the one most sensitive to strange quark effects in the nucleon.
The data point (3.77a) is plotted in Fig. (3.10). The model values for the preferred kaon asymptotics are the following possibilities (46 Electromagnetic form factors +0

| \( \beta(Q^2, \theta) \) | 0.392 | 0.285 (Model) | 0.392 | 0.285 (Model) |
| \( \mu \) | \( \pi - K \) | \( \pi - K \) | \( \pi - K \) | \( \pi - K \) |
| \( M = 420 \text{ MeV} \) | \( 1.433 - 1.695 \) | \( 1.363 - 1.605 \) | \( 0.103 - 0.071 \) | \( 0.100 - 0.066 \) |
| HAPPEX | \( 1.527 \pm 0.048 \pm 0.027 \pm 0.011 \) | \( 0.025 \pm 0.020 \pm 0.014 \) |

Table 3.5: The combinations \((G_E^s + \beta(Q^2, \theta)G_M^s)/(G_M^s/\mu_p)\) and \(G_E^s + \beta(Q^2, \theta)G_M^s\) for the HAPPEX kinematics \(Q^2 = 0.477 \text{ GeV}^2\) and \(\theta = 12.3^\circ\). The experimental data are taken from HAPPEX [170].

form factors.

<table>
<thead>
<tr>
<th>Exp.</th>
<th>A4</th>
<th>HAPPEX II</th>
<th>A4</th>
<th>A4</th>
<th>HAPPEX</th>
<th>A4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q^2(\text{GeV}^2) )</td>
<td>0.10</td>
<td>0.11</td>
<td>0.23</td>
<td>0.23</td>
<td>0.48</td>
<td>0.48</td>
</tr>
<tr>
<td>( \theta(\circ) )</td>
<td>35</td>
<td>6</td>
<td>35</td>
<td>145</td>
<td>12.3</td>
<td>145</td>
</tr>
<tr>
<td>( \beta ) (Model)</td>
<td>0.063</td>
<td>0.060</td>
<td>0.156</td>
<td>2.891</td>
<td>0.285</td>
<td>6.625</td>
</tr>
<tr>
<td>( \mu )</td>
<td>( \pi - K )</td>
<td>( \pi - K )</td>
<td>( \pi - K )</td>
<td>( \pi - K )</td>
<td>( \pi - K )</td>
<td></td>
</tr>
<tr>
<td>( M = 420 \text{ MeV} )</td>
<td>0.08-0.04</td>
<td>0.09-0.05</td>
<td>0.17-0.10</td>
<td>0.37-0.39</td>
<td>0.28-0.18</td>
<td>0.72-0.83</td>
</tr>
</tbody>
</table>

Table 3.6: The values of \((G_E^s + \beta(Q^2, \theta)G_M^s)/(G_M^s/\mu_p)\) at various \(Q^2\) and \(\theta\) for \(M = 420 \text{ MeV}\) with pion (\(\pi\)) and kaon (\(K\)) tails. All the experiments (Exp.) are still being performed except for HAPPEX.

Figure 3.10: Predictions for the G0 (left) and A4 (right) experiments.

Very recently first results of the experiment A4 were published [171]. These are

\[
(G_E^s + 0.225G_M^s)(Q^2 = 0.230) = 0.039 \pm 0.034, \quad (3.77a)
\]

\[
(F_1^s + 0.130F_2^s)(Q^2 = 0.230) = 0.032 \pm 0.028. \quad (3.77b)
\]

The data point (3.77a) is plotted in Fig. (3.10). The model values, for the preferred kaon asymptotics, seem to overestimate the data for the electron scattering angle of 35\(^\circ\). In Fig. (3.10) the \(\beta\) function is calculated with the form factors as calculated in the model in Section (3.5). In the model there are the following possibilities (\(Q^2 = 0.230\)):

\[
G_E^s + 0.225G_M^s = 0.039 \pm 0.034, \quad (A4) \tag{3.78a}
\]

\[
G_E^{(K)s} + 0.225G_M^{(K)s} = 0.063, \quad \text{(Model form factors and experimental \(\beta\))} \tag{3.78b}
\]

\[
G_E^{(K)s} + 0.156G_M^{(K)s} = 0.059, \quad \text{(Model form factors and model\(\beta\))} \tag{3.78c}
\]

\[
G_E^{(K)s} + 0.212G_M^{(K)s} = 0.062, \quad \text{("Soliton magneton")} \tag{3.78d}
\]
In all scenarios the CQSM result is at the top of the error bar. As to (3.77b), one similarly obtains:

\[ F_1^s + 0.130 F_2^s = 0.032 \pm 0.028, \quad \text{(A4)} \]
\[ F_1^{(K)s} + 0.130 F_2^{(K)s} = 0.052, \quad \text{(Model form factors and experimental \( \beta \))} \]
\[ F_1^{(K)s} + 0.079 F_2^{(K)s} = 0.051, \quad \text{(Model form factors and model/\( \beta \))} \]
\[ F_1^{(K)s} + 0.121 F_2^{(K)s} = 0.052, \quad \text{("Soliton magneton") (3.79d)} \]

Again the CQSM result is at the top of the error bar. The source for this apparent overestimation will only be clarified once information exists separately for electric and magnetic strange form factors. One, nevertheless, must note that the model is able to give a good description of these sensitive and small quantities entering the experimentally measured quantity.

### 3.7 Nucleon electromagnetic form factors

In the particular case of the form factors of the nucleon, by far the best studied experimentally, one may well start with a comparison between the flavor form factors for \( u \) and \( d \) quarks in flavors SU(2) and SU(3), as is shown in Fig. 3.11.

![Form factors](image)

**Figure 3.11:** The electric (left) and magnetic (right) form factors for flavors \( u \) and \( d \) as a function of \( Q^2 \). The constituent quark mass is \( M = 420 \text{ MeV} \) and the strange quark mass \( m_s = 180 \text{ MeV} \) in the SU(3) case.

The first observation from Fig. 3.11 is that the differences between the two flavor results are small. In particular, all the flavor form factors in SU(3)\( f \) are smaller than the corresponding ones in SU(2)\( f \). The differences between the form factors of the nucleon in the two flavor groups are thus not ascribable to the contribution of the \( s \) quark alone. Taking \( m_s \) to zero does not take from SU(3)\( f \) to SU(2)\( f \), which is to say that the collective matrix elements are close to each other, but not exactly the same.

The importance of the \( s \) quark to the form factors of the nucleon is better observable with the aid of the composite meson asymptotic description (3.58). Looking at Fig. 3.12, it is clear that such description shows how the electric form factor of the neutron in particular may sensitive to the strange quark. There were early indications, from vector meson dominance models [172] for instance, in this direction. The same kind of conclusion was found in heavy baryon CHPT [132]. In the case of the CQSM the electric form factor of the neutron as given using (3.58) still underestimates the experimental data. This means it is necessary to replace the phenomenological ansatz (3.58) by a better description of the effects it tries to mimic, namely those of the kaon meson cloud.

The general remarks which apply to the four form factors of the nucleon, Figs 3.12, 3.13, may
very massive and is almost at rest in the Breit-frame. In the limit of these degrees of freedom are vector mesons whose mass is around $m_V \approx 770$ MeV.

be summarized as follows: the flavor SU(3) with mixed asymptotics for the meson cloud is closer to phenomenology than flavor SU(2) at lower $Q^2$; the mixed asymptotics description coincides with the pion asymptotics in the case of the magnetic form factors, differs slightly in the proton electric form factor and is much larger in the case of the neutron electric form factor; the kaon asymptotics alone leads to harder form factors and deviates much from the phenomenological values and the flavor SU(2) form factors are always larger than the flavor SU(3) ones. These conclusions do not depend on model parameters, as Figs. H.11 and H.12 for the case of the constituent mass $M$ and Figs. H.13 and H.14 for the strange current quark mass $m_s$. In these figures the form factors are normalized to the dipole form (3.6a). Also from these figures, the nucleon form factors are seen to follow the dipole at low values of $Q^2$ with deviations growing with $Q^2$.

The charge and magnetization densities are depicted in Fig. (3.14) and the above comments still apply. In particular one does not see in the CQSM any evidence for the meson cloud effects discussed in [182].

As is the case with most low energy observables of the strong interactions, it is most interesting to see how the form factors behave as $Q^2$ increases, particularly when $Q^2$ comes close to the values for which perturbative QCD is able to give quantitative predictions. As referred before, the framework in which the foregoing results were obtained does not allow to go beyond $Q^2 = 1$ GeV$^2$ at best. At values of $Q^2$ approaching 1 GeV$^2$, and higher, the terms which were suppressed by $1/N_c$ in this calculation start to have large effects, e.g. recoil corrections. One should recall at this point that the translation zero modes are not relativistically treated. In fact, at large $N_c$ the nucleon is very massive and is almost at rest in the Breit-frame. In the limit $N_c \to 3$, in the mean field, one does not know neither the position of the nucleon center-of-mass nor its momentum. In order to have nucleon states with good linear momentum, one should project into these momentum operator eigenstates, in the same way the quantization of the zero modes of the soliton mean field solution corresponds to an angular momentum projection after variation (leading to mean field). To the best of our knowledge this kind of projection has only been carried out in the case of the color-dielectric model [183], where it was found to be important for the improvement of the description of form factors at higher values of $Q^2$, approaching 1 GeV$^2$. Also, as $Q^2$ increases it is less and less legitimate to take as frozen those degrees of freedom above the constituent quark mass. One of these degrees of freedom are vector mesons whose mass is around $m_V \approx 770$ MeV.

Figure 3.12: Electric form factors for proton and neutron. The proton experimental data (◦) comes from $p(e,e')$ [116,173,174] and (△) from $d(e,e'p)$ [175]. For the neutron the experimental (◦) data comes from $d(\bar{e},e'n)p$ [107,176,177], △ from $\bar{d}(\bar{e},e'n)p$ [178], and ◯ from $^3\bar{H}_e(\bar{e},e'n)$ [106,179].
Figure 3.13: Magnetic form factors for proton and neutron. Proton experimental data (○) is from \( p(e,e') \) [174, 180], and (△) from \( d(e,e'p) \) [175]. For the neutron the experimental data (○) is from \( d(e,e'p) \) [109,119,181]. Model parameters as in Fig. 3.11. The model results are easily identified by their underestimation of the magnetic moments; the other set of curves with the same symbols corresponds to the model set scaled to the experimental value of the magnetic moments.

In order to get an indication about the importance of these two ingredients in the form factors one may follow previous work in the Skyrme model, ref. [184,185], in using a phenomenological extension of the form factors computed here. These extension consists in incorporating vector mesons through the ansatz \( \lambda(Q^2) \), which tries to summarize into a single factor the various parameters of vector mesons.

\[
\Lambda(Q^2) = \lambda \left( \frac{m_V^2}{m_V^2 + Q^2} \right) + (1 - \lambda), \tag{3.80}
\]

The free parameter \( \lambda \) is used to interpolate between complete vector meson dominance (\( \lambda = 1 \)) and the model with pion asymptotics (or mixed asymptotics, as will be the case here) (\( \lambda = 0 \)). The densities entering the form factors are calculated in the soliton rest frame in terms of the profile of the soliton \( P_s(r) \). This frame only coincides with the Breit frame in the large \( N_c \) limit, since the nucleon mass ismof order \( N_c \). For values of \( Q^2 \) approaching, and above, the soliton mass one should include corrections for the boost of the soliton to the Breit frame, where it moves with velocity \( v \)

\[
\gamma^2 = \frac{1}{1 - v^2} = 1 + \frac{Q^2}{4M_S^2} \tag{3.81}
\]

The approximate way to deal with these recoil (rec) corrections is based on form factors being Lorentz scalars [186,187]. Therefore one may write

\[
G_E^{\text{rec}}(Q^2) = G_E^{\text{nr}} \left( \frac{Q^2}{\gamma^2} \right), \quad G_M^{\text{rec}}(Q^2) = \frac{1}{\gamma^2} G_M^{\text{nr}} \left( \frac{Q^2}{\gamma^2} \right) \tag{3.82}
\]

For the complete form factors, these aspects lead to

\[
G_{E,M}(Q^2) = \Lambda(q^2) G_{E,M}^{\text{rec}}(Q^2). \tag{3.83}
\]

Now ansatz (3.83) allows to plot the form factors up to higher \( Q^2 \) values. Using a soliton mass of about \( M_S = 1.2 \) GeV, the form factors thus obtained seem, without exploring the other model parameters, to be best described by low \( \lambda \) values. Fig. 3.15 shows the ratio \( \mu_p G_E^p / G_M^p (\mu_p \text{ is } G_M^p(0)) \), as a function of \( Q^2 \), together with recent experimental results for this quantity. The model results seem to overestimate the absolute value of this ratio.
Figure 3.14: Charge and magnetization densities of the nucleon. Model parameters as in Fig. 3.11.

Figure 3.15: The ratio $\mu_p G_E^p / G_M^p$. Experimental data (○) is from $p(\vec{e}, e'\vec{p})$ (○) [101–103, 188]) (△ second value from [103]). Model parameters as in Fig. 3.11.
The nucleon-$\Delta(1232)$ transition

The electromagnetic current is not only involved in elastic processes, as was the case in the previous chapter regarding form factors and their related properties like radii and magnetic moments. The electromagnetic current participates also in transition processes, i.e. related to off-forward matrix elements where the final hadronic state differs in constitution from the target state, as electromagnetic excitation and decay, for instance. In this chapter use is made of the representation of the electromagnetic current developed in the previous chapter to address the electromagnetic excitation of the delta, $\Delta(1232)$, from the nucleon.

There are two different approaches: One of them is based on form factors, along the same lines as the electromagnetic form factors, and the other is based on transition amplitudes [189]. They can be easily related to each other, as they are based on similar transition matrix elements. In this chapter, the approach based on transition amplitudes is preferred because the main interest lies in the multipole components of the helicity amplitudes. Earlier results [190], using the constituent quark model, in a SU(6), symmetric and nonrelativistic formulation, indicated that the nucleon to delta transition was a pure magnetic dipole ($M_1$), i.e. it proceeds via a single spin flip of one of the quarks in the nucleon, and that the electric quadrupole ($E_2$) vanished. In this approach the orbital angular momentum of the quarks is zero, both in the nucleon and the delta. It was soon experimentally found that it is not so. It was found [191] that, although $M_1$ dominates the transition, $E_2$ was nonvanishing. This nonvanishing of $E_2$ and also of the Coulomb quadrupole $C_2$ for virtual photons, are still the motivation for a lot of theoretical and experimental work.

The relevant quantities used to analyze the scattering $p + \gamma^* \rightarrow \Delta$ (usually in the context of the electroproduction of pions), are the ratios of the multipoles $E_2/M_1$ and $C_2/M_1$, which may be defined as ratios of amplitudes (4.4). Restricting to the most recent experimental data [124,192–195] the present situation is described by a small and negative $E_2/M_1$ up to around $Q^2 = 4$ GeV$^2$, and an also negative $C_2/M_1$, increasing in absolute value with $Q^2$.

These non vanishing quadrupole moments indicate a deformation either of the nucleon, the delta, or the transition density. The question of which of these possibilities lies at the origin of this deformation has not a clear answer, though.

In order to account for a nonvanishing $E_2$ it was initially proposed D-wave contributions to the nucleon and delta wave functions [196,197]. Another source for a nonvanishing $E_2$, still in the context of the quark model, is accounted by two-body exchange currents between the constituent quarks [198,199], according to which the $E_2$ transition proceeds via a two-quark spin-flip transition.

The calculation in Section 4.3 of the multipole ratios extends previous SU(2) CQSM calculations [200] to flavor SU(3), at the same time it repeats the SU(2) results. However, one considers only the flavor SU(3) symmetric case, i.e. all the quarks have the same current mass, thus excluding mass corrections. The rationale to take such tentative a choice is based on the results obtained for the form factors in the previous chapter where mass corrections are seen to be smaller that rotational $(1/N_c)$ corrections. Since both the scalar and electric quadrupole contributions to the amplitudes vanish at the leading orther in $N_c$, the main influence to the quadrupole contributions is expected to arise from the rotational corrections. The results, Section 4.4 and [201, 202], do seem to confirm this assertion.

The results correspond to a constituent quark mass of 420 MeV, which is chosen from the best description of the electromagnetic form factors in the previous chapter, and are thus free of any adjustable model parameters.
4.1 Definition of the transition amplitudes

The main matrix element of interest is the matrix element of the electromagnetic current between the nucleon ($N$) and delta ($\Delta$) states

$$\langle \Delta(p') | J_{EM}(0) | N(p) \rangle.$$  \hfill (4.1)

Although one may write the form factors for this matrix element, it is usually preferred to work with the transition amplitudes, defined below, and to use the reference frame in which the delta is produced at rest. This conforms with most of the theoretical and experimental literature on the subject.

In the one-photon exchange approximation, the $N - \Delta$ transition may proceed by either one of the transversal, scalar (or Coulomb) or longitudinal amplitudes, each defined on the basis of the corresponding components of the electromagnetic current. Current conservation may be used to relate the scalar and the longitudinal amplitudes. Since the current is conserved at the level of the model, the longitudinal amplitude was not explicitly calculated in this work.

The transverse amplitudes ($A_{\lambda}, \lambda = 1/2, 3/2$) and the scalar amplitude ($S_{1/2}$) are defined by

$$A_{\lambda} = -\frac{e}{\sqrt{2}\omega} \int d^3r \, e^{i q \cdot r} \tilde{\epsilon}_{+1} \cdot \langle \Delta^+(\frac{3}{2}, \lambda) | \overline{\Psi}Q\gamma_\lambda \Psi(r) | N(\frac{1}{2}, -\lambda - 1) \rangle,$$

$$S_{1/2} = -\frac{e}{\sqrt{2}\omega} \int d^3r \, e^{i q \cdot r} \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | \overline{\Psi}\gamma_0 \psi(r) | N(\frac{1}{2}, \frac{1}{2}) \rangle.$$ \hfill (4.2a)

where $q = p_\Delta - p_N$ is the three-momentum of the photon and $\tilde{\epsilon}_{+1} = -1/\sqrt{2}(1, +i, 0)$. In (4.2) $\omega$ is the time component of the exchanged photon four-momentum and the matrix $Q$ is the charge matrix$^1$.

It is the purpose of this chapter to calculate, in the multipole expansion of the helicity amplitudes (4.2), the ratios of the quadrupolar components, electric and scalar, to the magnetic dipole component. The multipole decomposition of the transverse amplitudes is

$$A_{\frac{1}{2}} = A_{\frac{1}{2}}(M1) + A_{\frac{1}{2}}(E2) = -\frac{1}{2} (M1 + 3E2),$$ \hfill (4.3a)

$$A_{\frac{3}{2}} = A_{\frac{3}{2}}(M1) + A_{\frac{3}{2}}(E2) = \sqrt{3} A_{\frac{1}{2}}(M1) - \frac{1}{\sqrt{3}} A_{\frac{1}{2}}(E2) = -\frac{\sqrt{3}}{2} (M1 - E2),$$ \hfill (4.3b)

which defines the multipole quantities $M1$ and $E2$. The scalar multipole is simply $C2 = -S_{1/2}$. From the transverse amplitudes (4.3) it follows that $M1 = -2A_{1/2}(M1)$ and $E2 = -2A_{1/2}(E2)/3$, so the ratios to compute are defined by

$$\frac{E2}{M1} = \frac{1}{3} \frac{A_{1/2}(E2)}{A_{1/2}(M1)}, \quad \frac{C2}{M1} = \frac{1}{2} \frac{S_{1/2}(C2)}{A_{1/2}(M1)}.$$ \hfill (4.4)

Regarding the kinematics, the chosen reference frame is the one where the $\Delta$ is produced at rest, Fig. 4.1. The kinematics is specified in this frame by the initial four-momenta of the nucleon and virtual photon: $(E_N, -q)$ and $(\omega, q)$, respectively. In terms of the photon virtuality,

$$\omega^2 - q^2 = q^2 < 0,$$

$$\omega^2 = (M_\Delta - E_N)^2 = q^2 + q^2$$ \hfill (4.5)

$^1$One should note that only the isovector part of the current enters the $N$-$\Delta$ transition.
one obtains, with $E_N^2 = M_N^2 + q^2$ and $Q^2 = -q^2$, that
\[
|q|^2 = \left( \frac{M_N^2 + M_N^2 + Q^2}{2 M_\Delta} \right)^2 - M_N^2, \tag{4.6a}
\]
\[
\omega = \frac{M_N^2 - M_N^2 - Q^2}{2 M_\Delta} \tag{4.6b}
\]
in the $\Delta$ rest frame.

### 4.2 Multipole expansions of the amplitudes

Angular momentum and parity are conserved in pion electroproduction. Due to the spin values of the $N$ and $\Delta$ the angular momentum of the exchanged photon can only, therefore, take the values $l = 1, 2$. The consequence for the multipole expansion is that the only multipoles that contribute to the $N - \Delta$ transition are the multipoles $M1, E2$ and $C2$ [203–205].

For the transverse amplitudes the multipole expansion reads [206]
\[
\hat{e}_{l+1} e^{i q \cdot r} = \sqrt{2 \pi} \sum_l \sqrt{2l + 1} \left[ A_{l1}(r; M) + i A_{l1}(r; E) \right]
\]
in terms of electric and magnetic multipole components\(^2\) $A_{l1}$ given by
\[
A_{lm}(r; M) = j_l(|q|r) Y_{lm}(\hat{r}) = \frac{1}{\sqrt{l(l+1)}} \hat{r} \cdot j_l(|q|r) Y_{lm}(\hat{r}) \tag{4.8a}
\]
\[
A_{lm}(r; E) = \frac{1}{i |q|} \nabla \times A_{lm}(r; M) \tag{4.8b}
\]
with $r = |r|$, $\hat{r} = r/|r|$ and the vector-spherical harmonics defined by
\[
Y_{lj;M}(\hat{r}) = \sum_{\mu} C_{lM-\mu}^j Y_{lM-\mu}(\hat{r}) \hat{e}_\mu. \tag{4.9}
\]

Since the photon has negative parity and the $\delta$ states positive parity, only the electric quadrupolar and magnetic dipolar components contribute to the transverse transition amplitude and we can write
\[
A_{1/2} = -\frac{e}{\sqrt{2\omega}} \int d^3 r \left\{ \Delta^+ \left( \frac{1}{2} \frac{1}{2} \right) \left[ i \sqrt{6\pi} A_{11}(r; M) - i \sqrt{10\pi} A_{21}(r; E) \right] \cdot \bar{\psi} Q \gamma_5 \gamma(r) |N(\frac{1}{2}, -\frac{1}{2}) \right\}
\]
\[
= -\frac{e}{\sqrt{2\omega}} \left[ \mathcal{M}^{M1} + \mathcal{M}^{E2} \right] \tag{4.10}
\]
splitting $A_{1/2}$ into the relevant multipoles for the calculation of the ratios: A magnetic dipole
\[
\mathcal{M}^{M1} = i \sqrt{6\pi} \int d^3 r A_{11}(r; M) \cdot \left\langle \Delta^+ \left( \frac{1}{2} \frac{1}{2} \right) | \bar{\psi} Q \gamma_5 \gamma(r) |N(\frac{1}{2}, -\frac{1}{2}) \right\rangle, \tag{4.11}
\]
and an electric quadrupole
\[
\mathcal{M}^{E2} = -i \sqrt{10\pi} \int d^3 r A_{21}(r; E) \cdot \left\langle \Delta^+ \left( \frac{1}{2} \frac{1}{2} \right) | \bar{\psi} Q \gamma_5 \gamma(r) |N(\frac{1}{2}, -\frac{1}{2}) \right\rangle. \tag{4.12}
\]

The $A_{11}(r; M)$ component is given from (4.8a) and (4.9) by
\[
A_{11}(r; M) = j_1(|q|r) \sum_{\mu} C_{11-\mu}^{11} Y_{1-\mu}(\hat{r}) \hat{e}_\mu. \tag{4.13}
\]
The relation between spherical coordinates and spherical harmonics
\[
Y_{1\mu}(\hat{r}) = \sqrt{\frac{3}{4\pi}} \hat{r}_{1\mu}, \tag{4.14}
\]
and the relation for the tensor product
\[
i \frac{1}{\sqrt{2}} (A \times B)_M = \sum_{\mu} C_{1M-\mu\mu}^{1} A_{M-\mu} B_{\mu} = \{A \otimes B\}_{1M} \tag{4.15}
\]
\(^2\)The parities of the components $A_{11}$ depend on $l$ according to $\Pi_{i1} = (-1)^{l+1}$, $\Pi_{iM} = (-1)^l$. 

allow to rewrite the scalar product of the spatial part of the electromagnetic current, here abbreviated by \( J(r) = \bar{\psi} Q \gamma^0 \psi(r) \), with \( A_{11}(r; M) \) as

\[
J \cdot A_{11}(r; M) = j_1(|q|r) \sqrt{\frac{3}{4\pi}} \{ \hat{r}_1 \otimes J_1 \}_1
\]

(4.16)

The magnetic dipolar component becomes then

\[
\mathcal{M}^{M1} = i \frac{3}{\sqrt{2}} \int d^3r \ j_1(|q|r) \left\langle \Delta^+(\frac{1}{2},\frac{1}{2}) \right| \{ \hat{r}_1 \otimes J_1 \}_1 \left| N(\frac{1}{2},-\frac{1}{2}) \right\rangle.
\]

(4.17)

In the case of the electric quadrupolar component, from (4.8), one has

\[
A_{21}(r; E) = \frac{1}{\sqrt{6}} r |q| j_2(|q|r) Y_{21}(\hat{r}) + \frac{1}{\sqrt{6}} |q| \nabla \left( \frac{\partial}{\partial r} r j_2(|q|r) \right) Y_{21}(\hat{r}).
\]

(4.18)

using

\[
\nabla \times \left( r \times \nabla \right) = r \nabla^2 - \nabla \frac{\partial}{\partial r} r
\]

(4.19a)

\[
\nabla^2 j_2(|q|r) Y_{21}(\hat{r}) = -|q|^2 j_2(|q|r) Y_{21}(\hat{r}).
\]

(4.19b)

Here, following [207, 208], only the dominant contribution is kept, i.e. only the term in \( 1/|q| \), which, for small values of \( Q^2 \) largely dominates the linear term in \( |q| \). It was further assumed that the \( N \) and \( \Delta \) are eigenstates of the Hamiltonian, so that the current conservation equation, \( \nabla \cdot J + \dot{\rho} = 0 \), allows to write the divergence of \( J \) in terms of the scalar density \( \rho = \bar{\psi}(r) Q \gamma^0 \psi(r) \) according to

\[
\left\langle \Delta | \nabla \cdot J \right| N \rangle = (E_\Delta - E_N) \left\langle \Delta | \rho \right| N \rangle = \omega \left\langle \Delta | \rho \right| N \rangle.
\]

(4.20)

The electric quadrupole quantity to evaluate is then given by

\[
\mathcal{M}^{E2} = -i \frac{\sqrt{5\pi}}{3} \int d^3r \ \left\langle \Delta^+(\frac{1}{2},\frac{1}{2}) \right| \frac{1}{|q|} \nabla \left( \frac{\partial}{\partial r} r j_2(|q|r) Y_{21}(\hat{r}) \right) \left| N(\frac{1}{2},-\frac{1}{2}) \right\rangle
\]

\[
= \sqrt{\frac{5\pi}{3}} \left| q \right| \int d^3r \ \left( \frac{\partial}{\partial r} r j_2(|q|r) \right) Y_{21}(\hat{r}) \left\langle \Delta^+(\frac{1}{2},\frac{1}{2}) \right| \bar{\psi} Q \gamma^0 \psi(r) \left| N(\frac{1}{2},-\frac{1}{2}) \right\rangle
\]

(4.21)

As to the scalar amplitude (4.2), using

\[
e^{i qr} = \sqrt{4\pi} \sum_l \sqrt{2l+1} j_l(|q|r) Y_{2l}(\hat{r}),
\]

(4.22)

and since only the case \( l = 2 \) is of interest, one obtains for the quadrupole amplitude

\[
S_{1/2}(C2) = \frac{e}{\sqrt{2\omega}} \sqrt{\frac{10\pi}{3}} \int d^3r \ j_2(|q|r) Y_{20}(\hat{r}) \left\langle \Delta^+(\frac{1}{2},\frac{1}{2}) \right| \bar{\psi}(r) Q \gamma^0 \psi(r) \left| N(\frac{1}{2},\frac{1}{2}) \right\rangle
\]

\[
= -\frac{e}{\sqrt{2\omega}} \frac{1}{\sqrt{20}} M^{C2},
\]

(4.23)

defining the scalar quadrupole quantity

\[
\mathcal{M}^{C2} = -\sqrt{\frac{20\pi}{3}} \int d^3r \ j_2(|q|r) Y_{20}(\hat{r}) \left\langle \Delta^+(\frac{1}{2},\frac{1}{2}) \right| \bar{\psi}(r) Q \gamma^0 \psi(r) \left| N(\frac{1}{2},\frac{1}{2}) \right\rangle.
\]

(4.24)

The multipole ratios (4.4) may be rewritten using the above defined multipole quantities \( \mathcal{M}^{E2} \), \( \mathcal{M}^{M1} \) and \( \mathcal{M}^{C2} \) as

\[
\frac{E2}{M1} = \frac{A_{1/2}(E2)}{3 A_{1/2}(M1)} = \frac{1}{3} \frac{\mathcal{M}^{E2}}{\mathcal{M}^{M1}},
\]

(4.25a)

\[
\frac{C2}{M1} = \frac{S_{1/2}(C2)}{2 A_{1/2}(M1)} = \frac{1}{2\sqrt{2}} \frac{\mathcal{M}^{C2}}{\mathcal{M}^{M1}}.
\]

(4.25b)

### 4.3 The matrix elements for nucleon-\( \Delta \) electroproduction

This section describes how the matrix elements entering the multipole quantities of interest, (4.17), (4.21), and (4.24), are obtained in the CQSM. They follow closely from the results for the form factors, particularly from eq. (E.137) for the electric and scalar case (\( \eta = -1 \)) and from eq. (E.142)
in the magnetic case ($\eta = 1$).

Some emphasis is given to the simplification of expressions (E.137) and (E.142) in the present situation, since that was not done, due to the lengthier expressions, in the case of form factors. The simplification consists mainly in separating the indices which connect the Dirac wave functions to the collective operators. Using the properties of the matrix elements of collective operators and the Wigner-Eckart theorem, together with the hedgehog symmetry, makes it possible to separate completely the products of Dirac one-particle wave functions from the collective operators.

### 4.3.1 Electric and scalar matrix elements

The matrix element entering these amplitudes, for which $\eta = -1$, is replaced by the one computed in the model, as in the case of the form factors,

$$\langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | \psi^1 \gamma^0 \gamma^0 \lambda^X \psi(r) | p(\frac{3}{2}, \frac{1}{2}) \rangle \rightarrow \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | \mathcal{F}^0X(r) | p(\frac{3}{2}, \frac{1}{2}) \rangle$$

The collective operator $\mathcal{F}^{\mu=0,X}$, which now represents the current, is given, similarly to the electric form factor, by

$$\frac{1}{N_c} \mathcal{F}^{0X}(r) = \frac{1}{\sqrt{3}} \frac{D^{(8)}_{s\phi}}{T_1} \left( \langle v|r|v \rangle \langle v|n|v \rangle \frac{1}{2} \sum_n sgn(\varepsilon_m) \langle n|r \rangle \langle n|v \rangle \right)$$

$$+ \frac{1}{2} \sum_n \left( \sum_{n\neq 0} \frac{1}{\varepsilon_V - \varepsilon_n} \langle v|r|n \rangle \langle n|v \rangle \langle n|v \rangle - \frac{1}{2} \sum_{n,m} \mathcal{R}_3(\varepsilon_n, \varepsilon_m) \langle m|r \rangle \langle n|v \rangle \langle n|v \rangle \right)$$

(4.26)

Taking into consideration the collective matrix elements of Tab. 4.1 results in a considerable simplification for the collective operator:

$$\mathcal{F}^{0X}(r) = \frac{1}{2T_1} \left( J_a, D^{(8)}_{s\phi} \right) \mathcal{G}_{ij}(r)$$

(4.27)

$$\text{Op} \quad \mathcal{D}_{33}^{(8)} \quad \text{Others}$$

<table>
<thead>
<tr>
<th>$\langle \Delta^+(\frac{3}{2}, \frac{1}{2})</th>
<th>\text{Op}</th>
<th>p(\frac{3}{2}, \frac{1}{2}) \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2\sqrt{3}}{15}$</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1: Collective matrix elements for the scalar $N-\Delta$ transition. Others are $D^{(8)}_{38}$, $D^{(8)}_{83}$, $D^{(8)}_{88}$, $D^{(8)}_\chi J_i$, $D^{(8)}_{3a} J_a$, $\frac{1}{2} \left( J_a, D^{(8)}_\chi \right) d_{\text{abs}}$, and $i f_{\text{abs}} \left( J_a, D^{(8)}_\chi \right)$

Scalar matrix element

The scalar amplitude can be expressed as

$$\mathcal{M}^{C2} = -\sqrt{20\pi} \int d^3r \ j_2(|q|r) Y_{20}(\hat{r}) \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | \overline{\psi} \gamma^0 \gamma^0 \psi(r) | p(\frac{3}{2}, \frac{1}{2}) \rangle$$

$$-\sqrt{20\pi} \sum_X Q_X \int dr \ r^2 \ j_2(|q|r) \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | \mathcal{M}^{X}_{C2}(r) | p(\frac{3}{2}, \frac{1}{2}) \rangle$$

(4.30)
where the second line makes emphasizes that the general expression of the first line is evaluated in the model according to the expression of the second line

\[ M_{C_2} (r) = \int d\Omega Y_{20}(\hat{r}) F^{0x} (r). \]  

(4.31)

Writing the anticommutator in (4.28) as

\[ \left\{ J_i, D^{(8)}_{\chi\lambda} \right\} = \left[ J_i, D^{(8)}_{\chi\lambda} \right] + 2D^{(8)}_{\chi\lambda} J_i \]  

(4.32)

it is easy to find, applying the Wigner-Eckart theorem, that the term with the commutator vanishes:

\[ \sum \int d\Omega Y_{20}(\hat{r}) \langle \Delta^+ (\frac{3}{2}, \frac{1}{2}) | i f_{ij\rho} D^{(8)}_{\chi\lambda} | p(\frac{3}{2}, \frac{1}{2}) \rangle \langle m | r \rangle \tau^j \langle r | n \rangle \langle n | \tau^i | m \rangle = 0. \]  

(4.33)

For the other term, using the Wigner-Eckart theorem one obtains

\[ \int d\Omega \sum_{G_n^2 G_m^2} \langle m | r \rangle Y_{20}(\hat{r}) \tau^j \langle r | n \rangle \langle n | \tau^i | m \rangle = \left( \sum_{\mu} U_{j\mu} U_{i-\mu} (-)^\mu C_{201\mu}^{1\mu} \right) \times (-)^{G_n-G_m} \frac{1}{3} \Delta (G_n J G_m) \langle \epsilon_1 \otimes \tau_1 \rangle \langle \rho | n \rangle \langle n | \tau_1 | m \rangle \]  

(4.34)

performing the sum of the Clebsh-Gordan coefficients. Using the definition (A.30) and the properties (A.33) of the matrix \( U \) one obtains

\[ \sum_{\mu} (-)^\mu U_{j\mu} U_{i-\mu} C_{201\mu}^{1\mu} = \frac{1}{\sqrt{10}} \delta_{ij} - \frac{3}{\sqrt{10}} \delta_{j0} \delta_{i0} \]  

(4.35)

which allows to rewrite (4.34) in the form

\[ \sum_{G_n^2 G_m^2} \int d\Omega Y_{20}(\hat{r}) \langle \Delta^+ (\frac{3}{2}, \frac{1}{2}) | 2D^{(8)}_{\chi\lambda} J_i | p(\frac{3}{2}, \frac{1}{2}) \rangle \langle m | r \rangle \tau^j \langle r | n \rangle \langle n | \tau^i | m \rangle = -\frac{3}{\sqrt{10}} \langle \Delta^+ (\frac{3}{2}, \frac{1}{2}) | D^{(8)}_{\chi\lambda} | p(\frac{3}{2}, \frac{1}{2}) \rangle \sum_{G_n^2 G_m^2} \langle m | r \rangle \{ Y_2 \otimes \tau_1 \} \langle r | n \rangle \cdot \langle n | \tau | m \rangle \]  

(4.36)

since \( \langle \Delta^+ (\frac{3}{2}, \frac{1}{2}) | D^{(8)}_{\chi\lambda} J_i | p(\frac{3}{2}, \frac{1}{2}) \rangle = 0 \).

Then, the matrix element of the collective operator simplifies to

\[ \langle \Delta^+ (\frac{3}{2}, \frac{1}{2}) | M_{C_2}^n (r) | p(\frac{3}{2}, \frac{1}{2}) \rangle = -\delta_{ij} \frac{3}{\sqrt{10}} \frac{1}{2I_1} \langle \Delta^+ (\frac{3}{2}, \frac{1}{2}) | D^{(8)}_{\chi\lambda} | p(\frac{3}{2}, \frac{1}{2}) \rangle \int d\Omega \frac{1}{\sqrt{4\pi}} g^{C2} (r) \]  

(4.37)

with

\[ \frac{1}{N_c} g^{C2} (r) = \sum_{n \neq 0} \frac{1}{\varepsilon_V - \varepsilon_n} \langle \varepsilon | r \rangle \left\{ \sqrt{4\pi Y_2 \otimes \tau_1} \right\} \langle r | m \rangle \cdot \langle m | \tau | n \rangle \ \frac{1}{2} \sum_{n,m} R_3(\varepsilon_n, \varepsilon_m) \langle m | r \rangle \left\{ \sqrt{4\pi Y_2 \otimes \tau_1} \right\} \langle r | n \rangle \cdot \langle n | \tau | m \rangle \]  

(4.38)

and the regularization function \( R_3(\varepsilon_n, \varepsilon_m) \) given in eq. (E.101).

The final result for \( M^{C2} \) in flavor-symmetric SU(3) is, finally,

\[ M^{C2} = -\sqrt{20\pi} \sum Q_x \int dr r^2 j_2(|q| r) \langle \Delta^+ (\frac{3}{2}, \frac{1}{2}) | M_{C_2}^n (r) | N(\frac{3}{2}, \frac{1}{2}) \rangle \]  

\[ = \frac{1}{2\sqrt{10}I_1} \int d^3 r j_2(|q| r) g^{C2} (r). \]  

(4.39)

and in flavor SU(2)

\[ M_{SU(2)}^{C2} = -\langle \Delta^+ (\frac{3}{2}, \frac{1}{2}) | D^{(1)}_{33} | p(\frac{3}{2}, -\frac{1}{2}) \rangle \frac{1}{8I_1} \int d^3 r j_2(|q| r) g^{C2} (r). \]  

(4.40)
Electric matrix element

As in the case of the scalar amplitude, the electric quadrupole term is given by

\[
\mathcal{M}^{E2} \approx \sqrt{\frac{5\pi}{3}} \frac{\omega}{|q|} \int d^3r \left( \frac{\partial}{\partial r} r_j \langle q | r \rangle \right) Y_{21}(\hat{r}) \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | \mathcal{O}_0^2(r) | p(\frac{1}{2}, -\frac{1}{2}) \rangle
\]

\[
= \sqrt{\frac{5\pi}{3}} \frac{\omega}{|q|} \sum_{\chi} Q_{\chi} \int d^3r \left( \frac{\partial}{\partial r} r_j \langle q | r \rangle \right) Y_{21}(\hat{r}) \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | M_{E2}^X(r) | p(\frac{1}{2}, -\frac{1}{2}) \rangle
\]

where again the general expression of the amplitude in the first line is calculated in the model following the second line and

\[
M_{E2}^X(r) = \int d\Omega Y_{21}(\hat{r}) \mathcal{F}^{0X}(r)
\]

with \( \mathcal{F}^{0X}(r) \) the same as in the scalar case (4.28) and \( G_{ij}(r) \) is given by (4.29). Expressing the anticommutator in (4.28) as in (4.32) leads, as in the scalar amplitude, to a vanishing contribution from the commutator. As to the second term from (4.32), after using the Wigner-Eckart theorem, one finds

\[
\int d\Omega \sum_{G_{i}^3 G_{h}^3} \langle m | r \rangle Y_{21}(\hat{r}) \tau^i \langle r | n \rangle \langle n | \tau^i | m \rangle \left( \sum_{\mu} U_{j\mu} U_{i\mu} - \mu (-1)^{1+\mu} C_{111\mu}^{111\mu} \right)
\]

\[= (-1)^{G_m-G_n} \frac{1}{3} \Delta (G_{m}JG_{m}) \langle m | r \rangle \{ \Sigma_1 \otimes \tau_1 \} J \langle r | n \rangle \langle n | \tau_1 | m \rangle . \]

The product of the quantity

\[
\sum_{\mu} U_{j\mu} U_{i\mu} (-1)^{1+\mu} C_{111\mu}^{111\mu} = \sqrt{\frac{3}{10}} (U_{j-1} U_{i0} + U_{j0} U_{i-1})
\]

with the matrix element of the collective operator results in

\[
\langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | 2D^{(8)}_{3j} J_2 | p(\frac{1}{2}, -\frac{1}{2}) \rangle \sqrt{\frac{3}{10}} (U_{j-1} U_{i0} + U_{j0} U_{i-1}) = \]

\[-\delta \chi^2 \frac{3}{10} \left( \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | D^{(8)}_{3j} | p(\frac{1}{2}, -\frac{1}{2}) \rangle + \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | 2D^{(8)}_{3j} J_1 | p(\frac{1}{2}, -\frac{1}{2}) \rangle \right) = \delta \chi^2 \frac{\sqrt{3}}{5}
\]

with the collective matrix elements from Tab. 4.2.

<table>
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<th>( D^{(8)}_{3j+1} )</th>
<th>( D^{(8)}_{3j+1} )</th>
<th>( D^{(8)}_{3j+1} J_2 )</th>
<th>( D^{(8)}_{3j+1} J_1 )</th>
</tr>
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<td>N(\frac{1}{2}, -\frac{1}{2}) \rangle )</td>
<td>(-\delta \chi^2 \frac{3}{10} )</td>
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<td>(-\delta \chi^2 \frac{3}{10} )</td>
</tr>
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</table>

Table 4.2: Collective matrix elements for the electric and magnetic amplitudes in the N-Δ transition.

The matrix element finally reads

\[
\langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | M_{E2}^X(r) | p(\frac{1}{2}, -\frac{1}{2}) \rangle = \delta \chi^2 \frac{3}{4I_1} \sqrt{\frac{3}{5}} \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | D^{(8)}_{3j+1} | N(\frac{1}{2}, -\frac{1}{2}) \rangle \frac{1}{\sqrt{4\pi}} G^{E2}(r)
\]

with \( G^{E2}(r) = G^{C2}(r) \), eq. (4.38). For the electric quadrupole term the final result is

\[
\mathcal{M}^{E2} = \sqrt{\frac{5\pi}{3}} \frac{\omega}{|q|} \sum_{\chi} Q_{\chi} \int d^3r \left( \frac{\partial}{\partial r} r_j \langle q | r \rangle \right) Y_{21}(\hat{r}) \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | M_{E2}^X(r) | p(\frac{1}{2}, -\frac{1}{2}) \rangle
\]

\[= \sqrt{\frac{5\pi}{3}} \frac{\omega}{|q|} \int d^3r \left( \frac{\partial}{\partial r} r_j \langle q | r \rangle \right) G^{E2}(r) .
\]

In flavor SU(2) the same term is given by

\[
\mathcal{M}^{E2}_{SU(2)} = -\langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | D^{(1)}_{3j} | p(\frac{1}{2}, -\frac{1}{2}) \rangle \frac{1}{16\sqrt{2}} \frac{\omega}{|q|} \int d^3r \left( \frac{\partial}{\partial r} \langle q | r \rangle \right) G^{E2}(r) .
\]
4.3.2 Magnetic matrix element

The definition of the magnetic dipole quantity $M^{M1}$ is

$$
M^{M1} = \frac{i}{\sqrt{2}} \int d^3r \ j_1(|q| r) \left\langle \Delta^+ \left( \frac{\hat{r}}{2} \right) \right| \{\hat{r} \otimes J_1\}_{11} | p(\frac{\hat{r}}{2}) \rangle
$$

$$
\rightarrow \frac{3}{\sqrt{2}} \sum_{\chi} Q_{\chi} \int d^3r \ j_1(|q| r) \left\langle \Delta^+ \left( \frac{\hat{r}}{2} \right) \right| M^{M1}(r) | p(\frac{\hat{r}}{2}) \rangle
$$

(4.49)

with

$$
M^{M1}_M(r) = \int d\Omega \ i \gamma^5 \{\hat{r} \otimes F_{\chi}^M (r)\}_{11}
$$

(4.50)

and from (E.142), which is closely related to the magnetic form factors ($\eta = 1$),

$$
\frac{1}{N_c} i \gamma^5 \{\hat{r} \otimes F_{\chi}^M (r)\}_{11} = D^{(8)}_{\chi}(v | r) i \gamma^5 \{\hat{r} \otimes \sigma_1\}_{11} \tau^j (r | v) \langle r | n \rangle + \frac{1}{2} \sum_{n, m} \mathcal{R}_5(\sigma_n, \sigma_m) \langle m | n \rangle \langle n | \tau^j | m \rangle
$$

(4.51a)

$$
+ \frac{1}{E} \left\{ \frac{J_n}{2L_1}, D^{(8)}_{\chi} \right\} \left( \sum_{n \neq 0} \frac{1}{\nu - \nu_n} \langle v | r \rangle i \gamma^5 \{\hat{r} \otimes \sigma_1\}_{11} \tau^j (r | n) \langle n | \tau^j | v \rangle - \frac{1}{2} \sum_{n, m} \mathcal{R}_5(\sigma_n, \sigma_m) \langle m | n \rangle \langle n | \tau^j | m \rangle \right)
$$

(4.51b)

$$
+ \left\{ \frac{J_n}{2L_2}, D^{(8)}_{\chi} \right\} d_{abi} \left( \sum_{n \neq 0} \frac{1}{\nu - \nu_n} \langle v | r \rangle i \gamma^5 \{\hat{r} \otimes \sigma_1\}_{11} \tau^j (r | n) \langle n | \tau^j | v \rangle - \frac{1}{2} \sum_{n, m} \mathcal{R}_5(\sigma_n, \sigma_m) \langle m | n \rangle \langle n | \tau^j | m \rangle \right)
$$

(4.51c)

To simplify this expression, one should look at the densities and collective matrix elements at the same time. The intermediate results are summarized in the following: The Wigner-Eckart theorem leads in the first term (4.51a) to

$$
\left\langle \Delta^+ \left( \frac{\hat{r}}{2} \right) \right| D^{(8)}_{\chi} | p(\frac{\hat{r}}{2}) \rangle \sum_{G_3} \langle n | r \rangle i \gamma^5 \{\hat{r} \otimes \sigma_1\}_{11} \tau^j (r | n) =
$$

$$
- \left\langle \Delta^+ \left( \frac{\hat{r}}{2} \right) \right| D^{(8)}_{\chi+1} | p(\frac{\hat{r}}{2}) \rangle \frac{1}{\sqrt{2G_n + 1}} \sqrt{2G_n + 1} \langle n | r \rangle i \gamma^5 \{\hat{r} \otimes \sigma_1\}_{11} \tau^j (r | n)
$$

$$
- \delta_{\chi+1} \sqrt{\frac{5}{45}} \langle \Delta^+ \left( \frac{\hat{r}}{2} \right) \right| D^{(8)}_{\chi+1} | p(\frac{\hat{r}}{2}) \rangle \sum_{G_3} \langle n | r \rangle i \gamma^5 \{\hat{r} \otimes \sigma_1\}_{11} \tau^j (r | n)
$$

(4.52)

using \( \{\hat{r} \otimes \sigma_1\}_{11} \tau^j (r | n) = -i(\hat{r} \times \sigma) \cdot \tau / \sqrt{6} \). The second term (4.51b), vanishes because

$$
\left\langle \Delta^+ \left( \frac{\hat{r}}{2} \right) \right| 2D^{(8)}_{\chi} | p(\frac{\hat{r}}{2}) \rangle = 0.
$$

(4.53)

For the third term (4.51c)

$$
\langle \Delta^+ \left( \frac{\hat{r}}{2} \right) \rangle \left\{ J_n, D^{(8)}_{\chi} \right\} d_{abi} | p(\frac{\hat{r}}{2}) \rangle \sum_{G_3} \langle m | r \rangle i \gamma^5 \{\hat{r} \otimes \sigma_1\}_{11} \tau^j (r | n) \langle n | m \rangle =
$$
\[ \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | A_j | p(\frac{1}{2}, -\frac{1}{2}) \rangle = \frac{1}{\sqrt{3}} \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | A_{+1} | p(\frac{1}{2}, -\frac{1}{2}) \rangle \langle m|r \rangle i\gamma^5 \langle \hat{r}_1 \otimes \sigma_1 \rangle_{11} \tau_{1j} \langle r|n^0 \rangle \langle n^0 | m \rangle \]

where the \( A_i \) are defined from

\[ \left\{ J_a, D^{(8)}_{\chi} \right\} d_{abi} = D^{(8)}_{\chi} J_a d_{abi} = A_j \delta_{ij} \]

\[ A_{+1} = -\frac{1}{\sqrt{2}}(A_1 + iA_2) = \frac{1}{2} (D^{(8)}_{\chi_{4-i7}} (J_4 + iJ_5) + D^{(8)}_{\chi_{4+i5}} (J_6 - iJ_7)) \]

For the proton with spin down

\[ \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | A_{+1} | p(\frac{1}{2}, -\frac{1}{2}) \rangle = -\frac{1}{2} \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | D^{(8)}_{\chi_{6-i7}} (J_4 + iJ_5) | p(\frac{1}{2}, -\frac{1}{2}) \rangle \]

\[ + \frac{1}{2} \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | D^{(8)}_{\chi_{4+i5}} (J_6 - iJ_7) | p(\frac{1}{2}, -\frac{1}{2}) \rangle = -\frac{2\sqrt{10}}{15} \delta_{33} \]

leading to

\[ \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | J_a, D^{(8)}_{\chi} \right\} d_{abi} | p(\frac{1}{2}, -\frac{1}{2}) \rangle \sum_{G^p_3, G^0_3} \langle m|r \rangle i\gamma^5 \langle \hat{r}_1 \otimes \sigma_1 \rangle_{11} \tau^i \langle r|n^0 \rangle \langle n^0 | m \rangle = \]

\[ = \frac{1}{\sqrt{3}} \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | A_{+1} | p(\frac{1}{2}, -\frac{1}{2}) \rangle \langle m|r \rangle i\gamma^5 \langle \hat{r}_1 \otimes \sigma_1 \rangle_{11} \tau_{10} \langle r|n^0 \rangle \langle n^0 | m \rangle \]

\[ = \delta_{33} \frac{\sqrt{20}}{40} \sum_{G^p_3, G^0_3} \langle m|r \rangle \gamma^5 (\hat{r} \times \sigma) \cdot \tau \langle r|n^0 \rangle \langle n^0 | m \rangle \]

The fourth term (4.51d)) is treated similarly

\[ \sum_{G^p_3, G^0_3} \left[ J_a, D^{(8)}_{\chi} \right\} \langle m|r \rangle i\gamma^5 \langle \hat{r}_1 \otimes \sigma_1 \rangle_{11} \tau^i \langle r|n \rangle \langle n | r^i | m \rangle = \left( \sum_{\nu} (-\nu) ^{J_{\mu}, D^{(8)}_{\chi_{\nu}}} C^{1I}_{11I} \right) \times (-)^{G_n - G_m} \frac{1}{3} \langle m|r \rangle i\gamma^5 \langle \hat{r}_1 \otimes \sigma_1 \rangle_{11} \tau_1 \langle r|n \rangle \langle n | \tau_1 | m \rangle \]

with

\[ \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | J_a, D^{(8)}_{\chi} \right\} | p(\frac{1}{2}, -\frac{1}{2}) \rangle \sum_{G^p_3, G^0_3} \langle m|r \rangle i\gamma^5 \langle \hat{r}_1 \otimes \sigma_1 \rangle_{11} \tau^i \langle r|n \rangle \langle n | r^i | m \rangle = \]

\[ = \frac{1}{2\sqrt{2}} \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | D^{(8)}_{\chi_{4+i5}} | p(\frac{1}{2}, -\frac{1}{2}) \rangle \sum_{G^p_3, G^0_3} \langle m|r \rangle i\gamma^5 (\hat{r} \times \sigma) \cdot \tau \langle r|n \rangle \cdot \langle n | \tau | m \rangle \]

using \( \{ \hat{r}_1 \otimes \sigma_1 \}_1 \otimes \tau_1 \} = - (\hat{r} \times \sigma) \times \tau / 2 \). The last term (4.51e) is found to vanish on account of the collective matrix elements\(^3\).

The final form for the dipolar magnetic quantity \( \mathcal{M}^{M1} \) is

\[ \mathcal{M}^{M1} = \int d^3r \, j_1(|q|r) G_{N-\Delta}^{M} (r) \]

with the combined density \( G_{N-\Delta}^{M} (r) \) given by

\[ G_{N-\Delta}^{M} (r) = \langle \Delta^+(\frac{3}{2}, \frac{1}{2}) | y^i \rangle \langle \hat{r}_1 \otimes f_{\eta=1}^{11} (r) \rangle_{11} \langle p(\frac{1}{2}, -\frac{1}{2}) \rangle \]

\[ = \frac{\sqrt{5}}{15} G_{1}^{M} (r) + \frac{\sqrt{5}}{15} G_{2}^{M} (r) + \frac{\sqrt{5}}{20 I_1} \tilde{G}_{3}^{M} (r) \].

\(^3\)Both the terms (4.27e) and (4.51e) come from the quantization offending terms (E.137c) and (E.142e), respectively. In the \( N-\Delta \) transition they both vanish.
The regularization functions \( R \) in the model is shown in Fig. 4.3 [201] as function of one corresponding to the photoproduction of the \( \Delta \) resonance, i.e. \( Q \) corrections were not taken into account. It is thus not clear if the origin for the underestimation of the nucleon-\( \Delta \) transition dashed line. The experimental data can be traced from [201].

The ratio between the electric quadrupole and magnetic dipole amplitudes, \( M_1 \) and \( C_2/M_1 \), in the chiral quark-soliton model for a constituent quark mass of 420 MeV: SU(3), solid line, and SU(2), dashed line. The experimental data can be traced from [201].

The expressions for the densities are

\[
\frac{1}{N_c}g_1^M (r) = \langle v | r \rangle \gamma^5 (\hat{r} \times \sigma) \cdot \tau \langle r | v \rangle + \sum_n \langle n | r \rangle \gamma^5 (\hat{r} \times \sigma) \cdot \tau \langle r | n \rangle R_1 (\varepsilon_n), \tag{4.63a}
\]

\[
\frac{1}{N_c}g_2^M (r) = \sum_{n^0} \frac{1}{\varepsilon_n - \varepsilon_{n^0}} \langle v | r \rangle \gamma^5 (\hat{r} \times \sigma) \cdot \tau \langle r | n^0 \rangle \langle n^0 | v \rangle - \sum_{m, n^0} R_5 (\varepsilon_m, \varepsilon_{n^0}) \langle m | r \rangle \gamma^5 (\hat{r} \times \sigma) \cdot \tau \langle r | n^0 \rangle \langle n^0 | m \rangle, \tag{4.63b}
\]

\[
\frac{1}{N_c}g_3^M (r) = \sum_{n \neq 0} \frac{\text{sgn} (\varepsilon_n)}{\varepsilon_n - \varepsilon_N} \langle v | r \rangle i \gamma^5 (\hat{r} \times \sigma) \times \tau \langle r | n \rangle \langle n | \tau | v \rangle + \frac{1}{2} \sum_{n, m} R_4 (\varepsilon_n, \varepsilon_m) \langle m | r \rangle i \gamma^5 (\hat{x} \times \sigma) \times \tau \langle r | n \rangle \langle n | \tau | m \rangle \tag{4.63c}
\]

The regularization functions \( R_1 (\varepsilon_n) \), \( R_4 (\varepsilon_n, \varepsilon_m) \) are given in (E.70) and (E.114) and the function \( R_5 (\varepsilon_m, \varepsilon_{n^0}) \) is defined in (E.123).

The previous calculation allows an easy extraction of the SU(2) flavor result, which reads

\[
M_{SU(2)}^M (|q|) = \langle \Delta^+ (\frac{3}{2}, \frac{1}{2}) | D_{33}^{(1)} | p (\frac{3}{2}, -\frac{1}{2}) \rangle \int d^3r \, j_1 (|q| r) \left( -\frac{\sqrt{3}}{2} g_1^M (r) + \frac{\sqrt{3}}{8} g_3^M (r) \right). \tag{4.64}
\]

4.4 Results for the ratios \( E2/M1 \) and \( C2/M1 \)

The CQSM results for the dipole magnetic amplitude \( M_1 \) as a function of \( Q^2 \) are presented in Fig. 4.2 for the cases of flavor SU(2) (4.64) and flavor-symmetric SU(3) (4.61). It is clear from Fig. 4.2 that the model results underestimate the experimental values. This is in agreement with what was observed with the magnetic moments in the previous chapter. Here, however, mass corrections were not taken into account. It is thus not clear if the origin for the underestimation of \( M_1 \) in the model has the same origin as that of the magnetic moments.

![Figure 4.2: The magnetic dipole amplitude \( M_1 = -[3A_{3/2} + \sqrt{3}A_{1/2}]/(2\sqrt{3}) \), in units of \( 10^{-3} \text{ GeV}^{-1/2} \), in the chiral quark-soliton model for a constituent quark mass of 420 MeV: SU(3), solid line, and SU(2), dashed line. The experimental data can be traced from [201].](image)

The ratio between the electric quadrupole and magnetic dipole amplitudes, \( E2/M1 \), obtained in the model is shown in Fig. 4.3 [201] as function of \( Q^2 \) and compared with the experimental data. Emphasis is given to the most recent experimental results. An important \( Q^2 \) point is the one corresponding to the photoproduction of the \( \Delta \) resonance, i.e. \( Q^2 = 0 \). The value obtained
is $-2.1\ %$ in flavor SU(2) and $-1.4\ %$ in flavor symmetric SU(3), meaning the existence of a spatial deformation in the $N$-$\Delta$ transition. It is not easy, however, to ascribe the deformation to the nucleon, the delta or the transition density. The model result is smaller then the value $-2.5 \pm 0.5\ %$ quoted by the Particle Data Group [124], although it falls within the experimental errors. For a review on the current theoretical and experimental status of photoproduction of the $\Delta$ see [209]. There are nevertheless other analysis that allow for higher values for this ratio, up to $-3.5\ %$ [210]. The result quoted in [124] is based on the existing experimental data for the photoproduction of the $\Delta$ resonance, and seems presently well established. For finite $Q^2$ the ratio $E2/M1$ increases with $Q^2$ reaching almost zero, for $Q^2 \geq 0.6\ GeV^2$. This behaviour is qualitatively in agreement with the most recent experimental data [192,193]: the sign is the same and the dependence with $Q^2$ is qualitatively similar. The fact that the ratio reaches 0 for $Q^2 \geq 0.6\ GeV^2$ does not have the slope as found in [192]. However, data from [192] does not completely agree with other data taken at higher $Q^2$ which seems to indicate [193,211,212] that the ratio $E2/M1$ only reaches 0 at values of $Q^2$ closer to, or higher than, 4 $GeV^2$. In this case, although the model prediction for $Q^2 = 0$ is reasonably well reproduced, the $Q^2$ behaviour of the ratio is not. Also the reason for the model result to approach 0 is not easy to identify, particularly why it occurs at low $Q^2$.

One should recall, in this respect, that the model predictions are as more reliable as the values of $Q^2$ are smaller then the nucleon mass. There are also some pertinent comments regarding the comparison between the model results and the experimental values. In the case of the model calculation the formalism used corresponds to a stable $\Delta$: the $\Delta$ decay is of higher order in $1/N_c$ and to incorporate it would imply a deep change in the formalism. Although one can get a correct qualitatively description, which is important in terms of testing the model, one loses the advantage of studying the experimental data at a much lower level, where some models must be used to guide the study of the data. This assumes even more relevance since the analysis of the experimental data is very difficult even apart from this model dependence: there is still ongoing debate about the analysis of the data on which recent results are based.

![Figure 4.3: The E2/M1 ratio in the chiral quark-soliton model for a constituent quark mass of 420 MeV : SU(3), solid line, and SU(2), dashed line. The references for the data with the open symbols can be traced from [201]. Solid symbols represent recent data: * [124], □ [192], ○ [193].](image)

In what regards the comparison for $E2/M1$ between these results with other model calculations, pursued in a more extensive manner in Ref. [201], one concludes that the results obtained in the CQSM lie between the values predicted by constituent quark models and the Skyrme model. This is not a surprise since, as was mentioned in Chapter 2, the CQSM is expected to interpolate between those kinds of models. Values for $E2/M1$ in quark models range from $-0.2\ %$, [213] in a
non-relativistic approach, up to $-3.5\%$, [198] taking into account exchange currents. A modified version of the Skyrme model [214] yielded an $E2/M1$ ratio between $-2.6\%$ and $-4.9\%$ while, with the inclusion of deformations due to the rotation of the soliton and recoil corrections, the Skyrme model ratio was $-2.6\%$ [215]. The results here presented are similar in general behaviour to those obtained in the context of the linear sigma model with quarks and the chromodielectric model [207,208], which, however, achieve better general quantitative agreement with the data.

Fig. 4.4 presents the ratio $C2/M1$ as a function of $Q^2$, compared with the experimental data. The model prediction underestimates the most recent experimental results, specially for values of $Q^2$ above $Q^2 = 0.3$. The general behaviour is nevertheless in agreement with the phenomenological one [193]. It is negative and decreases with $Q^2$, almost linearly. Taking into account the fact that the ability to predict small quadrupole components in the nucleon-$\Delta$ transition with the right signs is considered to be a criterion that a good model of the nucleon must meet, it is an important point that the CQSM is able to achieve this prediction. A recent experimental value for $C2/M1$ near $Q^2 = 0$ is $-6.4 \pm 1.5\%$ [104] which is exceeds the model results that fall below 5%. Other models achieve a better agreement for this ratio [207,215].

![Figure 4.4](image)

Figure 4.4: The $C2/M1$ ratio in the chiral quark-soliton model for a constituent quark mass of 420 MeV: SU(3), solid line, and SU(2), dashed line. The references for the data with the open symbols can be traced from [201]. Solid symbols represent recent data: ■ [192], • [194], ▲ [195], ♦ [193].

The model calculation of this chapter is found thus not able to provide information on one of the interesting points of these ratios, namely that of the transition to perturbative QCD. Work in perturbative QCD power counting [216] predicts that, in the limit $Q^2 \to \infty$ and up to logarithmic QCD corrections, $E2/M1 \to 100\%$ and $C2/M1 \to$ constant. In a recent work [217], a lower limit for this constant is found to be $-20\%$ at $Q^2$ of order 10 GeV$^2$. Regarding the ratio $E2/M1$ matching the experimental at the photon point and the perturbative QCD results implies a crossover of the ratio. Experimentally this is not observed up to $Q^2$ of order 4 GeV$^2$.

The conclusions regarding the CQSM results are the the ratios are both negative in the studied $Q^2$ range, in agreement with experiment. The model results are also similar to experiment regarding the evolution in $Q^2$. The absolute values underestimate, however, the phenomenological ones. Within the model it is found that the SU(2) result is closer to the data than the SU(3) result. This seems to indicate that the SU(3) should include the mass corrections.
5 Axial and neutral weak form factors

The description of the axial coupling constant of the nucleon $g_A$ is probably the next most important quantity a model of baryon structure must reproduce after the magnetic moment (and perhaps the charge radius), since spin and charge are usually fixed from the framework. For a review on the present understanding and experimental status of the axial structure of the nucleon see [218]. The emphasis on $g_A$ is evident from the attention given to it in the context of the chiral quark-soliton model (CQSM). Early work in the Nambu–Jona-Lasinio model (NJLM) [219] and the Skyrme model (SM) [148] pointed towards a very small $g_A$ when compared to experiment. The inclusion of sea quarks [220] and rotational corrections [221–223], later scrutinized in [224, 225], helped to diminish the difference to the experimental value. The flavor SU(2) axial form factors and related static quantities were worked out in the CQSM in [226].

The flavor SU(3) version of the CQSM was first applied to the axial coupling constants in [227,228]. The present work extends the analysis to the full axial form factors up to 1 GeV$^2$, taking into account for the first time the symmetry conserving quantization prescription and strange mass corrections.

A similar treatment to the case of the electromagnetic form factors of Chapter 3 could also be made in the case of the axial form factors. The densities are the same for the all baryon octet and the collective matrix elements are the same as in the electromagnetic case. This would lead to a lot more information than the available experimental data, which again privileges the nucleon. In this work we will thus restrict the shown results to the nucleon case for briefness.

The chapter includes the form factors for the weak neutral current, since they are related to both the vector and axial-vector currents. They allow, in particular, to access the parity-violating in electron proton scattering, which is the quantity actually measured in the study of the matrix elements of the strange vector currents in the nucleon.  

5.1 Axial-vector currents and their form factors

The general decomposition of the matrix element of an axial-vector current based on Lorentz covariance, time reversal and hermiticity, Section C.5 eq. C.72, is given by

$$ \langle B(p', s')| A^{\alpha \chi}(0) | B(p, s) \rangle = \bar{u}(p', S_3') \left( G_A^{(\chi)}(q^2) \gamma_\mu \gamma_5 + \frac{G_P^{(\chi)}(q^2)}{2 M} \gamma_\mu \right) u(p, S_3). $$  \hspace{1cm} (5.1)

where the $G_A^{(\chi)}$ are the axial form factors and $G_P^{(\chi)}$ the induced pseudoscalar form factors pertaining to the axial-vector current $\chi$. In this work only the axial form factors will be considered.

The case $\chi = 3$ is by far the best studied case since the axial charge $g_A \equiv g_3 = G_3(Q^2 = 0)$ is well known from neutron beta decay, $g_A = 1.2673 \pm 0.0035$ [124]. The methods to determine the axial form factor are twofold. One is (anti)neutrino scattering off protons or nuclei and the other is charged pion electroproduction. The induced pseudoscalar axial form factor is experimentally accessible in ordinary and radiative muon capture and pion electroproduction. The experimental data covers in both cases several $Q^2$ values. This is not the case with the singlet and octet form factors, which are less known.

Indeed, while $A^3$ and $A^8$ are flavor nonsinglet axial-vector currents, $A^0$ is flavor singlet. As consequence, the first two axial currents, $g^3$ and $g^8$, may be determined from neutron and hyperon semileptonic decays while $g^0$ cannot, since the singlet axial current plays no role in the weak interactions. In this way, one may access $g^8$ by hyperon semileptonic decays constants $F$ and $D$, albeit hampered by the use of exact SU(3) flavor symmetry in extracting these constants.
In the case of the singlet axial current on has to rely on the determination of the nucleon structure functions to access $g^0 = G^0(Q^2 = 0)$. In the naive quark model $a_0 = g^0$ is related to the fraction of the proton spin carried by quarks. The experimental finding [229] of a small value for $a_0$ and thus for an indication that the contribution of the quarks to the proton spin was small was termed “spin crisis” and prompted several studies on the axial singlet current. The experimental details did not yet lead to the knowledge of the singlet form factor.

### 5.2 Proton axial form factors

The present determination of the axial form factors is done on the Breit frame, where the time-components of the axial currents vanish and one extracts therefore the axial form factors from the spatial part of the axial currents. This method is also dictated by large $N_c$ arguments, namely the suppression in $N_c$ of the time component with respect to the spatial ones. The expression for the axial form factor, as shown in Section C.5, is, from (C.85),

$$G_A^{(\chi,B)}(q^2) = \frac{M}{E} \int d^3x \left( j_0(|q| r) \langle B(S_3)| \mathcal{A}_{(\chi)}(x) \rangle \right)$$

$$- \sqrt{2\pi j_2}(|q| r) \langle N(S_3)| \{Y_2(\hat{r}) \otimes A_{1}^{(\chi)}(x)\} \rangle_{10} \langle B(S_3) \rangle$$

where $B$ is the baryon state. Here the only baryon state is restricted to the nucleon case although the expression is valid for hyperons. The radial density functions are $(\chi = 3, 8)$

$$A_0^{(\chi,B)}(r) = \frac{1}{\sqrt{2\pi}} \int dr \left( j_0(|q| r) A_0^{(\chi,B)}(r) \right)$$

$$A_2^{(\chi,B)}(r) = \frac{1}{\sqrt{2\pi}} \int dr \left( j_0(|q| r) A_2^{(\chi,B)}(r) \right)$$

The matrix elements in (5.3) are calculated in the CQSM up to linear order in the angular velocity and strange mass corrections, exactly as in the case of the electromagnetic form factors, from the results (E.52) and (E.136) considering the case $\eta = 1$, $\mu = 3$, and $\chi = 3, 8$, combined as in Section E.4.3.

In the triplet and octet cases $(\chi = 3, 8)$, for $A_0^{(\chi,B)}(r)$ one finds\(^1\) that

$$\langle B(S_3)| \mathcal{A}_{(\chi)}(z) \rangle \langle B(S_3) \rangle =$$

$$\frac{3}{2} \left( D_{\chi}^{(8)} \right)_{B} A_0(z) + \frac{1}{6\sqrt{3}l_1} \left( \langle J_0 D_{\chi}^{(8)} \rangle_{B} - 2K_1 M_8 \langle D_{\chi}^{(8)} D_{\chi}^{(8)} \rangle_{B} \right) B_0(z)$$

$$+ \frac{1}{3l_2} \left( \langle d_{ab} D_{\chi}^{(8)} \rangle_{B} - 2K_2 M_8 \langle D_{ab}^{(8)} D_{ab}^{(8)} \rangle_{B} \right) C_0(z) - \frac{i}{6l_1} \langle D_{\chi}^{(8)} \rangle_{B} D_0(z)$$

$$+ \frac{2}{3} \left( M_1 \langle D_{\chi}^{(8)} \rangle_{B} + \frac{1}{\sqrt{3}} M_8 \langle D_{\chi}^{(8)} D_{\chi}^{(8)} \rangle_{B} \right) \mathcal{H}_0(z)$$

$$+ \frac{2}{3\sqrt{3}} \left( M_8 \langle D_{\chi}^{(8)} D_{\chi}^{(8)} \rangle_{B} \right) \mathcal{I}_0(z) + \frac{2}{3} M_8 \langle D_{ab}^{(8)} D_{ab}^{(8)} \rangle_{B} \mathcal{J}_0(z)$$

(5.4)

with the densities\(^2\) given by:

$$\frac{1}{N_c} A_0(z) = \langle \nu | \pi \cdot \tau(z) | \nu \rangle + \sum_n R_1(\varepsilon_n) \langle n | z \rangle \pi \cdot \tau(z) | n \rangle$$

$$\frac{1}{N_c} B_0(z) = \sum_{m \neq 0} \frac{1}{\sqrt{5 - \varepsilon_m}} \langle \nu | \pi \cdot \tau(z) | m \rangle - \frac{1}{2} \sum_{n,m} R_2(\varepsilon_n, \varepsilon_m) \langle n | z \rangle \pi \cdot \tau(z) | m \rangle$$

$$\frac{1}{N_c} C_0(z) = \sum_{m \neq 0} \frac{1}{\sqrt{5 - \varepsilon_m}} \langle \nu | m \rangle \pi \cdot \tau(z) | m \rangle - \sum_{n,m} R_3(\varepsilon_n, \varepsilon_m) \langle n | z \rangle \pi \cdot \tau(z) | m \rangle

(5.5a)

(5.5b)

(5.5c)

---

\(^1\)The notation is $\langle D_{\chi}^{(8)} \rangle_{B} = \langle B(S_3)| D_{\chi}^{(8)} | B(S_3) \rangle$

\(^2\)Note the absence of the density (E.144e) due to the symmetry conserving quantization.
5.2 Proton axial form factors

\[ \frac{1}{N_e} D_0(z) = \sum_n \sum_{\mathbf{E}, \mathbf{E}'} \langle n | \mathbf{P} \rangle \langle \mathbf{P}| n \rangle \sigma \cdot \tau \langle \mathbf{P} | \mathbf{V}| \mathbf{P} \rangle + \frac{1}{2} \sum_{m,m'} \mathcal{R}_4(\mathbf{x}, \mathbf{y}) \langle m| \mathbf{P} \rangle \langle \mathbf{P}| m \rangle \sigma \cdot \tau \langle \mathbf{P} | \mathbf{V}| \mathbf{P} \rangle \]  \hspace{1cm} (5.5d)

\[ \frac{1}{N_e} \mathcal{H}_0(z) = \sum_n \sum_{\mathbf{E}, \mathbf{E}'} \langle \mathbf{V} | \mathbf{P} \rangle \langle \mathbf{P}| n \rangle \gamma^0 \langle \mathbf{V}| \mathbf{P} \rangle + \frac{1}{2} \sum_{m,m'} \mathcal{R}_2(\mathbf{x}, \mathbf{y}) \langle m| \mathbf{P} \rangle \langle \mathbf{P}| m \rangle \sigma \cdot \tau \langle \mathbf{P} | \mathbf{V}| \mathbf{P} \rangle \]  \hspace{1cm} (5.5e)

\[ \frac{1}{N_e} I_0(z) = \sum_n \sum_{\mathbf{E}, \mathbf{E}'} \langle \mathbf{V} | \mathbf{P} \rangle \langle \mathbf{P}| n \rangle \gamma^0 \langle \mathbf{V}| \mathbf{P} \rangle + \frac{1}{2} \sum_{m,m'} \mathcal{R}_2(\mathbf{x}, \mathbf{y}) \langle m| \mathbf{P} \rangle \langle \mathbf{P}| m \rangle \sigma \cdot \tau \langle \mathbf{P} | \mathbf{V}| \mathbf{P} \rangle \]  \hspace{1cm} (5.5f)

\[ \frac{1}{N_e} J_0(z) = \sum_n \sum_{\mathbf{E}, \mathbf{E}'} \langle \mathbf{V} | \mathbf{P} \rangle \langle \mathbf{P}| n \rangle \gamma^0 \langle \mathbf{V}| \mathbf{P} \rangle + \sum_{m,m'} \mathcal{R}_2(\mathbf{x}, \mathbf{y}) \langle m| \mathbf{P} \rangle \langle \mathbf{P}| m \rangle \sigma \cdot \tau \langle \mathbf{P} | \mathbf{V}| \mathbf{P} \rangle \]  \hspace{1cm} (5.5g)

The matrix element \( \langle B(S_3) | \{ Y_2(r) \times A_1^3(x) \} | B(S_3) \rangle \) entering the function \( A_2^{X+B}(r) \) has an identical expression to the matrix element in \( A_0^{X+B}(r) \), eq. 5.4, except for the change in the subscripts \( 0 \to 2 \). This change is accompanied with just the change of the operators in the first matrix element in the densities. The replacement of operators from densities 0 to densities 2 are shown in Tab. 5.1. The densities are otherwise the same.

<table>
<thead>
<tr>
<th>Density</th>
<th>Operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_0, I_0 )</td>
<td>( \sigma )</td>
</tr>
<tr>
<td>( B_2, I_2 )</td>
<td>( \sqrt{4\pi} Y_2 \otimes \sigma_1 )</td>
</tr>
<tr>
<td>( A_0, C_0, H_0, J_0 )</td>
<td>( \sigma \cdot \tau )</td>
</tr>
<tr>
<td>( A_2, C_2, H_2, J_2 )</td>
<td>( \frac{1}{2} { \sqrt{4\pi} Y_2 \otimes \sigma_1 } \otimes \tau_1 )</td>
</tr>
<tr>
<td>( D_0 )</td>
<td>( \sigma \times \tau )</td>
</tr>
<tr>
<td>( D_2 )</td>
<td>( -i\sqrt{2} { \sqrt{4\pi} Y_2 \otimes \sigma_1 } \otimes \tau_1 )</td>
</tr>
</tbody>
</table>

Table 5.1: Operator replacements to obtain \( A_2^{X+B}(r) \). The factor \(-\sqrt{3}\) comes from \( \sigma \cdot \tau = -\sqrt{3} \{ \sigma_1 \otimes \tau_1 \} \) and the factor \(-i\sqrt{2}\) from \( \sigma \times \tau = -i\sqrt{2} \{ \sigma_1 \otimes \tau_1 \} \).

The matrix element for the singlet form factor can be extracted from this result as explained in Section E.2.1. It is given by

\[ \langle B(S_3) | A^{X=0}(x) | B(S_3) \rangle = \frac{2}{M_1} \left( \langle J_3 \rangle - 2K_1 M_8 \langle D_8^{(8)} \rangle \right) B_0(z) + \frac{2}{3} M_8 \langle D_8^{(8)} \rangle I_0(z) \]  \hspace{1cm} (5.6)

Again the matrix element \( \langle B(S_3) | \{ Y_2(r) \times A_1^{X=0}(x) \} | B(S_3) \rangle \) has the same expression, except for the operator replacements in Tab. 5.1.

In flavor SU(2) one finds, from the SU(3) expressions,

\[ A_0^{X=0}(r) = \frac{1}{3M_1} \langle J_3 \rangle B_0(r) \]  \hspace{1cm} (5.7a)

\[ A_0^{X=3}(r) = -\frac{1}{\sqrt{3}} \langle D_3^{(3)} \rangle A_0(r) - \frac{1}{3\sqrt{2} M_1} \langle D_3^{(3)} \rangle D_0(r) \]  \hspace{1cm} (5.7b)

where the operator replacements in Tab. 5.1 still apply in order to get the functions \( A_2(r) \).

The results and experimental data are presented in Fig. 5.1 for the axial form factor of the nucleon \((X = 3)\) and in Fig. 5.2 for the singlet and octet form factors. We chose to present both the results for the pion and kaon asymptotics since it makes thus possible to see the importance of these effect, Section 2.6.2, on the axial form factors, even though it is not applicable in this case an analysis in the spirit of (3.58). In Fig. 5.1 the experimental information is summarized by the dipole formula

\[ G_A(Q^2) = \frac{g_A}{1 + Q^2/M_A^2} \]  \hspace{1cm} (5.8)

using the experimental value \( g_A = 1.2673 \pm 0.0035 \) from \( \beta \)-decay experiments [124]. For the axial mass \( M_A \), the circles correspond to the world average, \( M_A = 1.032 \pm 0.036 \text{ GeV} \), from neutrino (antineutrino) scattering experiments on protons and nuclei [230]; the triangles correspond to the world average in pion electroproduction, \( M_A = 1.069 \pm 0.016 \text{ GeV} \); and the squares to the value
observed at the mass parameter in the dipole behaviour. The axial radii are defined as in the electromagnetic value falls between the model results for pion and kaon asymptotics. The pion asymptotics result 

\[ M_A = 1.077 \pm 0.039 \]

obtained in charged pion electroproduction [231].

\[ \langle r^2 \rangle_A = -6 \frac{1}{G^{(A)}_A(0)} \left( \frac{dG_A(Q^2)}{dQ^2} \right)_{Q^2=0} \]  

(5.9)

The values for the radii and the dipole mass \( M_A \) are listed in Tab. 5.2. The dipole masses are obtained by inserting the dipole formula (5.8) in the expression for the radius (5.9) with the result

\[ M_A^2 = 12/\langle r^2 \rangle_A \]  

(5.10)

One of the main aspects of the results in Fig. 5.1 is the underestimation of the axial charge by 7 % of the experimental value. Furthermore, the difference between the model results and the experimental dipole curves is almost independent of \( Q^2 \), i.e. the difference stays close to the value observed at \( Q^2 = 0 \). This is clearly seen by comparing the dipole masses. The experimental dipole mass falls between the values for the pion and kaon asymptotics, with the pion asymptotics being equal to 87 % of the experimental value. Concerning the axial radius, again the experimental value falls between the model results for pion and kaon asymptotics. The pion asymptotics result

Figure 5.1: The axial form factor of the proton \( G_A \) up to 1 GeV\(^2\). The constituent quark mass is 420 MeV and the strange quark mass 180 MeV in the shown flavor SU(3) results. The symbols follow the dipole formula (5.8) for experimental values of the axial mass \( M_A \) (see text). The two values of \( \mu \) correspond to pion and kaon asymptotics, Section 2.6.2.

Figure 5.2: Singlet and octet axial form factors. Model parameters as in Fig. 5.1.
5.2 Proton axial form factors

<table>
<thead>
<tr>
<th>SU(2)</th>
<th>SU(3)</th>
<th>Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_A^B$</td>
<td>$g_A^{SU(2)}$</td>
<td>$g_A^{SU(3)}$</td>
</tr>
<tr>
<td>0.358</td>
<td>0.367</td>
<td>0.497</td>
</tr>
<tr>
<td>$\langle r_f^2 \rangle^{1/2}$</td>
<td>0.662</td>
<td>0.844</td>
</tr>
<tr>
<td>$M_F^0$</td>
<td>1.033</td>
<td>0.810</td>
</tr>
<tr>
<td>$g_A^B$</td>
<td>1.196</td>
<td>1.176</td>
</tr>
<tr>
<td>$\langle r_f^2 \rangle^{1/2}$</td>
<td>0.666</td>
<td>0.732</td>
</tr>
<tr>
<td>$M_F^0$</td>
<td>1.026</td>
<td>0.934</td>
</tr>
<tr>
<td>$g_A^B$</td>
<td>–</td>
<td>0.360</td>
</tr>
<tr>
<td>$\langle r_f^8 \rangle^{1/2}$</td>
<td>–</td>
<td>0.739</td>
</tr>
<tr>
<td>$M_F^8$</td>
<td>–</td>
<td>0.926</td>
</tr>
</tbody>
</table>

Table 5.2: Axial constants $g_A^B = G_A^B (Q^2 = 0)$, axial radii (in fm) (eq. (5.9) and dipole masses (in GeV) from the dipole expression. Model parameters are $M = 420$ MeV and $m_s = 180$ MeV.

overestimates the experimental radius by 15%, similar to the underestimation of the dipole mass. Comparing the model is the two flavor groups, the differences between SU(2)$_f$ and SU(3)$_f$ are very small. The fact that the SU(3)$_f$ for the kaon asymptotics (see Section 2.6) nicely describes the data is difficult to interpret and at a first looks mostly as a coincidence. One should nevertheless note that the underestimation of $g_A$ is not as large as in many model calculations.

In the case of the octet component of the axial currents, the available data is restricted to $Q^2 = 0$. The experimental results for $G^B (0)$ come from semileptonic decays (see next section) of the hyperons. Using fits to the experimental semileptonic decay data it is found that $g_8 = G^B (0) = 0.338 \pm 0.015 [232]$, which is smaller than any CQSM result, by 7% in the pion asymptotics case.

As for the singlet axial component, $a_0 = G^B (Q^2 = 0) \equiv \Delta \Sigma$, it is determined from deep inelastic scattering (DIS) due to the fact that this current plays no role in the weak interactions. For the sake of the discussion one may find useful to review the procedure which leads to the value for $a_0$. In DIS the pertinent measured quantity is the polarized structure function $g_1(x)$, where $x$ is the Bjorken variable$^3$. In terms of the parton model, the expression for $g_1(x)$ is, in the baryon state $B$,

$$g_1^B(x) = \frac{1}{2} \sum_j Q_{Bj}^2 (\Delta f_j(x) - \Delta \tilde{f}_j(x))$$  \hspace{1cm} (5.11a)

$$\Delta f_j(x) = f_j^B(x) - f_j^\perp(x)$$  \hspace{1cm} (5.11b)

where $\Delta f_j(x)$ represents the difference between the distributions $f_j(x)$ of quarks of flavor $j = u, d, s, \ldots$, polarized along the nucleon spin ($\uparrow$) and opposite ($\downarrow$). The first moment of the distribution function $g_1(x)$ is defined, for proton and neutron, by

$$\Gamma_1^B = \int_0^1 dx g_1^B(x).$$  \hspace{1cm} (5.12)

In the case of the quark parton model, which corresponds to the limit free quark fields, the moments of $g_1(x)$ for the nucleon are related to the axial constants according to

$$\Gamma^{p,n} = \frac{1}{12} \left( \pm a_3 + \frac{1}{\sqrt{3}} a_8 + \frac{4}{3} a_0 \right)$$  \hspace{1cm} (5.13)

where the $+$ sign refers to the proton case.

It is then a matter to obtain the first moment $\Gamma_1^{p,n}$ in DIS in order to be able to combine it with the knowledge of $a_3$ and $a_8$ to extract $a_0$. Experimentally, it has been found [233], at a

$^3$The dependence in $x$, although essential as part of the definition, will not be important for the present discussion for the quantities of interest correspond to integrals over $x$.  

[The rest of the text continues with further discussion and equations related to the topic.]
renormalization scale of $Q^2 = 5$ GeV$^2$, that
\[ \Delta \Sigma = 0.23 \pm 0.04 \text{(stat)} \pm 0.06 \text{(syst)}, \tag{5.14} \]
with $\Delta \Sigma$ standing for the quark contribution to the nucleon spin. The importance of $a_0$ comes from its connection with $\Delta \Sigma$. In the parton model, $a_0$ may be written as
\[ a_0 = \int_0^1 dx \left( \Delta u(x) + \Delta \bar{u}(x) + \Delta d(x) + \Delta \bar{d}(x) + \Delta s(x) + \Delta \bar{s}(x) \right) \]
\[ = \Delta u + \Delta \bar{u} + \Delta d + \Delta \bar{d} + \Delta s + \Delta \bar{s}. \tag{5.15} \]
Taking into consideration that, in the parton model, the contribution $S_j^a$ of the quark with flavor $j$ to the spin is given (similarly for antiquarks) by
\[ S_j^a = \frac{1}{2} \int_0^1 dx \left( f_j^a - f_j^{ar{a}} \right) = \frac{1}{2} \int_0^1 dx \Delta f_j(x), \tag{5.16} \]
the total contribution of the quarks to the nucleon spin $S_{\text{quarks}}^z$ is simply
\[ S_{\text{quarks}}^z = \Delta \Sigma = \frac{a_0}{2}. \tag{5.17} \]
Combining this relation with the experimental result (5.14) results in a very small contribution of quarks to the nucleon spin, a situation which was termed “spin crisis”. The “crisis” was due to the fact that it deviated much from the expected value of the simple constituent quark model, according to which the quarks should carry all the spin of the nucleon, i.e. \( a_0 = 1 \). The identification of \( a_0 \) with $\Delta \Sigma$ is modified when going beyond the parton model. The relation between these quantities becomes
\[ \Delta \Sigma = a_0 + \frac{1}{3} \frac{\alpha_s}{2\pi} \Delta g \tag{5.18} \]
with $\Delta g$ the spin contribution of the gluons and $\alpha_s$ the strong coupling constant. This means that a small $a_0$ does not mean necessarily a small $\delta \Sigma$. In spite of the fact that $\Delta g$ is not very well known, it is accepted that the contribution of the quarks to the total nucleon spin is nevertheless small. In the case of the model, $a_0$ and $\Delta \Sigma$ indeed coincide for there are no explicit gluons in the model. This might account for the larger value of the model $a_0$ as compared to the experimental value.

### 5.3 Semileptonic decays of hyperons

The weak current also induces semileptonic decays of hyperons
\[ B_i \rightarrow B_j \ell^- \bar{\nu}_\ell, \quad \text{or} \quad B_i \rightarrow B_j \ell^+ \nu_\ell \tag{5.19} \]
where $B_i$ and $B_j$ are baryon states differing by one unit of hypercharge, e.g. $\Lambda \rightarrow p$, $\Xi \rightarrow \Lambda$. SU(3) flavor symmetry is violated by the different baryon masses. However, since the masses in the octet differ by at most 10 %, one may expect that symmetry breaking in hyperon semileptonic decays is small. Considering the matrix element of the axial current (which is part of the weak current) between baryon states $B^\beta$, $B^\gamma$ ($\beta, \gamma = 1, \cdots, 8$), one has, on the grounds of Lorentz invariance (see Section C.2), that
\[ \langle B^\beta(p') | A^{\mu \alpha} | B^\gamma(p) \rangle = \overline{B}^{\beta}(p') \left( g^\alpha_{\beta \gamma} \gamma^\mu g^5 + g^\alpha_{\gamma \beta} g^\mu g^5 + g^\alpha_{\beta \gamma} P^\mu g^5 \right) B^\gamma(p) \quad (\text{no sums}) \tag{5.20} \]
The $g^\alpha_{\beta \gamma}$ are transition form factors. In terms of the general invariance of the matrix element under time reversal and charge symmetry (i.e. along the same lines as in Section C.2), one may show that the form factors are real and that $g_3 = 0$.

In the case of SU(3) symmetry, the octet of axial currents and the field operators $B^\beta$ transform according to, respectively,
\[ [F^\alpha, A^\beta] = i f^{\alpha \beta \gamma} A^\gamma, \tag{5.21a} \]
\[ F^\alpha |B^\beta\rangle = i f^{\alpha \beta \gamma} |B^\gamma\rangle. \tag{5.21b} \]
Transformations (5.21) constrain the matrix elements of the axial currents between baryon states. Indeed, using (5.21a) one has
\[ \langle B^\beta|A^\alpha(0)|B^\gamma \rangle - \langle B^\beta|A^\alpha F^\delta(0)|B^\gamma \rangle = i f^{\delta\alpha\beta} \langle B^\beta|A^\alpha|B^\gamma \rangle \]
Taking now (5.21b) and imposing the preceding condition upon the form factors, instead of the full matrix elements, leads to
\[ f^{\delta\alpha\beta} g_i^{\delta\alpha\gamma} + f^{\delta\beta\gamma} g_i^{\delta\alpha\gamma} + f^{\delta\alpha\beta} g_i^{\delta\beta\gamma} = 0, \quad (i = 1, 2, 3). \]

Using the Jacoby identities and the group SU(3) structure constants, Tab. A.1,
\[
\begin{align*}
\{[\lambda_\alpha, \lambda_\beta], \lambda_\gamma \} + \{[\lambda_\beta, \lambda_\gamma], \lambda_\alpha \} + \{[\lambda_\gamma, \lambda_\alpha], \lambda_\beta \} &= 0, \\
\{[\lambda_\alpha, \lambda_\beta], \lambda_\gamma \} + \{[\lambda_\beta, \lambda_\gamma], \lambda_\alpha \} + \{[\lambda_\gamma, \lambda_\alpha], \lambda_\beta \} &= 0,
\end{align*}
\]
one finds that each of the identities indicates one solution to (5.23), respectively, i.e.
\[ g_i^{\alpha\beta\gamma} \sim F_i(q^2) f^{\alpha\beta\gamma}, \quad g_i^{\alpha\beta\gamma} \sim D_i(q^2) d^{\alpha\beta\gamma}. \]

In fact, these are the only solutions of (5.23). \( F_i \) and \( D_i \) are just functions of the squared transferred four-momentum, \( q^2 \).

Restricting attention to \( g_1^{\alpha\beta\gamma} \), one obtains \( (F_1 \equiv F, D_1 \equiv D) \)
\[ g_1^{\alpha\beta\gamma} = -i F(q^2) f^{\alpha\beta\gamma} + D(q^2) d^{\alpha\beta\gamma}. \]
The constants \( F \) and \( D \), which are the values at \( Q^2 = 0 \) of the corresponding form factors, completely characterize the axial-vector matrix elements of all the baryon semileptonic decays.

Replacing (5.26) in (5.20) and comparing the proton matrix elements \( \langle p|A^3|p \rangle \) and \( \langle p|A^8|p \rangle \) with the neutron \( \beta \) decay matrix element \( \langle n|A^{1+2}n \rangle \) one finds, with
\[ |p\rangle = (|B^3\rangle + i|B^7\rangle)/\sqrt{2}, \quad |n\rangle = (|B^3\rangle + i|B^7\rangle)/\sqrt{2}, \quad A^{1+2} = (A^1 + iA^2)/\sqrt{2} \]
that the axial form factors are related to \( F \) and \( D \) according to
\[ G_A^{(3)} = F + D, \quad G_A^{(8)} = \frac{1}{\sqrt{3}}(3F - D). \]

At \( Q^2 = 0 \), the values of \( F \) and \( D \) thus obtained from the axial form factors is presented in Tab. 5.3. The fitting, done in [232], of \( F \) and \( D \) to the experimental data on semileptonic decays is indicated in Tab. 5.3 as the experimental data on \( F \) and \( D \).

<table>
<thead>
<tr>
<th>( \mu = 140 \text{ MeV} )</th>
<th>( \mu = 490 \text{ MeV} )</th>
<th>Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F )</td>
<td>0.385</td>
<td>0.417</td>
</tr>
<tr>
<td>( D )</td>
<td>0.726</td>
<td>0.758</td>
</tr>
<tr>
<td>( F/D )</td>
<td>0.529</td>
<td>0.550</td>
</tr>
<tr>
<td>( F + D )</td>
<td>1.091</td>
<td>1.172</td>
</tr>
<tr>
<td>( g_1(0)/f_1(0) )</td>
<td>1.176</td>
<td>1.251</td>
</tr>
</tbody>
</table>

Table 5.3: Model predictions for the constants \( F \) and \( D \) in the case of flavor symmetry and of \( g_1(0)/f_1(0) \) in the case of symmetry breaking. Experimental data is from [232]. Model parameters as in Tab. 5.2.

In order to calculate \( F \) and \( D \), expressions (5.28) are used, but the input consists in the axial form factors calculated in the symmetric limit, which are not presented here. From Tab. 5.3 it is clear that the CQSM results for \( F \) and \( D \) fall below the fits to the experimental semileptonic decay data. The discrepancy is of the order of 16% for \( F \), 10% for \( D \) and 8% for \( F/D \). It is also seen from Tab. 5.3 that the discrepancy of the results of the CQSM to the experimental ones is smaller in the case of kaon asymptotics. One finds, in this case, the discrepancies of about 10% for \( F \), 6% for \( D \) and 5% for \( F/D \). One should remark, however, that considering the kaon asymptotics is less justified here than it was in the case of the electromagnetic form factors, since we apply the formalism only to the nucleon. The reason is the flavor separation, which is not done in the case of the axial form factors, since we apply the formalism only to the nucleon. The better description
provided by the kaon asymptotics indicates the axial quantities tend to favour a smaller soliton.

Relations (5.28), as is already the case with Tab. 5.3, are used to determine $F$ and $D$. This is done by calculating the form factors $G_A^{(3)}$ and $G_A^{(8)}$ in the flavor symmetric case. This is necessary in order to conform with the definition of the antisymmetric $F$ and symmetric $D$ components of $g_1$, equivalent to the Wigner-Eckart theorem in SU(3), which is based on exact SU(3) symmetry. Fig. 5.3 shows the dependence with $Q^2$ of the form factors $F$ and $D$. The main aspects from Fig. 5.3 are the similar behaviour of $F$ and $D$ as compared to the axial form factors, already expected from (5.28), and the almost independence of the ratio $F(Q^2)/D(Q^2)$ on $Q^2$.

![Graph showing $F$, $D$, and the ratio $F/D$ as functions of $Q^2$ according to (5.28). Model parameters as in Fig. 5.1.](image)

While the question of the importance of SU(3) breaking effects in the extraction of $F$ and $D$ from experimental data is still an open one, the CQSM results are different whether one works in the symmetric case or not [234]. From Tab. 5.3 one may find that the flavor symmetry breaking effects are expected to be small, on the basis of the relative small difference between $F + D$ and $g_1(0)/f_1(0)$ obtained in the CQSM. This statement lacks support from the study of symmetry breaking for the remaining semileptonic decays.

### 5.4 Flavor axial-vector form factors

The singlet, triplet and octet components of the octet of axial currents may be written in terms of the axial-vector flavor currents. The corresponding relations involving form factors are:

\[
G_A^{(0)} = G_A^s + G_A^d + G_A^A, \quad (5.29a)
\]

\[
G_A^{(3)} = G_A^s - G_A^d, \quad (5.29b)
\]

\[
\sqrt{3} G_A^{(8)} = G_A^s + G_A^d - 2G_A^A. \quad (5.29c)
\]

These relations may be inverted leading to a flavor-like decomposition of these form factors, i.e. to the form factors for each axial-vector flavor current:

\[
G_A^s = \frac{1}{3} G_A^{(0)} + \frac{1}{2} G_A^{(3)} + \frac{1}{2\sqrt{3}} G_A^{(8)}, \quad (5.30a)
\]

\[
G_A^d = \frac{1}{3} G_A^{(0)} - \frac{1}{2} G_A^{(3)} + \frac{1}{2\sqrt{3}} G_A^{(8)}, \quad (5.30b)
\]

\[
G_A^A = \frac{1}{3} G_A^{(0)} - \frac{1}{\sqrt{3}} G_A^{(8)}. \quad (5.30c)
\]

Similarly, in SU(2), inverting the relations now following from the Pauli matrices

\[
G_A^{(0)SU(2)} = G_A^sSU(2) + G_A^dSU(2), \quad G_A^{(3)SU(2)} = G_A^sSU(2) - G_A^dSU(2), \quad (5.31)
\]

\[
G_A^{(8)SU(2)} = \frac{1}{\sqrt{3}} G_A^{(8)}SU(2). \quad (5.31)
\]
one obtains the flavor decomposed axial form factors

\[ G_{A}^{u} \text{SU}(2) = \frac{1}{2} G_{A}^{(0)} \text{SU}(2) + \frac{1}{2} G_{A}^{(3)} \text{SU}(2), \quad G_{A}^{d} \text{SU}(2) = \frac{1}{2} G_{A}^{(0)} \text{SU}(2) - \frac{1}{2} G_{A}^{(3)} \text{SU}(2). \]  \hspace{1cm} (5.32)

As already pointed out in the case of \( a_0 \), (5.15), the axial constants may be written in terms of the spin contribution of each flavor:

\[ a_0 = g_{A}^{(0)} = \Delta u + \Delta \bar{u} + \Delta d + \Delta \bar{d} + \Delta s + \Delta \bar{s}, \]  \hspace{1cm} (5.33a)

\[ a_3 = g_{A}^{(3)} = \Delta u + \Delta \bar{u} - \Delta d - \Delta \bar{d}, \]  \hspace{1cm} (5.33b)

\[ a_8 = \sqrt{3} g_{A}^{(8)} = \Delta u + \Delta \bar{u} + \Delta d + \Delta \bar{d} - 2 (\Delta s + \Delta \bar{s}). \]  \hspace{1cm} (5.33c)

It is possible from these relations to find each of the flavor contributions. In our case they are simply given by

\[ \Delta u + \Delta \bar{u} = G_{A}^{u}(Q^2 = 0), \quad \Delta d + \Delta \bar{d} = G_{A}^{d}(Q^2 = 0), \quad \Delta s + \Delta \bar{s} = G_{A}^{s}(Q^2 = 0). \]  \hspace{1cm} (5.34)

The results for these quantities, as computed in the model, are listed in Tab. 5.4, together with the results of the analysis of the experimental data from ref. [134].

<table>
<thead>
<tr>
<th></th>
<th>SU(2)</th>
<th>SU(3)</th>
<th>( \mu = 140 \text{ MeV} )</th>
<th>( \mu = 490 \text{ MeV} )</th>
<th>[134]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta u + \Delta \bar{u} )</td>
<td>0.777</td>
<td>0.814</td>
<td>0.911</td>
<td>0.78 ± 0.03</td>
<td></td>
</tr>
<tr>
<td>( \langle r_{\Delta u}^2 \rangle^{1/2} )</td>
<td>0.665</td>
<td>0.751</td>
<td>0.598</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta d + \Delta \bar{d} )</td>
<td>-0.419</td>
<td>-0.362</td>
<td>-0.340</td>
<td>-0.48 ± 0.03</td>
<td></td>
</tr>
<tr>
<td>( \langle r_{\Delta d}^2 \rangle^{1/2} )</td>
<td>0.668</td>
<td>0.688</td>
<td>0.480</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta s + \Delta \bar{s} )</td>
<td>-</td>
<td>-0.086</td>
<td>-0.075</td>
<td>-0.14 ± 0.03</td>
<td></td>
</tr>
<tr>
<td>( \langle r_{\Delta s}^2 \rangle^{1/2} )</td>
<td>-</td>
<td>0.554</td>
<td>0.172</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.4: Values for \( \Delta u + \Delta \bar{u} \), \( \Delta d + \Delta \bar{d} \), and \( \Delta s + \Delta \bar{s} \) and their “radii”. Model parameters as in Tab. 5.2.

The CQSM results of Tab. 5.4 show a slight overestimation of the values from [134] in the case of the \( u \) quark and a somewhat larger underestimation in the case of \( s \) quarks and \( d \) quarks. It is interesting, nevertheless, that the model succeeds in predicting a negative polarization of the strange sea in the nucleon, as is believed to be the experimental case [235].

![Image](image.png)

Figure 5.4: The strange axial-vector form factor. Model parameters as in Fig. 5.1.

The full axial-vector form factors for each flavor are presented in Fig. 5.4 for the strange quark and in Fig. 5.5 for the \( u \) and \( d \) quarks. The knowledge of the form factors allows to calculate their slope at the origin. These are indicated in Tab. 5.4 using the same expression for the radii as in the electromagnetic case (C.90). These quantities do not have, however, an analog interpretation as in
the electromagnetic case. In the present context they may be used to relate the relative behaviour of different flavors. One concludes that, while such “radius” is practically the same in SU(2) for $u$ and $d$ quarks, it decreases in SU(3) from the $u$ quark to the $s$ quark, i.e.

$$\langle r_u^2 \rangle^{1/2} > \langle r_d^2 \rangle^{1/2} > \langle r_s^2 \rangle^{1/2}.$$  \hspace{1cm} (5.35)

It is also found that the kaon asymptotics does not improve the agreement with the experimental values.

### 5.5 PCAC

Before proceeding one should consider the nonconservation of the axial currents. Indeed in the case of explicit symmetry breaking, which is unavoidable as long as the quarks have distinct masses, the axial currents are not conserved beyond the chiral limit (B.18), not to mention the anomalies, already at the chiral limit. According to the hypothesis of partial conservation of the axial current (PCAC), the divergence of the current (here we restrict the analysis to the case of the nonsinglet current in SU(2)) is related to the pion field

$$\partial_\mu A^{\mu} = \overline{\psi}(x)i\gamma_5 \tau^a \psi(x) = m^2_{\pi}f_{\pi}\pi^a(x).$$ \hspace{1cm} (5.36)

However, these relations do not necessarily hold if one computes the matrix elements for the different terms of (5.36) in a given model. These terms are, considering the matrix elements in the nucleon, given by

$$q_u \langle N(p)|A^{\mu}|N(p)\rangle, \quad q_d \langle N(p)|\overline{\psi}(x)i\gamma_5 \tau^a \psi(x)|N(p)\rangle,$$

$$m^2_{\pi}f_{\pi}\langle N(p)|\pi^a(x)|N(p)\rangle.$$ \hspace{1cm} (5.37)

The first of these terms, the value of the divergence of the current, poses immediately some difficulties. On one hand, $q_u \langle A^0 \rangle$ is suppressed by two orders of $1/N_c$ as compared to $q_d \langle A^1 \rangle$. On the other hand, the determination of $q_u \langle A^1 \rangle$ is hampered by the fact that the Dirac equation must be used to calculate $q^i$, which will lack the rotational corrections entering the other terms of (5.37).

It is actually considered [39] that the differences among the terms in (5.37) give an estimate of the theoretical error which is introduced by the restriction to the zero modes in the $1/N_c$ corrections. The mismatch of terms in (5.37) is not restricted to the CQSM and has been studied in the context of the linear sigma model using a virial theorem [236,237]. In reference [39] it is concluded that PCAC is fulfilled in flavor SU(2) CQSM within 15%, following from the calculation of $g_A$. The Goldberger-Treiman relation [238], which is closely related to PCAC, is not, therefore, well described in the model.

### 5.6 Weak neutral current form factors of the nucleon

The standard electroweak couplings to the quarks are listed in Tab. 5.5. Very much in the same
way the electromagnetic vector current is given by
\[ J^\mu_v = \frac{2}{3} \bar{u} \gamma^\mu u - \frac{1}{3} \bar{d} \gamma^\mu d - \frac{1}{3} \bar{s} \gamma^\mu s \] (5.38)
Tab. 5.5 leads analogously to
\[ J^\mu_Z = \left( 1 - \frac{8}{3} \sin^2 \theta_W \right) \bar{u} \gamma^\mu u + \left( -1 + \frac{4}{3} \sin^2 \theta_W \right) \bar{d} \gamma^\mu d + \left( -1 + \frac{4}{3} \sin^2 \theta_W \right) \bar{s} \gamma^\mu s, \] (5.39)
the neutral weak current, which, in lowest order, corresponds to the \( Z \) exchange of Fig. 5.7a. The coefficients of this current depend now on the weak mixing angle,
\[ \sin^2 \theta_W = 0.23117 \pm 0.00016 \] (5.40)
whose value is taken from ref. \[124\]. The flavor structure contained in (5.38,5.39) allows to study each flavor separately and is therefore the basis for the present experimental program to find the flavor structure of the vector form factors, with emphasis on strangeness.

The form factors for the neutral weak current (5.39) are given by
\[ G^Z_{E,M} = \left( 1 - \frac{8}{3} \sin^2 \theta_W \right) G^u_{E,M} + \left( -1 + \frac{4}{3} \sin^2 \theta_W \right) G^d_{E,M} + \left( -1 + \frac{4}{3} \sin^2 \theta_W \right) G^s_{E,M} \] (5.41a)
\[ \quad = \left( 1 - 4 \sin^2 \theta_W \right) G^u_{E,M} - G^d_{E,M} - G^s_{E,M} \] (5.41b)
where the flavor form factors are exactly the same as the ones found in the study of the electromagnetic form factors in Chapter 3. The second line in (5.41) follows from using isospin symmetry to replace the flavor form factors for \( u \) and \( d \) quarks by the form factors of the proton and neutron. It shows clearly that the neutral weak form factors are related to the electromagnetic form factors and to the strange form factors. In particular it shows that to access the strange form factors one has just to measure the neutral weak ones, since the electromagnetic form factors have been studied since a long time.

In the present work, it is a rather simple task to use (5.41) and the previous form factors results from Chapter 3 to find the neutral weak form factors in the CQSM. They are shown in Fig. 5.6, where the experimental point is that of \[142\]
\[ G^Z_{M}(Q^2 = 0.1) = 1.49 \pm 0.29 \text{ (stat.)} \pm 0.31 \text{ (syst.) n.m.} \] (5.42)
Again the CQSM result falls within the (admitidly large) error bars.

The coefficients of the form factors in (5.41) of the current (5.39) are modified by higher order corrections. These radiative corrections are related to the \( \gamma - Z \) box Fig. 5.7c and the \( \gamma - Z \) mixing Fig. 5.7b, also called anapole moment. These corrections lead to [239,240]
\[ G^Z_{E,M} = \left( 1 - 4 \sin^2 \theta_W \right) (1 + R^p_V) G^u_{E,M} - \left( 1 + R^p_V \right) G^d_{E,M} - G^s_{E,M}, \] (5.43a)
\[ R^p_V = -0.054 \pm 0.033, \quad R^s_V = -0.0143 \pm 0.0004. \] (5.43b)

In the case of the axial-vector currents, the electroweak model relates charge-changing and neutral processes. Examples of these processes are the neutron \( \beta \) decay, \( G^{(3)}_A \), and neutrino scattering, \( G^\nu_A \), respectively. Due to the effect of the strange quark-antiquark pairs there is an isoscalar neutral weak form factor. In neutrino scattering, which has no electromagnetic interaction in lowest order, the form factor reads
\[ G^Z_A(Q^2) = G^\nu_A(Q^2) = -G^{(3)}_A(Q^2)\tau^3 + G^s_A(Q^2), \] (5.44)

<table>
<thead>
<tr>
<th>( q^3 )</th>
<th>( q^s )</th>
<th>( a^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-1 + 4 \sin^2 \theta_W</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>-1 + 4 \sin^2 \theta_W</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1 + 4 \sin^2 \theta_W</td>
</tr>
</tbody>
</table>

Table 5.5: Electroweak couplings for the fundamental charged particles.
where $\Delta s$ stands for the polarization of the strange quarks, as in Section 5.2, and $\tau^3 = 1(-1)$ for the proton(neutron). The form factor (5.44) is modified by an additive term $R_\nu$ when higher order electroweak radiative corrections are taken into account.

In the case of electron scattering, the relevant case in parity-violating electron proton scattering, the neutral weak axial form factor is given by [241]

$$\frac{G_A^e(Q^2)}{G_A^0(Q^2)} = 1 + 2\eta F_A + R_e$$

where $F_A$ is the anapole form factor, $\eta = 8\pi\sqrt{2}\alpha/(1 - 4\sin^2\theta_W)$, and $R_e$ stands for other radiative corrections. The anapole and the other radiative corrections are more important in the axial case due to the fact that the leading order, tree-level, axial form factor is suppressed by a factor of $1 - 4\sin^2\theta_W << 1$, while $F_A$ is enhanced by $\eta \sim 3$. The actual separation of the radiative corrections into $F_A$ and $R_e$ is delicate and gauge dependent [240].

The results for the axial neutral weak form factors in the CQSM are presented in Fig. 5.8. The expression used for these form factors is, however, from [167],

$$G_A^e(Q^2) = -(1 + R_A^1)\tau^3 G_A^{(3)}(Q^2) + R_A^0 + G_A^e,$$

with the values for the radiative corrections

$$R_A^1 = -0.41 \pm 0.24, \quad R_A^0 = 0.06 \pm 0.14$$

obtained in chiral perturbation theory [242]. For the isovector component one has

$$G_A^{e(T=1)} = (G_A^{e_{\pi}} - G_A^{e_\nu})/2.$$

The value obtained in the CQSM for the isovector component of $G_A^e$ is

$$G_A^{e(T=1)}(Q^2 = 0) = -0.574,$$


5.7 Asymmetry in parity violating electron scattering

The usual method to determine the weak form factors is the measurement of the parity violating asymmetry in electron-proton scattering. At tree level, the total cross section would just be proportional to the square of the amplitudes for single $\gamma$, Fig. 5.7a, with a $\gamma$ instead of a $Z$, and a $Z$, Fig. 5.7a, exchange. It would be insensitive to the weak form factors due to the large suppression of the weak form factors, which explains the use of the asymmetry. Parity violation, due to the axial-vector part of the weak current, can be probed using longitudinally polarized electrons, since the two states of polarization of the electrons have opposite parities. The parity violating cross section asymmetry for the scattering of longitudinally polarized electrons on unpolarized protons is defined as the difference of the total cross sections for right and left electrons (or positive and negative helicities) divided by their sum,

$$A = \frac{\sigma_R - \sigma_L}{\sigma_R + \sigma_L}. \quad (5.50)$$

Denoting, at tree level, the amplitudes for $\gamma$ and $Z$ exchange by $T_\gamma$ and $T_Z$, respectively, the total cross section for either polarization is given by the square of the sum of the amplitudes

$$\sigma_{R,L} \sim |T_\gamma + T_Z|^2_{R,L} \quad (5.51)$$

The amplitude for $\gamma$ exchange is given by (C.3), since it is the amplitude leading to the Rosenbluth formula for unpolarized electron-proton scattering, see Appendix C. For the weak amplitude, one finds that it is a sum of a parity conserving piece $T_{PC}$ and a parity violating piece $T_{PV}$. The parity violating piece, in the notation of (C.3) reads

$$T_{PV} \equiv \langle p', k' | T_{PV} | p, k \rangle = \frac{G_F}{2\sqrt{2}} \left[ \langle U(k', s') \gamma_\mu U(k, s) JZ^{\mu\delta} | p, S \rangle + \langle U(k', s') \gamma_\mu \gamma_5 U(k, s) \langle p', S' | JZ^{\mu} | p, S \rangle \right]. \quad (5.52)$$

with $G_F$ the Fermi constant.

For the asymmetry (5.50) one finds

$$A = \frac{|T_\gamma + T_{PC} + T_{PV}|^2_R - |T_\gamma + T_{PC} + T_{PV}|^2_L}{|T_\gamma + T_{PC} + T_{PV}|^2_R + |T_\gamma + T_{PC} + T_{PV}|^2_L} \sim \frac{T_{\gamma}^* T_{PV}}{|T_\gamma|^2}, \quad (5.53)$$

which is smaller than the ChPT results from [242].
which may be reduced to
\[
A = A_0 \left( Q^2 \right) \frac{\varepsilon G_E^Z G_L^Z + \tau G_M^Z G_A^Z - \left(1 - 4 \sin^2 \theta_W \right) \varepsilon' G_M^Z G_A^Z}{\varepsilon (G_E^Z)^2 + \tau (G_M^Z)^2}.
\] (5.54)

The simplifications follow from using the definition of the form factors and discarding terms of order \(G_F\) in the denominator and of order \(G_L^2\) in the numerator. The asymmetry (5.54) is a function of \(Q^2\), with the \(Q^2\) dependence entering through
\[
A_0 \left( Q^2 \right) = -G_F Q^2 \left( \frac{4 \sqrt{2} \pi \alpha}{\varepsilon} \right) = -0.899 Q^2 / \text{GeV}^2
\] (5.55)
as well as through all the form factors in it (5.54) and the kinematic factors \(\tau = Q^2 / (4 M_N^2)\) and \(\varepsilon = 1 / \left[1 + 2 \left(1 + \tau \right) \tan^2 \left(\theta/2\right)\right]\), \(\varepsilon' = \sqrt{\tau \left(1 + \tau \right) \left(1 - \varepsilon^2\right)}\). (5.56)

The asymmetries as obtained by the use of (5.54) and the various form factors computed in the model are depicted in Fig. 5.9 for the kinematics of the three experiments currently measuring this asymmetry. The results denoted as model results include the form factors as they are obtained in the CQSM. These show a large deviation from the experimental values for the asymmetry. The main cause for this deviation is related to the difficulty of the model to reproduce the magnetic moments, see Chapter 3. Once the magnetic form factors of the nucleon are scaled to their experimental value, but keeping their \(Q^2\) dependence from the model, the results for the asymmetry come very close to the experimental values. Fig. 5.9 also shows the results for the asymmetry when one further takes into account the radiative corrections to the form factors, as described in the preceding section.
6 Summary and outlook

In this chapter we review the main achievements and physics aspects of this work on electroweak form factors of the baryon octet in the chiral quark-soliton model (CQSM). This summary covers the main aspects of the framework, Chapter 2, and the main results, presented in Chapters 3 to 6. It is the basis for a twofold discussion: First, a general qualitative overview of the results, in terms of their successes, shortcomings and conceptual and phenomenological relevance, addresses the physics encompassed by the CQSM. Second, in the outlook of this work, we discuss possible extensions and improvements of the results in this thesis and propose related topics for future investigation within the CQSM.

Overview

We start this overview by summarizing the formalism of Chapter 2 and appendices. The formalism used in this work included the self-consistent calculation of of the mean field equations of motion, the semiclassical quantization of the mean field in order to have the proper quantum numbers for the baryon octet and the computation of the electroweak form factors taking into account the so-called rotational $1/N_c$ corrections and the linear terms in the strange-to-nonstrange quark mass difference $\delta m$ (first order mass corrections).

The framework is thus found not to include: quark-meson loop effects at the level of observables; degrees of freedom other than pions, kaons and constituent quarks; effects of higher orders than one in $1/N_c$ and $\delta m$. Even though these ingredients are of conceptual theoretical interest, it is, nevertheless, a realization of the present work, as was the case with many previous similar studies in the CQSM, that these effects do not play a crucial role at the momentum transfers considered in the calculation of form factors. It is found, therefore, that the formalism, as it was used, is able to provide consistent descriptions of baryon properties even in the case of small and sensitive quantities like the electric form factor of the neutron and the strange form factors of the nucleon.

Starting with the meson loops, only the quark loop effects were taken into account at the level of the effective action, where the meson loop contribution is suppressed by one order of $1/N_c$. It means that the sources for the meson fields were not included in the path integral for observables and that, consequently, there are no meson-loop contributions to observables. These meson sources were only included in the vacuum sector in order to determine the meson masses and decay constants. Mesons are nevertheless present in the description of baryons, they provide the binding self-consistent field, and are regarded as collective excitations of quark-antiquark pairs. We have shown, particularly in Chapter 3, that mesons play an important role in observables, in special through the effects of the asymptotics of the meson profile function. At the level of observables, however, the meson-loop terms would be of the same order in $N_c$ as the rotational corrections, since the rotational corrections are of order $1/N_c$ but there are $N_c$ quarks. Meson-loop terms correspond to the inclusion of modes normal to the zero modes used in the collective quantization. The most important consequence would be a much more complicated calculation of observables, since the formalism would change from a simple Hartree mean field picture to a RPA-like one. Meson-loop contributions would also imply another cut-off [243] for the regularization of these loops, making the calculation of observables even more difficult. Nevertheless, since the meson loops are connected to vibrations of the soliton, which are believed to be much less important than the zero modes, the view adopted here is to prefer the exact treatment of the zero modes and neglect the meson-loop effects. This situation is called a zero-meson-loop approximation, which is the approximation used in all the calculations of observables found in the literature, with the exception of corrections to the baryon masses [244]. The effects of meson loops have already been study in the CQSM [243] at the level
of the regularization and meson properties, masses and decay constants.

The degrees of freedom of the model are massive constituent quarks and the lowest mesons: Pions in flavor SU(2) and pions and kaons in flavor SU(3). There are two sources for comment in this case: First, in the case of flavor SU(3), the embedding (2.37) makes all mesons degenerate in mass with the pions. One of the original aspects of this thesis was the phenomenological study concerning the effects of breaking such degeneracy. It has been found that a consistent description of the kaon degrees of freedom is necessary in the case of flavor SU(3), since they may have potentially sizeable effects at the level of some observables, like the electric form factor of the neutron. Second, higher energy degrees of freedom, e.g. vector mesons, are frozen. Such approximation, starting with the restriction to the chiral circle, mainly limits the $Q^2$ values up to which one may calculate form factors and does not affect the low energy quantities computed in this work. One must recall that the model, as deduced from the instanton liquid model description of SCSB, is suited for small momentum transfers only, e.g. it has a low cut-off. Moreover, from the large $N_c$, the momentum transfered is of order $N_c^0$ while the mass of the nucleon is of order $N_c$.

Finally, regarding the truncation in the $1/N_c$ and $\delta m$ expansions, the results of this work, with the exception of Chapter 4, all include rotational corrections, i.e. $1/N_c$ corrections, and linear terms in the perturbative treatment of $\delta m$, directly related to the difference in masses between strange and $u,d$ quarks. The next terms are of order $1/N_c^2$, $\delta m/N_c$ and $\delta m^2$. In any of these cases, their calculation would imply a growing number of terms involving the trace of a product of several one-particle matrix elements and, therefore, would be much more difficult to compute numerically. These corrections, on the basis on the values of the inertia parameters at the $1/N_c$ and $\delta m$ calculation, are expected to be small.

The overall precision, considering these theoretical approximations of the present framework, is characterized by a deviation from the experimental data below an upper limit of 20 %, for all observables analyzed in this work, summarized in the following. Since the considered observables correspond to very distinct physics, one may well take the abovementioned value as a conservative estimate for the accuracy attainable in the present framework.

We may next review the results obtained in this work. Starting with Chapter 3, its subject was the study of the vector currents, related to the electromagnetic interaction. The following list summarizes the results of Chapter 3:

- Form factors for the singlet, triplet and octet currents of the vector octet for the states of the baryon octet;
- Form factors for each of the flavor currents in the octet;
- Strange form factors of the nucleon and predictions for ongoing parity-violating electron-proton scattering experiments;
- Electromagnetic form factors of the baryon octet, based on the flavor form factors;
- Charge and magnetization radii, as well as magnetic moments, for all the aforementioned form factors;
- Study of the importance of the asymptotics of the meson fields in the description of form factors and related quantities;
- Electromagnetic form factors of the nucleon and the ratio $\mu p G_E^p / G_M^p$, at higher momentum transfers.

One of the most important points in the study of the form factors performed in Chapter 3 were the strange form factors of the nucleon. It was shown that, in the description of the form factors of the baryon octet, the flavor decomposed form factors in the CQSM allow for a description of the octet magnetic moments within 16% of the experimental values. Using this as a gauge for the value
of the strange magnetic moment of the nucleon one achieves a reasonable level of confidence in the
positive value found for this quantity in the CQSM, in agreement with recent experimental results
and in contradistinction to the majority of the models of hadron structure. The model results for
the strange form factors agree with all the available experimental data collected in parity-violating
electron scattering at forward angles. The data on backward angles, expected within a few years,
will allow to establish unambiguously how well the CQSM results [245] agrees with the data.

Chapter 4 considers the electromagnetic current from the point of view of transition processes
between baryons. The main subject of Chapter 4 was the electromagnetic transition of the nucleon
to the $\Delta(1232)$ accompanying topical experiments at ELSA (Bonn) and JLAB (Newport News,
USA). The results consist on the calculation of the:

- Magnetic dipole transition amplitude $M1$;
- Ratios of the quadrupole electric, $E2$, and scalar, $S2$, amplitudes to the dipole magnetic one.

This was the only case within the scope of this thesis, where strange mass corrections ($\delta m^1$)
were not taken into account. It has been seen that the model results underestimate the latest
experimental results, although the general behaviour for this sensitive observable is reproduced.
The inclusion of strange mass corrections is important in order to see whether their role narrows
the gap to phenomenology.

The study conducted for the vector flavor currents in Chapter 3 has also been carried out,
basically in the same manner, for the axial-vector currents in Chapter 5. Although Chapter 5
confines the results to the nucleon case, for simplicity, the results for the all octet have also been
computed in a flavor decomposed form. The presented results for the nucleon include:

- Form factors for the singlet, triplet and octet components of the octet of axial-vector currents;
- The axial-vector form factors for the flavor currents;
- Radii and axial constants for the above form factors;
- Contribution of each quark flavor to the spin of the nucleon, on the basis of axial constants;
- Semileptonic decay constants $F$, $D$ and $g_1(0)/f_1(0)$;
- Neutral weak form factors of the nucleon and their associated radii and magnetic moment;
- Form factor of the axial neutral weak current, in particular its isovector component in electron
  scattering;
- Parity-violating asymmetries in polarized electron-proton scattering presently performed at
  MAMI-A4 (Mainz) and JLAB-G0 (Newport News, USA).

The results for the axial-vector form factors are unfortunately hampered by an even smaller
basis for comparison with the octet phenomenology than the vector quantities. In most cases, the
CQSM compares favourably with phenomenology.

**Outlook**

From the point of view of the formalism, future work could go in the direction of introducing, into
the framework to compute observables, some of the physics ingredients mentioned at the beginning
of the previous section, e.g. meson loops, higher terms in $1/N_c$, etc. It is qualitatively found,
however, in the present, work that they are not absolutely essential to describe the observables in
the scope of this thesis. This is corroborated from other investigations, with the CQSM, for very
different quantities, like parton distribution functions [115, 246, 247]. At present, modifications of
the framework seem to be more of pure theoretical interest than of necessity for the calculation of
observables. In this theoretical interest one could, as well, include the study of the SU(3) flavor
symmetry breaking, eventually considering in the CQSM the approaches of Callan-Klebanov [87] or Yabu-Andu [88], instead of the perturbative treatment used here for $\delta m$.

In spite of the theoretical interest in these extensions of the formalism, there are, nevertheless, further quantities interesting to calculate in the framework used in this work.

Regarding Chapter 3, one such example is offered by the electric polarizabilities of the nucleon, already studied in the CQSM [249], although within a different approach. A straightforward continuation of Chapter 3 could be the study of the baryon decuplet, for which there are also experimental results. The most interesting development, however, would consist in making predictions for the electromagnetic properties of the recently found [2] exotic antidecuplet baryon, the pentaquark $\Theta^+$. The discovery of this state was very much prompted by the CQSM prediction for its mass and small decay width [3]. In fact, studies to calculate the electromagnetic form factors, radii and magnetic moments for the whole of the antidecuplet are under way.

As to Chapter 4, inclusion of strange mass corrections in the transition $N - \Delta$ is also a natural candidate for future calculations with the CQSM. The case of the nucleon to $\Delta$ transition is just one particular case of the electroproduction of baryons belonging to the baryon $J^\pi = 3/2^+$ decuplet. The formalism used in Chapter 4 may be easily modified in order to address the electromagnetic induced transitions from the octet to the decuplet, in terms of transition amplitudes and their ratios, e.g. $E2/M1$. Such study would also yield data for the radiative decays of the decuplet baryons, in particular the decay widths. Depending on the future experimental work, it may turn out necessary to study the $N - \Delta$ transition in terms of transition form factors and eventually to include recoil effects due to the $N - \Delta$ mass difference. A similar opportunity for future work are the electroproduction of the $\Theta^+$ and, eventually, all the remaining members of the antidecuplet.

Regarding Chapter 5, a simple application of the results for the axial-vector currents in the octet will be the study of the semileptonic decays, in particular the study of the effects of symmetry breaking, extending [250] for finite $Q^2$. Another open question, which may be eventually worth investigating, is the difficulty with PCAC in the framework of the model which produced the results here reported.

There is, of course, a lot of quantities, different from the ones studied both theoretical and numerically here, which are amenable to be calculated in the CQSM. Examples inspired in the present report are the axial transitions between octet and decuplet, or other multiplets. A starting point in this direction would be simply a straightforward extension of Chapter 4 to compute the axial amplitudes in the $N - \Delta$ transition, as was recently studied in the case of the linear sigma model [251]. The knowledge of the form factors for both the vector and axial-vector flavor currents opens up the possibility to study the various aspects of the weak interaction in transitions within and among baryon multiplets.

Completely new topics would most naturally include the study of the generalization of the matrix element for local currents (1.1), related to form factors, to the case of nonlocal operators, $\langle F|\bar{\psi}(z)\Gamma\Lambda\psi(0)|T\rangle$, between a target state $T$ a final state $F$. ($\Gamma$ and $\Lambda$ stand for Dirac-Lorentz and flavor operators.) Such matrix elements appear in the definition of various parton distributions and were first investigated in the CQSM in [114,115]. The framework is closely related to the one used here, but is has been mostly used in flavor SU(2), [246], although extensions to SU(3) exist [247]. In particular, some distributions of one class of these parton distributions, the so-called generalized parton distributions, coincide with the form factors in the limit of a local operator. Their calculation in the CQSM would mean a broader basis for understanding baryon structure, when compared with the simpler one based on form factors.

Finally, not exhausting the list of possible future directions for the work with the CQSM, one may refer the calculation of the nucleon form factors for the energy-momentum tensor. These form factors would bring information not only on the relative contributions of quark spin and orbital angular momentum to the spin of the nucleon, they would also yield information, e.g. on the mass distribution of the nucleon, as was exploratory done in [252].
A Notation and conventions

The conventions and notation followed throughout the text are the same as those in [253] for mathematical functions, [254] for field theory and [255] for angular momentum. Unless otherwise stated, natural units will be used $\hbar = c = 1$.

A.1 Short review of notation

Minkowski space is characterized by the metric $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, relating covariant and contravariant vectors in the usual way: $V^\mu = g^{\mu\nu}V_\nu$. Greek indices run from 0 to 3 (from 1 to 4 in Euclidian space) and Latin indices from 1 to 3, except in flavor space, in which case their meaning is explicitly given. The scalar product of two four-vectors $V$ and $W$ is given by

$$V \cdot W = V^\mu g_{\mu\nu}W_\nu = V^0W^0 - V \cdot W.$$  \hspace{1cm} (A.1)

Coordinate space three-vectors are displayed in boldface type $V = \{V_i, i = 1, 2, 3\}$, while three-vectors in isospin space are indicated with an arrow.

The Pauli matrices are:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$ \hspace{1cm} (A.2)

and obey commutation and anticommutation relations easy to extract from

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk} \sigma^k$$ \hspace{1cm} (A.3)

where $\epsilon^{ijk}$ is the Levi-Civita completely antisymmetric tensor.

A.1.1 The Euclidian space

In passing to the Euclidian space (denoted with $E$) the main changes, as chosen here, concern the time components of the four-vectors. The convention followed is based on the replacement

$$x^0 = -ix^4, \quad x_M^i = x_E^i. $$ \hspace{1cm} (A.4)

This means that $x_0 = ix_4, \quad x_M^i = x_E^i$, because the metric tensor in the Euclidian space has then signature $-4$. In fact from the scalar product of two four-vectors in Minkowski space

$$a \cdot b \equiv a^\mu b_\mu = a^\mu g^M_{\mu\nu}b^\nu \rightarrow (-ia^4)(-ib^4) - a_E^i b_E^i = a_E^\mu g_E^{\mu\nu}b_E^\nu = a_E^\mu b_E^\mu = a_E \cdot b_E$$ \hspace{1cm} (A.5)

one can thus obtain a metric in Euclidian space:

$$g^E_{\mu\nu} = g^\mu^\nu_E = -\delta_{\mu\nu}. $$ \hspace{1cm} (A.6)

For the gamma matrices the convection followed is the same:

$$\gamma^0 = -i\gamma^4, \quad \gamma^i_M = \gamma^i_E. $$ \hspace{1cm} (A.7)

This immediately makes the Euclidian matrices anti-hermitian, $\gamma^\mu_E = -\gamma^\mu_E$. The anticommutation relations are preserved in form

$$\{\gamma^{\mu}_E, \gamma^{\nu}_E\} = 2g^{\mu\nu}_E.$$ \hspace{1cm} (A.8)

For $\gamma^5$ we have

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \rightarrow -\gamma^1_E\gamma^2_E\gamma^3_E\gamma^4_E = \gamma^5_E \hspace{1cm} (A.9)$$

with the properties:

$$\gamma^{5}_E = \gamma^5_E, \quad (\gamma^5_E)^2 = 1, \quad \{\gamma^{i}_E, \gamma^{j}_E\} = 0. $$ \hspace{1cm} (A.10)
A.2 Flavor groups

Some quantities in SU(3) are best written in terms of the SU(3) Murray Gell-Mann matrices. These are given by

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\
\lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{align*}
\]

(A.11)

These matrices are traceless, hermitian and form a group with the multiplication table summarized by

\[
\lambda^\alpha \lambda^\beta = (if_{\alpha\beta\gamma} + d_{\alpha\beta\gamma}) \lambda^\gamma + 2/3\delta^{\alpha\beta},
\]

(A.12)

where \( \alpha \) and \( \beta \) belong to \( \{1, \ldots, 8\} \) and the normalization of the \( \lambda \) matrices is \( \text{tr}(\lambda^\alpha \lambda^\beta) = 2\delta^{\alpha\beta} \).

The totally antisymmetric structure constants \( f_{\alpha\beta\gamma} \) and the totally symmetric \( d_{\alpha\beta\gamma} \) are given in Tab. A.1.

From (A.12), the commutation and anticommutation relations between the \( \lambda \) matrices are:

\[
\begin{align*}
[\lambda^\alpha, \lambda^\beta] &= 2if_{\alpha\beta\gamma} \lambda^\gamma, \\
\{\lambda^\alpha, \lambda^\beta\} &= 2d_{\alpha\beta\gamma} \lambda^\gamma + 4/3\delta^{\alpha\beta}.
\end{align*}
\]

(A.13a, b)

It is often defined a matrix proportional to the identity, \( \lambda^0 = \sqrt{2/3} \mathbf{1} \), which obeys the previous commutation and anticommutation relations with \( d_{0\alpha\beta} = \delta^{\alpha\beta} \). In this work, the unit matrix in flavor space was preferred to \( \lambda^0 \), but flavor quantities related to the unit matrix retain the index 0, for convenience of notation.

Projectors into strange and non-strange subspaces

The projectors into strange \( (P_S) \) and non-strange \( (P_T) \) subspaces are given by

\[
\begin{align*}
P_S &= \text{diag} (0, 0, 1) = \frac{1}{3} \mathbf{1}_3 - \frac{1}{\sqrt{3}} \lambda^8, \\
P_T &= \text{diag} (1, 1, 0) = \frac{2}{3} \mathbf{1}_3 + \frac{1}{\sqrt{3}} \lambda^8.
\end{align*}
\]

(A.14a, b)

They obey \( P_SP_S = 1, P_TP_T = 1 \) and \( P_TP_S = P_SP_T = 0 \).

The effect of these projectors over the \( \lambda \) matrices are simply:

\[
P_T \lambda^\rho P_T = \delta_{\rho \in \{1, 2, 3\}} (\tau^\rho \oplus P_S) + \frac{1}{\sqrt{3}} \delta_{\rho 8} (\mathbf{1}_2 \oplus P_S)
\]

(A.15a)
\[ P_T \lambda^a P_S \lambda^b P_T = \delta_{\alpha,\beta \in \{4,5,6,7\}} \left[ (d_{\alpha,\beta \rho} + i f_{\alpha,\beta \rho}) P_T \lambda^\rho P_T + \frac{2}{3} \delta_{\alpha,\beta} P_T \right] \]
\[ = (d_{\alpha \beta \rho} + i f_{\alpha \beta \rho}) P_T \lambda^\rho P_T + \frac{2}{3} \delta_{\alpha \beta} P_T \]  
(A.15b)

Using (A.15a) the following result holds for the mass difference (A.23):
\[ P_T A^l \delta m A P_T = P_T A^l (M_1 + M_8 \lambda^8) A P_T = M_1 + M_8 D_{88}^{(8)} P_T \lambda^a P_T \]
\[ = M_1 + M_8 D_{88}^{(8)} \tau^1 + \frac{1}{\sqrt{3}} M_8 D_{88}^{(8)} \]  
(A.16)

**Mass matrices**

In SU(2) the mass matrix is
\[ m = \text{diag}(m_u, m_d) = m_1 \mathbf{1}_2 + m_3 \tau^3, \]  
(A.17)
with
\[ m_1 = \frac{m_u + m_d}{2}, \quad m_3 = \frac{m_u - m_d}{2}. \]  
(A.18)

In SU(3)
\[ m = \text{diag}(m_u, m_d, m_s) = m_1 \mathbf{1}_3 + m_3 \lambda^3 + m_8 \lambda^8, \]  
(A.19)
with, in the same notation as in SU(2),
\[ m_1 = \frac{m_u + m_d + m_s}{3}, \quad m_3 = \frac{m_u - m_d}{2}, \quad m_8 = \frac{m_u + m_d - 2 m_s}{2 \sqrt{3}}. \]  
(A.20)

In all applications the isospin breaking term \( m_3 \) is neglected, i.e. the isospin breaking effects are not taken into account. In this case
\[ m_u = m_d = \overline{m}, \]  
(A.21)
so that, in SU(3),
\[ m_1 = \frac{2 \overline{m} + m_s}{3}, \quad m_8 = \frac{\overline{m} - m_s}{\sqrt{3}}. \]  
(A.22)

In the calculation of the symmetry breaking induced by the strange quark mass, one performs an expansion in the difference between the strange quark mass and the mass of the up and down quarks. Such difference is given by
\[ \delta m = m - \overline{m} \mathbf{1}_3 = (m_1 - \overline{m}) \mathbf{1}_3 + m_8 \lambda^8 = M_1 \mathbf{1}_3 + M_8 \lambda^8, \]  
(A.23)
which defines:
\[ M_1 = \frac{m_8 - \overline{m}}{3}, \quad M_8 = m_8. \]  
(A.24)

**Charge matrices**

As to what concerns the charge matrices, they are
\[ Q_{SU(2)} = \text{diag} \left( \frac{2}{3}, -\frac{1}{3} \right) = \frac{1}{6} + \frac{1}{2} \tau^3 \]  
(A.25)
in SU(2) flavor and
\[ Q_{SU(3)} = \text{diag} \left( \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right) = \frac{1}{2} \left( \lambda^3 + \frac{1}{\sqrt{3}} \lambda^8 \right) \]  
(A.26)
in SU(3). The flavor decomposition rests on writing the charge according to
\[ Q = Q_u + Q_d + Q_s, \]  
(A.27)
with individual flavor charges given by
\[ Q_u = \text{diag} \left( \frac{2}{3}, 0, 0 \right) = \frac{2}{9} \mathbf{1}_3 + \frac{1}{3} \lambda^3 + \frac{1}{3 \sqrt{3}} \lambda^8, \]  
(A.28a)
\[ Q_d = \text{diag} \left( 0, \frac{1}{3}, 0 \right) = -\frac{1}{9} \mathbf{1}_3 + \frac{1}{6} \lambda^3 - \frac{1}{6 \sqrt{3}} \lambda^8, \]  
(A.28b)
\[ Q_s = \text{diag} \left( 0, 0, -\frac{1}{3} \right) = \frac{1}{9} I_3 + \frac{1}{3\sqrt{3}} \chi_8. \]  

(A.28c)

This choice has the advantage of including both the information on how many quarks of a given flavor there are in a given baryon together with the charge of the quark. This allows for an easy normalization, when desired.

### A.2.1 Spherical basis

The relations between the Cartesian basis and the spherical basis are

\[ A_x = \sum_{\nu} U_{x\nu} A_{\nu} = -\frac{1}{\sqrt{2}} A_{+1} + \frac{1}{\sqrt{2}} A_{-1} \]  

(A.29a)

\[ A_y = \sum_{\nu} U_{y\nu} A_{\nu} = \frac{i}{\sqrt{2}} A_{+1} + \frac{i}{\sqrt{2}} A_{-1} \]  

(A.29b)

\[ A_{+1} = \sum_{a} U_{+1a} A_{a} = -\frac{1}{\sqrt{2}} A_x - \frac{i}{\sqrt{2}} A_y \]  

(A.29c)

\[ A_{-1} = \sum_{a} U_{-1a} A_{a} = \frac{1}{\sqrt{2}} A_x - \frac{i}{\sqrt{2}} A_y \]  

(A.29d)

which define the matrices

\[ U_{aa} = 
\begin{pmatrix}
-1 & 0 & +1 \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0
\end{pmatrix} \]  

(A.30)

and

\[ U_{aa} = 
\begin{pmatrix}
-1 & x & y & z \\
\frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\
0 & 1 & 0 & 1 \\
+1 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0
\end{pmatrix} \]  

(A.31)

with the properties

\[ U_{aa} = U_{aa}^*, \quad U_{aa}^* = (-)^\omega U_{-aa}, \quad U_{aa}^* = (-)^\alpha U_{a-a} \]  

(A.32)

and obeying

\[ \sum_{a} U_{aa} U_{ab}^\dagger = \sum_{a} U_{aa} U_{ba}^* = \delta_{ab} \]  

(A.33a)

\[ \sum_{a} U_{aa} U_{a\beta}^\dagger = \sum_{a} U_{aa} U_{\beta a}^* = \delta_{a\beta}. \]  

(A.33b)
B Quantum Chromodynamics (QCD)

The chapter starts by reviewing the general properties of QCD, in particular how they account for the difficulties with QCD at low energies. The listing of the general properties of QCD is relevant due to the fact that the difficulties of QCD at low energies are in many cases bypassed by the use of models, of which the CQSM is one example, and that these models are in some cases constructed using some of these properties as a guide.

B.1 Definition of QCD

The degrees of freedom of Quantum Chromodynamics QCD [13, 16, 256], a non-Abelian Young-Mills field theory, are the current quarks \( q_f \) in the triplet representation (antiquarks belonging to the complex conjugate triplet) and gluons \( A_\mu^a \) (\( a \) is the color index) in the adjoint representation of a local SU(3) color group. All of the strong interactions, structure and interactions from the lightest hadron to the heaviest nucleus, are believed to be encompassed by, although no necessarily computable from, the QCD Lagrangian

\[
\mathcal{L}_{\text{QCD}} = \sum_{f} N_f \left( i \gamma^\mu D_\mu - m_f \right) q_f - \frac{1}{2} \text{tr} G^{\mu\nu} G^{\mu\nu},
\]

where

\[
G^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu], \quad A_\mu = \sum_{a=1}^{8} A_\mu^a \frac{\lambda^a}{2},
\]

\[
D_\mu q_f = (\partial_\mu - ig A_\mu) q_f.
\]

The index \( f \) represents the flavor, \( q_f : u, d, s, c, b, t, \cdots \), where the dots represent the fact that experiment does not exclude higher flavors. The number of quark flavors is \( N_f \). In a general context the Lagrangian should include all of the known quarks, from the up quart to the top quark. The \( \lambda^a \) are the Gell-Mann matrices of Appendix A. However, the masses of the quarks vary tremendously: While the up and down quarks have masses below 10 MeV, the mass of the charm quark is already above 1 GeV, with the mass of the strange quark around 160 MeV. The difference in the masses is the reason why one may ignore the heavier quarks, in favor of either of the doublet of up and down quarks or the triplet made by adding the strange quark, when addressing the dynamics of the vacuum and low-lying hadronic states. Absent from (B.1) are the gauge fixing terms and the ghost terms necessary to quantize the theory [254, 257, 258], i.e. to obtain the Feynman rules.

B.2 Properties of QCD

One of the most striking properties entailed in (B.1) is that of asymptotic freedom [16–19]. Technically in order to understand what asymptotic freedom is, one may start from the fact that any process corresponds in field theory to a function of the set of momenta involved, \( p_i \), and of the parameters of the theory, \( g_i \) (like coupling constants and masses.) One can then scale all the momenta, \( p_i \rightarrow \lambda p_i \), and see what happens when \( \lambda \) increases, in particular, what happens in the limit \( \lambda \rightarrow \infty \). Using the renormalization group equations it is possible to shift the dependence on \( \lambda \) from the momenta to new effective parameters of the theory, \( \tilde{g}_i (\lambda) \), that thus become scale dependent. These functions are solutions to some differential equations, which are uniquely determined by the Lagrangian, with its original parameters appearing just as initial conditions, \( \tilde{g}_i (\lambda = 1) = g_i \) of such differential equations. The simplest behaviour one can find corresponds to finite fixed points for
the effective parameters, i.e. \( \lim_{\lambda \to \infty} \bar{g}_i(\lambda) = g_i^* \). An interacting field theory for which all of the fixed points vanish, \( g_i^* = 0 \), is said to be asymptotically free. QCD is an example of such theory. The usual statement regarding the asymptotic freedom of QCD is that the coupling constant of QCD becomes small at large momenta, or, equivalently, at short distances.

Asymptotic freedom was very important to establish QCD as the theory of the strong interactions because it made possible the use of perturbation theory at high enough momenta and energies, thus allowing to look directly for quark and gluon manifestations at short distances. This possibility soon offered an explanation for the experimentally observed scaling in deep-inelastic scattering. Simultaneously one finds that the coupling constant grows at small momentum, large distances, rendering the perturbation theory useless in that domain. This is the reason behind the distinction between a perturbative, high-energy regime and a nonperturbative, low-energy regime of QCD. The lowest hadron states belong thus to the nonperturbative regime and are not computable within QCD in a perturbative way. The current difficulties of QCD at low energies stem from the lack of a manageable nonperturbative approach.

The other striking property of QCD is confinement (of color). This is a postulated property of QCD for which there is yet no theoretical explanation. The postulate says that all hadron states and physical observables (currents, energies, masses, ...) are color-singlets, i.e. they carry no net color. Thus phrased it looks like a kinematical constraint to eliminate colored states from the asymptotically free ones. Theoretically it is still a challenge to find, if that is the case, how it arises from the dynamics of the theory.

Apart from these two properties, symmetries of the Lagrangian are in general responsible for much of the information one is able to extract from the theory. This is the case of QCD at low energies. Apart from the obvious continuous space-time symmetries, the QCD Lagrangian is also invariant, in the absence of the \( \theta \)-term, under the discrete symmetries of space inversion (P), time reversal (T) and charge conjugation (C).

However, the symmetries with further reaching implications are related to global symmetries of the QCD Lagrangian. QCD possesses a global \( U(N)_L \times U(N)_R \) chiral symmetry in the limit \( m_i = 0, i = 1, 2, \ldots, N < N_f \) when \( N \) of the quark mass parameters vanish. In order to make this symmetry more evident, one may write the quark fields as, for \( N = 2 \), \( q = (u d)^T \), with a mass matrix \( m = \text{diag}(m_u, m_d) \), or, for \( N = 3 \), \( q = (u d s)^T \) with \( m = \text{diag}(m_u, m_d, m_s) \). Thus, separating the matrix with the quark mass parameters,

\[
L_{QCD}^{(N)} = L_{QCD}^0 + L_{QCD}'^N, \quad (B.3a)
\]

\[
L_{QCD}^0 = \bar{q}_R \gamma^\mu D_\mu q_R - \frac{1}{2} \text{tr} G_{\mu \nu} G^{\mu \nu}, \quad (B.3b)
\]

\[
L_{QCD}' = -\bar{q} m q. \quad (B.3c)
\]

The physics allowing the diagonal mass matrices in the non-Abelian Young-Mills theory with local color \( SU(3) \) symmetry is the flavor-neutrality and spin 1 (vector) of the gluons.

### B.3 Chiral symmetry

#### B.3.1 Chiral limit

In this limit one takes \( m = 0 \), i.e. the Lagrangian restricts to \( L_{QCD}^0 \). Using the projectors into left- and right-hand projectors of the quark fields

\[
P_L = \frac{1 - \gamma^5}{2}, \quad P_R = \frac{1 + \gamma^5}{2} \quad (B.4)
\]

the massless part of the QCD Lagrangian becomes

\[
L_{QCD}^0 = \bar{q}_R i \gamma^\mu D_\mu q_R + \bar{q}_L i \gamma^\mu D_\mu q_L - \frac{1}{2} \text{tr} G_{\mu \nu} G^{\mu \nu}. \quad (B.5)
\]

The right-, \( q_R = P_R q \), and left-handed, \( q_L = P_L q \), fields have definite chirality, i.e. they are eigenstates with eigenvalues +1, −1, respectively, of the chirality operator \( \gamma^5 \). For massless particles chirality and helicity become equal. This Lagrangian is readily seen to be invariant under the
B.3 Chiral symmetry

transformation

\[
SU(N)_R: \quad q_L \rightarrow q_L; \quad q_R \rightarrow e^{i\gamma^5 t}q_R
\]

\[
SU(N)_L: \quad q_L \rightarrow e^{i\gamma^5 t}q_L; \quad q_R \rightarrow q_R
\]

\[
U(1)_L: \quad q_L \rightarrow q_L; \quad q_R \rightarrow e^{i\gamma^5 t}q_R
\]

\[
U(1)_R: \quad q_L \rightarrow q_L; \quad q_R \rightarrow q_R
\]

where \( u \cdot t = \sum_{a=1}^{N^2-1} u^a t^a \), with \( u \) a vector and \( t^a \) the matrices in the fundamental representation of the flavor group \( SU(N) \) (Pauli, \( t^a = \tau^a/2 \), and Gell-Mann, \( t^a = \lambda^a/2 \), matrices for \( N = 2 \), \( 3 \), respectively).

The chiral symmetry of the Lagrangian can thus be written as a direct product

\[
\delta(2N^2-1) \rightarrow \delta(2N^2-1)
\]

in a different form:

\[
SU(N)_V: \quad q \rightarrow e^{i\gamma^5 t}q; \quad q_{R,L} \rightarrow e^{i\gamma^5 t}q_{R,L}
\]

\[
SU(N)_A: \quad q \rightarrow e^{i\gamma^5 t}q; \quad q_R \rightarrow e^{-i\alpha t}q_L
\]

\[
U(1)_V: \quad q \rightarrow e^{i\gamma^5 t}q; \quad q_{R,L} \rightarrow e^{i\gamma^5 t}q_{R,L}
\]

\[
U(1)_A: \quad q \rightarrow e^{i\gamma^5 t}q; \quad q_R \rightarrow e^{-i\alpha t}q_L
\]

where \( V \) stands for vector and \( A \) for axial transformations. While \( SU(N)_L \) and \( SU(N)_R \) commute, \( SU(N)_V \) and \( SU(N)_A \) do not, which follows from \([t^a, \gamma^5 b t^b]\) being not necessarily always vanishing while \([P_R^a, P_L^b t^b] = 0\). The group \( SU(N)_L \times SU(N)_R \times G_\chi \) is known as the chiral group.

The Noether theorem then states that for every Lagrangian symmetry there is a corresponding conserved current. Furthermore, the three-dimensional integral of the time component of this current is a charge generator and is conserved. The Noether currents associated with the global transformations (B.7b) and the notation for the corresponding charge generators are given in Tab. (B.1)

<table>
<thead>
<tr>
<th>(a = 1, \ldots, N^2 - 1)</th>
<th>( SU(N)_V )</th>
<th>( SU(N)_A )</th>
<th>( U(1)_V )</th>
<th>( U(1)_A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current</td>
<td>( J^a_\mu = q^\gamma^5 t^a q )</td>
<td>( J^a_\mu = q^\gamma^5 t^a q )</td>
<td>( J^N_\mu = q^\gamma^5 q )</td>
<td>( J^N_\mu = q^\gamma^5 q )</td>
</tr>
<tr>
<td>Charge</td>
<td>( Q^a )</td>
<td>( Q^a )</td>
<td>( Q^a )</td>
<td>( Q^a )</td>
</tr>
</tbody>
</table>

Table B.1: Noether currents and charges for QCD in the chiral limit.

The general formula\(^1\)

\[
\delta(x_0) \left[ q^I \Gamma_1 \Lambda_1 q(x) , q^I \Gamma_2 \Lambda_2 q(0) \right] = \frac{1}{2} \delta^3(x) q^I \left( \{ \Gamma_1, \Gamma_2 \} \{ \Lambda_1, \Lambda_2 \} + \{ \Gamma_1, \Gamma_2 \} \{ \Lambda_1, \Lambda_2 \} \right) q(0)
\]

where \( \Gamma_1 \) are Dirac matrices and \( \Lambda_1 \) are flavor matrices, makes it easy to verify that the charges satisfy the algebra associated with the (equal-time)commutation relations:

\[
\left[ Q^a, Q^b \right] = i f^{abc} Q^c, \quad \left[ Q^a, Q^b_5 \right] = i f^{abc} Q^c_5, \quad \left[ Q^a_5, Q^b_5 \right] = i f^{abc} Q^c_5, \quad \left[ Q^a_N, Q^b_N \right] = i f^{abc} Q^c_N
\]

where \( [Q^a_N, Q^a_N] = 0 \). The right-hand side then is simplified with the relation \( 2(ABCD - BADC) = [A,B] [C,D] + [B,A] [D,C] \).

\(^1\)Using the on the left-hand side of the expression the relation

\[
[AB,CD] = A \{ B, C \} D - AC \{ B, D \} + \{ A, C \} DB - C \{ A, D \} B
\]

it can be expressed in terms of quark field anticommutators which obey \( \delta(x_0) \{ q^I (x), q^I (0) \} = 0 \), and \( \delta(x_0) \{ q^I (x), (0) q^I \} = \delta^3(x) \delta_{x_0} \delta_{x_0} \). The right-hand side then is simplified with the relation

\[
2(ABCD - BADC) = [A,B] [C,D] + [B,A] [D,C]
\]
### B.3.2 Manifest symmetry

Considering some conserved charge generator $Q^a$ ($Q^a = Q^a, Q^5$)

$$\dot{Q}^a = i[H, Q^a] = 0$$

(B.11)

and some particle eigenstate $|n(p)\rangle$ of the Hamiltonian, $E_n = \sqrt{m_n^2 + p^2}$,

$$H|n\rangle = E_n|n\rangle,$$  

(B.12)

one finds that the state $Q^a|n(p)\rangle$ is also an eigenstate (with the same energy and mass):

$$H Q^a|n\rangle = Q^a H|n\rangle = E_n Q^a|n\rangle.$$

(B.13)

Manifest symmetry implies that $Q^a|n\rangle$ is a one particle eigenstate of the Hamiltonian (possibly different from $|n\rangle$). Since in

$$Q^a|n\rangle = Q^a a_n^\dagger|0\rangle = \left[Q^a, a_n^\dagger\right]|0\rangle + Q^a|0\rangle$$

(B.14)

the commutator is of the same form of the creation operator, the state $Q^a|n\rangle$ generated from $|n\rangle$ is presumably a one particle state provided the charges annihilate the vacuum

$$Q^a|0\rangle = 0.$$  

(B.15)

This means that the vacuum is invariant under the symmetry transformations generated by these charges and is nondegenerate. The algebra of the charges $Q^a$ imply the segregation of the spectrum into multiplets of particles with the same mass, generated by application of the charge generators. Since this realization of the symmetry is recognizable from the spectrum, it is termed manifest or as the Wigner-Weyl mode.

In the case of the vector charges, this is approximately the case. Indeed the spectrum is observed to be composed of multiplets: isodoublet, isotriplet, ..., in flavor SU(2); octet, decuplet, ..., in flavor SU(3). The symmetry is nevertheless only approximate, as will be seen below. On the contrary, the manifest realization of the axial-vector symmetries would imply the existence for all hadron states of an equal mass state with opposite parity, since $Q^5|n\rangle$ and $|n\rangle$ have opposite parity. This parity, or chiral, partner to an hadron state is not, if existent, easily found in the hadron spectrum and thus the symmetry is not manifest. Another consequence of manifest chiral symmetry would be the existence of massless, or near massless, baryon states, as a consequence of the Goldberger-Treiman relations [238, 259].

### B.3.3 Hidden chiral symmetry

In this case the charges do not annihilate the vacuum state

$$Q^a|0\rangle \neq 0.$$  

(B.16)

From (B.14), it follows that this nonvanishing of the last term, makes the states generated from $|n\rangle$ to differ from $|n\rangle$ and thus making it so that the symmetry is not obvious in the hadron spectrum.

As the charge is still conserved, $[Q^a, H] = 0$, the vacuum is degenerate, i.e. all the states generated by

$$|\nu\rangle = e^{-i\nu Q}|0\rangle$$

(B.17)

are all zero energy eigenstates of the Hamiltonian. Any of these vacua is a valid candidate for the physical vacuum. The symmetry is said to be spontaneously broken or that the symmetry is realized in the Nambu-Goldstone mode. Instead of spontaneously broken one can speak of hidden symmetry, since the symmetry still is present at the level of the Lagrangian, but not of the vacuum and is thus hidden in the spectrum [260].

Since all the states (B.17) have zero energy, it is expectable that the charges create and annihilate zero energy and momentum (therefore massless) particles from the vacuum. This is the case and is summarized in Goldstone’s theorem [261, 262]. The theorem asserts that this mode of realization is associated with the appearance of zero mass bosons, the Nambu-Goldstone, or Goldstone bosons. These are the pions ($\pi^-, \pi^0, \pi^+$) in SU(2) and the lowest lying pseudoscalar octet ($\pi^-, \pi^0, \pi^+, K^-, K^0, K^0, K^+, \eta$) in flavor SU(3). These pseudoscalar massless particles explain the parity doubling in the case of manifest axial symmetry: they can be added to any state changing
its parity but no its energy.

This symmetry realization is signaled, according to the Goldstone’s theorem, by the quark condensate, i.e. the nonvanishing of the vacuum expectation value of the quark bilinear $\bar{q}q$, the quark condensate $\langle \bar{q}q \rangle$. It is one of the main parameters of the QCD vacuum phenomenology. It is easy to see that it connects quarks with different helicity: This quantity would vanish for a symmetric vacuum. Because this quantity vanishes at all orders of perturbation theory, SCSB is a nonperturbative phenomenon.

**B.3.4 Explicit symmetry breaking**

Taking into consideration (B.3c), $L_{QCD} = -\bar{q}mq$, the currents may not be conserved anymore. The Noether theorem states then that the divergence of the current is given by the variation the Lagrangian under the symmetry transformation ($\varepsilon$ is the generic parameter of the transformation), $-\varepsilon^a \partial^\mu J^a_\mu = \delta L$. Considering the transformations (B.7a), ..., (B.7d) and the normalizations in Tab. (B.1) ($a = 1, \cdots, N^2 - 1$) the divergences of the currents read

$$\partial^\mu J^a_\mu = i\bar{q} [m, t^a] q = i \sum_{i,j=1}^N \bar{q}_i (m_i - m_j) (t^a)_{ij} q_j \quad \text{(B.18a)}$$

$$\partial^\mu J^a_{\mu 5} = i\bar{q} [m, t^a] q = i \sum_{i,j=1}^N \bar{q}_i (m_i + m_j) \gamma_5 (t^a)_{ij} q_j \quad \text{(B.18b)}$$

$$\partial^\mu J^N_\mu = 0; \quad \partial^\mu J^N_{\mu 5} = 2i\bar{q}m\gamma_5 q. \quad \text{(B.18c)}$$

It is clear that the axial currents are not conserved as long as the current quark masses deviate from zero while the conservation of the vector current depends on the current quark masses being equal.

The consequences of the explicit chiral symmetry breaking for the manifest vector symmetries is that the charges corresponding to currents which are not conserved do not annihilate the vacuum. This destroys the degeneracy of mass within the multiplets: The mass splittings within the multiplets resulting from the violation of these symmetries is expected to be proportional to the quark mass differences, hence small, but still visible in the spectrum. For the axial-vector symmetries the Nambu-Goldstone bosons are expected to acquire a mass $m_\pi^2 = \langle \pi | h | \pi \rangle = K (m_u + m_d) + O \left( m_{u,d}^2 \right)$ in lowest order in the perturbation $H' = -L_{QCD} \ (N = 2)$ using the soft-pion techniques, with $K$ depending just on the pion decay constant and the quark condensate.

**B.4 The Goldberger-Treiman relation and PCAC**

In this section we follow [273] in deriving the Goldberger-Treiman [238] relation and in arriving at the PCAC relation. We consider the simplest case of SU(2)$\times$SU(2) and start by considering the axial current $A^a_\mu(x)$ is conserved, $\partial^\mu A^a_\mu(x) = 0$. This axial current is identified with the corresponding one in the weak interactions which couples to the pion such that

$$\langle 0 | A^a_\mu(0) | \pi^b(q) \rangle = if_{\pi} q_{\mu} \delta^{ab}, \quad \text{(B.19)}$$

where $f_{\pi}$ is the pion decay constant $f_{\pi} = 93$ MeV. Current conservation implies then that

$$\langle 0 | \partial^\mu A^a_\mu(0) | \pi^b(q) \rangle = if_{\pi} m_\pi^2 \delta^{ab}, \quad \text{(B.20)}$$

that is to say, that

$$f_{\pi} m_\pi^2 = 0, \quad \text{(B.21)}$$

which is the Goldstone theorem: Since the axial charge does not annihilate the vacuum, the $f_{\pi} \neq 0$, it is required that $m_\pi = 0$.

Considering matrix elements of the axial current between nucleon states, leads, under the same general assumptions of Section C.5, see Eq. (C.72), to

$$\langle N(p')|A^a_\mu(0)|N(p)\rangle = \frac{\pi(p')}{2}\gamma_\mu(\gamma^5 | ^A G_A(q^2) + q_\mu \gamma_5 h(q^2)| u(p), \quad q = p' - p. \quad \text{(B.22)}$$
In this case, current conservation requires
\[ 2MG_A(q^2) + q^2 h(q^2) = 0 \]  
(B.23)
where \( M \) is the nucleon mass and \( H \) is related to the pseudoscalar form factor. Since \( G_A(0) = g_A = 1.267 \neq 0 \), it follows that \( h(q^2) \) must have a pole at \( q^2 = 0 \) due to the Goldstone pion. Using (B.19), the residue of this pole can be computed and one finds that \( q^2 h(q^2) = -2f_\pi g_\pi N \) when \( q^2 \to 0 \), with \( g_\pi N \) the pion-nucleon coupling constant. Using relation (B.23) one obtains the Goldberger-Treiman relation
\[ Mg_A = f_\pi g_\pi N. \]  
(B.24)

Eq. (B.23) could also be solved in a second way, namely by the solution \( M = 0 \). This would be the sign of manifest symmetry realization, while the previous solution is a sign for the spontaneous breaking of SU(2)\(_A\). One sees then, that this relation is first of all a consequence of symmetry. It is remarkable in relating constants which are related to weak and strong interactions. In nature,
\[ 1 - g_A M / g_\pi N \simeq 0.07 \]  
(B.25)
which shows that SU(2)\( \times \)SU(2) is good symmetry within 7%.

Considering chiral symmetry breaking, the Hamiltonian reads
\[ H = H_0 + \epsilon^0 \sigma_0 \]  
(B.26)
where \( H_0 \) is SU(2)\( \times \)SU(2) invariant and \( \epsilon \) is the unrenormalized symmetry breaking parameter. Assuming that the unrenormalized fields \( (\sigma_0, \pi^a_0) \) transform as a chiral quadruplet, one has the equal-time commutation relations
\[ [Q^a_5, \sigma_0(x)] = -i\pi^a_0(x), \quad [Q^a_5, \pi^b_0(x)] = i\delta^{ab}\sigma_0(x). \]  
(B.27)
The renormalized pion field is defined by
\[ \langle 0|\pi^a_0(0)|\pi^b(k)\rangle = \delta^{ab} Z_\pi^{1/2}. \]  
(B.28)
The consequence of symmetry breaking is that
\[ \partial^\mu A^a_\mu = i [H(x), Q^a_5] = -\epsilon^0 \pi^a_0(x). \]  
(B.29)
It follows then from the pion decay matrix element (B.19) that
\[ \epsilon = \epsilon^0 Z_\pi^{-1/2} = -m_\pi^2 f_\pi \]  
(B.30)
where \( \epsilon \) is now the renormalized symmetry breaking parameter. In the Nambu-Goldstone realization of the symmetry, as \( \epsilon \) vanishes, also \( m_\pi \) vanishes. Writing the renormalized pion field as \( \pi = Z_\pi^{-1/2} \pi_0 \) one then arrives at the relation known as the partial conservation of the axial current (PCAC)
\[ \partial^\mu A^a_\mu = m_\pi^2 f_\pi \pi^a_0(x). \]  
(B.31)
As it stands it is just the definition of the “pion” field \( \pi^a(x) \). The physical consequences of this relation follow mostly from considering that the pion field defined in this way is a good extrapolating field.
C General aspects of form factors

C.1 Elastic electron-proton scattering

In electron-proton scattering the simplest situation consists in detecting the outgoing electron. In the case the polarizations are not observed one has to sum over the final polarizations and average over the initial polarizations. The cross section is \([254]\) given by

$$d\sigma = \frac{mM}{\sqrt{p \cdot k - m^2 M^2}} \frac{d^3k'}{2(2\pi)^3} (2\pi)^4 \delta^{(4)} (p' + k' - p - k) \left| \langle p', k' | T | p, k \rangle \right|^2$$  \(\text{(C.1)}\)

where the averaged, \(\langle \rangle = (2S + 1)(2s + 1)\), amplitude over the polarizations is

$$\left| \langle p', k' | T | p, k \rangle \right|^2 = \frac{1}{4} \sum_{s,s'} \sum_{s,s'} \left| \langle p', k' | T | p, k \rangle \right|^2$$  \(\text{(C.2)}\)

with the amplitude given to lowest order (one photon exchange) by

$$\langle p', k' | T | p, k \rangle = \frac{e^4}{q^4} U (k', s') \gamma_\mu U (k, s) \langle p', S' | J^\mu | p, S \rangle .$$  \(\text{(C.3)}\)

The kinematics is described by the momenta of the electrons, incoming \(k\) and outgoing \(k'\), the corresponding momenta for the proton, \(p\) and \(p'\), respectively, with \(q = p' - p\) the four-momentum transferred. It is worth noting here the difference between the treatment of a fundamental particle as the electron and a composite particle as the proton. In fact, while \(U (k, s)\) is a spinor of the free Dirac equation, the details of the proton structure are still encoded in \(\langle p', S' | J^\mu | p, S \rangle\).

For the averaged amplitude one finds

$$\left| \langle p', k' | T | p, k \rangle \right|^2 = \frac{e^4}{q^4} L_{\mu\nu} (k', k) \frac{1}{2} \sum_{s,s'} \langle p', S' | J^\mu | p, S \rangle \langle p, S | J^{\nu\dagger} | p', S' \rangle$$  \(\text{(C.4)}\)

defining the leptonic tensor (\(k = \gamma_\mu k_\mu\))

$$L_{\mu\nu} (k', k) = \frac{1}{2} (2m)^2 \text{tr} \left[ \gamma_\mu (k + m) \gamma_\nu (k' + m) \right]$$  \(\text{(C.5)}\)

and using the fact that for the free Dirac equation the spinors fulfill

$$\sum_s U (k, s) \overline{U} (k, s) = \frac{k + m}{2m} .$$  \(\text{(C.6)}\)

Evaluating the traces the result for the leptonic tensor is

$$L_{\mu\nu} (k', k) = \frac{1}{2m^2} (k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu} k \cdot k') .$$  \(\text{(C.7)}\)

One may write, including the integral over the proton final momentum

$$\int \frac{d^3p'}{p^{00}} \delta^{(4)} (p' + k' - p - k) \left| \langle p', k' | T | p, k \rangle \right|^2 = \frac{e^4}{q^4} L_{\mu\nu} (k', k) W^{\mu\nu} (p, q)$$  \(\text{(C.8)}\)

which defines the hadronic tensor

$$W^{\mu\nu} (p, q) = \sum_{s,s'} \int \frac{d^3p'}{2p'^0} \delta^{(4)} (p + q - p') \langle p', S' | J^\mu | p, S \rangle \langle p, S | J^{\nu\dagger} | p', S' \rangle$$  \(\text{(C.9)}\)

The kinematical variables are

\begin{align*}
p' - p & = k - k' = q \quad \text{(C.10a)} \\
p' + p & = P \quad \text{(C.10b)} \\
p' & = \frac{P + q}{2}, \quad p = \frac{P - q}{2} \quad \text{(C.10c)}
\end{align*}
where write

$$ q^2 = -2p \cdot q = -2p \cdot (k - k') = -2M (E - E') = -2\nu M $$

(C.10d)

$$ p \cdot p' = p \cdot (p + q) = M^2 + p \cdot q = M^2 - \frac{q^2}{2} $$

with the transferred energy $\nu$

$$ \nu = E - E' = -\frac{q^2}{2M}. $$

(C.11)

The proton matrix element of the electromagnetic current is written as

$$ \langle p', S' | J^\mu(0) | p, S \rangle = \bar{u} (p', S') \Gamma(p', p) u (p, S) $$

(C.12)

which corresponds to nucleons described by free spinors away from the interaction region, interacting non-locally with the external electromagnetic field with a strength depending on the momentum of the exchanged photon. With this definition, the hadronic tensor becomes

$$ W^{\mu\nu} (p, q) = \sum_{S'} \int \frac{d^3 p'}{2q^0} \delta^{(4)} (p + q - p') \bar{u} (p', S') \Gamma^\mu \sum_S u (p, S) \bar{u} (p, S) \Gamma^\nu u (p', S') $$

(C.13)

taking into consideration that the integral, with

$$ \frac{1}{2p^0} = \int d p^0 \delta (p^0) \delta (p^2 - M^2), $$

is given by

$$ \frac{1}{2 p^0} \int \frac{d^3 p'}{2q^0} \delta^{(4)} (p + q - p') \int d^4 p' \delta^{(4)} (p + q - p') \theta (p^0) \delta (p^2 - M^2) = \frac{1}{2M} \delta \left( \nu + \frac{q^2}{2M} \right) $$

(C.14)

(C.15)

In order to evaluate the (C.1) one needs to make further assumptions regarding the matrices $\Gamma$ in (C.13). These matrices encode the information on what is understood as the electromagnetic structure of the proton, since this is the target in the present context.

### C.2 General decomposition of matrix elements of vector currents

The aim of this section is to study the constraints on the $\Gamma$ matrices defined in (C.12) following from general principles. In this section one generalizes to the case of any baryon state $B$ since the conclusions are the same irrespective of this state. It will be thus shown how a matrix element of the electromagnetic current between 1/2-spin states of different momenta (for instance the octet baryons), $\langle B(p') | J^\mu | B(p) \rangle$, may be written in terms of form factors, which are thus expected to carry the information about the current in that process. The matrix element in (C.12) is a Lorentz vector, as the current, and depends on the two momenta $p$ and $p'$ only. The first step in order to find a general expression for this matrix element is to start by its Lorentz character. It may be expanded [263] in the complete set of gamma matrices according to

$$ \bar{u} (p', s') \Gamma^\alpha (p', p) u (p, s) = \bar{u} (p', s') \left( A^\alpha + B^\alpha_\beta \gamma^\beta + C^\alpha_\beta \sigma^{\beta\rho} + D^\alpha_\beta \gamma^\beta \gamma^5 + E^\alpha \gamma^5 \right) u (p, s) $$

(C.16)

where $A^\alpha$ is a vector, $B^\alpha_\beta$ a second-rank tensor, $C^\alpha_\beta\rho = -C^\alpha_{\beta\rho}$ a third rank tensor, $D^\alpha_\beta$ second-rank pseudotensor, and $E^\alpha$ a pseudovector. These coefficients depend on $p'$ and $p$. In this way, one may write

$$ A^\alpha = a_1 p^\alpha + a_2 p'^\alpha $$

(C.17a)

$$ B^\alpha_\beta = b_1 p^\alpha p_\beta + b_2 p'^\alpha p_\beta + b_3 p^\alpha p_\beta' + b_4 p'^\alpha p_\beta' + b_5 \delta^\alpha_\beta, $$

(C.17b)

$$ C^\alpha_\beta\rho = (c_1 p^\alpha + c_2 p'^\alpha) (p_\beta p_\rho - p_\rho p_\beta') + c_3 (\delta^\alpha_\beta p_\rho - \delta^\alpha_\rho p_\beta) + c_4 (\delta^\alpha_\beta p_\rho - \delta^\alpha_\rho p_\beta'), $$

(C.17c)

$$ D^\alpha_\beta = d^\alpha_\beta\rho \rho \rho', $$

(C.17d)

$$ E^\alpha = 0 $$

(C.17e)
The twelve coefficient functions \(a_1, \cdots, a_d\) are scalars and, hence, depend on \(p \cdot p'\) and \(q^2 = (p' - p)^2\) only, since \(p^2 = p'^2 = M^2\), with \(M\) the mass of the spinors. These twelve functions characterize the matrix \(\Gamma^\alpha(p', p)\), which transforms as a vector. The number of functions which characterize \(\bar{u} (p', s') \Gamma^\alpha(p', p) u (p, s)\) is smaller, however, for the spinors satisfy the free Dirac equation

\[
(\not{p'} - M) u (p', s') = (\not{p} - M) u (p, s) = 0
\]

(C.18a)

\[
\bar{u} (p, s) (\not{p} - M) = \bar{u} (p', s') (\not{p'} - M) = 0
\]

(C.18b)

For instance, one finds that from (C.17b)

\[
(b_1 p^\alpha p_\beta + b_2 p^\alpha p_\beta') \bar{u} (p', s') \gamma^\beta u (p, s) = (b_1 + b_2) p^\alpha \bar{u} (p', s') M u (p, s)
\]

(C.19)

which shows that this part of \(B^\alpha_\beta\) reduces to the term with \(a_1\) in (C.17a). Likewise, the terms with \(b_3, b_4\) of \(B^\alpha_\beta\) reduce to the term with \(a_2\) in (C.17a), leaving just the last term in \(b_5\) of (C.17b) to take into account. Taking into account that

\[
(p_{\beta p}' - p_{\beta p}') \sigma^{\beta p} = 2p_{\beta p}' \sigma^{\beta p} = i [\not{p}', \not{p}'] = \Gamma_{\beta p}' - \not{p}' \not{p}' = 2i (p \cdot p' - \not{p}' \not{p}'),
\]

(C.20a)

\[
\Gamma_{\beta p}' = 2p \cdot p'
\]

(C.20b)

\[
(\delta_{\beta p}' \delta_{\beta p} - \delta_{\beta p} \delta_{\beta p}') \sigma^{\beta p} = 2\sigma^{\alpha p} p_{\rho} = i (\gamma^\alpha \not{p}' - \not{p}' \gamma^\alpha) = 2i (\gamma^\alpha \not{p}' - p^\alpha)
\]

(C.20c)

and the Dirac equation, one finds

\[
(p_{\beta p}' - p_{\beta p}') \bar{u} (p', s') \sigma^{\beta p} u (p, s) = 2i (p \cdot p' - M^2) \bar{u} (p', s') u (p, s)
\]

(C.21a)

\[
(\delta_{\beta p}' \delta_{\beta p} - \delta_{\beta p} \delta_{\beta p}') \bar{u} (p', s') \sigma^{\beta p} u (p, s) = \bar{u} (p', s') 2i (\gamma^\alpha \not{p}' - p^\alpha) u (p, s)
\]

(C.21b)

which shows that the terms in \(C_{\beta p}^\alpha\) also reduce to the terms in \(A^\alpha\) (C.17a) and the last term of \(B_{\beta p}^\alpha\) (C.17b). Also in the case of \(D_{\beta p}^\alpha\) a similar result holds. Using the result

\[
i\varepsilon_{\mu \nu \lambda \alpha} \gamma^\mu \gamma^\nu \gamma^\lambda = (\gamma_\mu g_{\nu \lambda} - \gamma_\nu g_{\mu \lambda} + \gamma_\lambda g_{\mu \nu})
\]

(C.22)

the pseudotensor \(D_{\beta p}^\alpha\)

\[
\bar{u} (p', s') \varepsilon_{\beta p} a p' \not{p}' \gamma^\beta \gamma^5 u (p, s) = i\bar{u} (p', s') \left[ M^2 \gamma^\alpha - p \cdot p' + M (p^\alpha - p'^\alpha) \right] u (p, s)
\]

(C.23)

is also found to reduce to the terms in \(A^\alpha\) (C.17a) and the last term of \(B_{\beta p}^\alpha\) (C.17b).

From Lorentz covariance alone and the use of the Dirac equation constrains the matrix element of the electromagnetic current to the general form

\[
\bar{u} (p', s') \Gamma^\alpha (p', p) u (p, s) = \bar{u} (p', s') \left( a \left( q^2 \right) \gamma^\alpha + b \left( q^2 \right) p^\alpha + c \left( q^2 \right) q^\alpha \right) u (p, s).
\]

(C.24)

where \(P\) and \(q\) are given by (C.10a) and (C.10b), respectively. Form factors \(a, b\) and \(c\) transform differently under \(G\) parity: \(a\) and \(b\) are of the first kind and \(c\) of the second kind. \(c\) is related to the so-called second class currents which are related to violations of the symmetries obeyed by the interaction. In the case of \(c\) it is easy to show that it vanishes identically for all values of \(Q^2\) if conservation of the vector current (CVC) is verified. Indeed, current conservation

\[
\partial_\alpha J^\alpha (x) = 0
\]

(C.25)

implies for the matrix element that

\[
\partial_\alpha \left< p' \big| J^\alpha (x) \big| p \right> = \partial_\alpha \left< p' \big| e^{ix(p' - p)} J^\alpha (0) e^{-ix(p - p')} \big| p \right>
\]

\[
= i e^{ix(p' - p)} (p' - p) \alpha \left< p' \big| J^\alpha (0) \big| p \right> = 0
\]

(C.26)

which is then equivalent to

\[
q_\alpha \bar{u} (p', s') \Gamma^\alpha (p', p) u (p, s) = 0.
\]

(C.27)

Since the Dirac equation guarantees

\[
q_\alpha \bar{u} (p', s') \gamma^\alpha u (p, s) = 0,
\]

\[
q_\alpha p^\alpha = (p' - p)_\alpha (p' + p)^\alpha = 0,
\]

(C.28a)

current conservation implies,

\[
q_\alpha \bar{u} (p', s') c \left( q^2 \right) q^\alpha u (p, s) = c \left( q^2 \right) q^2 \bar{u} (p', s') u (p, s) = 0,
\]

(C.29)

that \(c \left( q^2 \right)\) vanishes identically for all values of \(Q^2\).
Eq. C.24 may be written in a more common form using the Gordon identity
\[ \bar{u}(p', s') \gamma^\alpha u(p, s) = \frac{1}{2M} \bar{u}(p', s') \left( P^\alpha + i\sigma^{\alpha\beta} q_\beta \right) u(p, s). \] (C.30)

The general form for the matrix element of interest is
\[ \bar{u}(p', s') \Gamma^\alpha (p', p) u(p, s) = \bar{u}(p', s') \left( F_1(q^2) \gamma^\alpha + i \frac{F_2(q^2)}{2M} \sigma^{\alpha\beta} q_\beta \right) u(p, s). \] (C.31)

The functions \( F_1(q^2) \) and \( F_2(q^2) \) characterize the one-nucleon matrix element of the hadron electromagnetic current operator and are called electromagnetic form factors of the nucleon. \( F_1(q^2) \) is the Dirac form factor and \( F_2(q^2) \) the Pauli form factor.

These form factors are real functions of \( Q^2 \). To verify this, one may start by defining the operator \( T \) as [263]
\[ T = S - 1 \] (C.32)
from the S-matrix, the unitarity of the S-matrix \( S^\dagger S = 1 \) implies
\[ T + T^\dagger = -T^\dagger T. \] (C.33)

Considering initial and final states \(|i\rangle \) and \(|f\rangle \) and inserting a complete set of states
\[ \langle f|T|i\rangle + \langle i|T|f\rangle^* = -\sum_n \langle f|T^\dagger|n\rangle \langle n|T|i\rangle. \] (C.34)

In the case of elastic electron-proton scattering the left- and right-hand sides of C.34 differ in magnitude. While in the left-hand side the transition amplitude is of the order of \( e^3 \), the right-hand side is of order \( e^4 \) since the state \(|n\rangle \) contains at least an electron. In the one-photon approximation one has then that
\[ \langle f|T|i\rangle \approx -\langle i|T|f\rangle^*. \] (C.35)

Assuming further that time reversal invariance holds, one has
\[ \langle i|T|f\rangle = \langle fT|i\rangle, \] (C.36)
with the states \(|iT\rangle \) and \(|fT\rangle \) containing the same particles as states \(|i\rangle \) and \(|f\rangle \) but opposite momenta and spin projections. Unitary of the S-matrix and time reversal invariance lead to the relation
\[ \langle f|T|i\rangle = -\langle fT|iT\rangle^*. \] (C.37)

Applying this expression to C.3 gives
\[ \bar{U}(k', s') \gamma_\mu U(k, s) \langle p', S'|J^\mu |p, S \rangle = (\bar{U}(k_T', -s') \gamma_\mu U(k_T, -s) \langle p_T', -S'|J^\mu |p_T, -S \rangle)^*, \] (C.38)

where the index \( T \) recalls that the time reversal changes the sign of the momenta, \( k_T = (k^0, -\mathbf{k}) \).

One may next use the matrix element of the vector current as in (C.31) and the time reversal operator \( T \), which acts on the spinors according to
\[ TU(k, s) = e^{\imath \alpha(k, s) U^*(k_T, -s)}. \] (C.39)

One is thus able to write, not including the phases for simplicity,
\[ \bar{U}(k', s') \gamma_\alpha U(k, s) \bar{u}(p', s') F_1(q^2) \gamma^\alpha u(p, s) \]
\[ = \bar{U}^* (k_T', -s') \gamma_\alpha T^{-1} U^* (k_T, -s) \bar{u}^* (p_T', -s') F_1(q^2) T \gamma^\alpha T^{-1} u^* (p_T, -s) \]
\[ = \bar{U}^* (k_T', -s') \gamma^\alpha U^* (k_T, -s) \bar{u}^* (p_T', -s') F_1(q^2) \gamma^\alpha u^* (p_T, -s))^* \]
\[ = (\bar{U} (k_T', -s') \gamma_\alpha U (k_T, -s) \bar{u} (p_T', -s') F_1(q^2) \gamma^\alpha u (p_T, -s))^* \] (C.40)

using the transformation properties of the \( \gamma \) matrices under time reversal
\[ T\gamma_\mu T^{-1} = \gamma^{\mu*}, \] (C.41)
with the conclusion that \( F_1 \) is real, \( F_1^* = F_1 \). For \( F_2 \), taking into account that
\[ T\sigma^{\mu\nu} T^{-1} = -\sigma^{\mu\nu}_{*}, \] (C.42)
one finds, similarly,
\[ \bar{u}(p', s') i \frac{F_2(q^2)}{2M} \delta^{\alpha\beta} q_\beta u(p, s) = \bar{u}^*(p_T', -s') i \frac{F_2(q^2)}{2M} T\sigma^{\alpha\beta} T^{-1} q_\beta u^*(p_T, -s) \]
\[ = \bar{u}^*(p_T', -s') (-i) \frac{F_2(q^2)}{2M} (\sigma^{\alpha 0} q^0 - \sigma^{\alpha i} q^i) u^*(p_T, -s) \]
\[ = \bar{u}^*(p_T', -s') (-i) \frac{F_2(q^2)}{2M} \sigma^{\alpha i} q^i u^*(p_T, -s) \]
\[ = \left( \bar{u}^*(p_T', -s') \frac{F_2(q^2)}{2M} \sigma^{\alpha i} q^i u(p_T, -s) \right)^* \]
which also indicates that \( F_2 \) is real.

C.3 The Rosenbluth formula and Sachs form factors

In order to evaluate the traces in the hadronic tensor (C.13), it is better to have as few as possible \( \gamma \) matrices. One may thus use the Gordon identity (C.30) to obtain
\[ \bar{u}(p', s') \Gamma^\mu u(p, s) = \bar{u}(p', s') \left[ F_1 \gamma^\mu + \frac{1}{2M} F_2 i \sigma^{\mu\nu} q_\nu \right] u(p, s) \]
\[ = \bar{u}(p', s') \left( G_M \gamma^\mu + F (p' + p)^\mu \right) u(p, s) \]
(C.44)
defining the quantities
\[ G_M = F_1 + F_2, \quad F = -\frac{F_2}{2M} \]  
(C.45)

Using (C.44), the evaluation of the traces in the hadronic tensor lead to
\[ \frac{1}{4} \text{tr} \left[ \Gamma^\mu \left( \not{p} + M \right) \Gamma^\nu \left( \not{p'} + M \right) \right] = \frac{1}{2} G_M^2 q^2 \left( g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \]
\[ + \rho^{\mu\nu} \left[ \frac{1}{2} G_M^2 + 2 M G_M F + 2 M^2 F^2 \left( 1 - \frac{q^2}{4 M^2} \right) \right] \]
(C.46)
since \( p^\mu p'^\nu + p' \rho^{\mu\nu} = (P^\mu P'^\nu - q^\mu q^\nu) / 2 \). Making the definitions
\[ G_E = F_1 + \frac{q^2}{4M} F_2 = F_1 - \tau F_2, \]
\[ \tau = \frac{-q^2}{4M} \]
(C.47a)
it is then possible to make the simplifications
\[ 2 M F = -F_2 = -\frac{G_M - G_E}{1 + \tau} = \frac{G_E - G_M}{1 + \tau} \]
(C.48)
and
\[ \frac{1}{2} G_M^2 + 2 M N G_M F + 2 M^2 F^2 \left( 1 - \frac{q^2}{4 M^2} \right) = \frac{G_E^2 + \tau G_M^2}{2(1 + \tau)} \]
(C.49)

With the definition (C.47a) the trace part of the hadronic tensor becomes
\[ \frac{1}{4} \text{tr} \left[ \Gamma^\mu \left( \gamma^\alpha \not{p}_a + M_N \right) \Gamma^\nu \left( \gamma^\beta \not{p} + q \right)_\beta + M_N \right] = \]
\[ = \frac{1}{2} G_M^2 q^2 \left( g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) + \left( p^\mu - \frac{p \cdot q}{q^2} q^\mu \right) \left( p'^\nu - \frac{p' \cdot q}{q^2} q'^\nu \right) \]
\[ \frac{G_E^2 + \tau G_M^2}{2(1 + \tau)} \]
(C.50)
since \( q^2 = -2 p \cdot q \) and
\[ P^\mu = (p + p')^\mu = 2 \left( p + \frac{1}{2} q \right)^\mu = 2 \left( p^\mu - \frac{p \cdot q}{q^2} q^\mu \right) \]
(C.51)

In the laboratory reference frame, neglecting the electron mass, with the initial and final states are on mass shell, the momenta are
\[ k = (E, k), \quad k' = (E', k'), \quad p = (M, 0), \]
(C.52a)
\[ k^2 = E^2 - k^2 = m^2 \simeq 0 \rightarrow E^2 \simeq k^2, \quad E'^2 \simeq k'^2, \]  
(C.52b)

and the products of momenta are

\[ p \cdot k = ME, \quad p \cdot k' = ME', \quad p \cdot q = p \cdot (k - k') = M \left( E - E' \right) \]  
(C.53a)

\[ k \cdot k' = EE' - |k||k'| \cos \theta \simeq EE' (1 - \cos \theta) = 2EE' \sin^2 \frac{\theta}{2}, \]  
(C.53b)

\[ k \cdot q = = k - k \cdot k' \simeq -k \cdot k', \quad k' \cdot q \simeq k \cdot k', \]  
(C.53c)

\[ q^2 = (k - k')^2 = k^2 + k'^2 - 2k \cdot k' \simeq -4EE' \sin^2 \frac{\theta}{2}, \]  
(C.53d)

with \( \theta \) is the angle between in- and outgoing electron in the laboratory system.

For the averaged amplitude, in terms of the form factors,

\[
\int \frac{d^4 p'}{p^0} \delta^{(4)} (p' + k' - p - k) |\langle p', k' | T | p, k \rangle|^2 = \frac{e^4}{q^4} L_{\mu \nu} (k', k) \frac{W^{\mu \nu} (p, q)}{M^2} \sin^2 \frac{\theta}{2}, \]  
(C.54)

using the following results in the Breit frame

\[
(k_{\mu} k_{\mu}' + g_{\nu \mu} k_{\nu} k'_{\mu} - g_{\nu \mu} k_{\nu} k'_\mu) \left( g^{\mu \nu} - \frac{q^{\mu} q^{\nu}}{q^2} \right) \simeq q^2 = -4EE' \sin^2 \frac{\theta}{2} \]  
(C.55a)

\[
(k_{\mu} k_{\mu}' + g_{\nu \mu} k_{\nu} k'_{\mu} - g_{\nu \mu} k_{\nu} k'_\mu) \left( p^\mu - \frac{p \cdot q}{q^2} q^\mu \right) \left( p'^\nu - \frac{p \cdot q}{q^2} q'^\nu \right) \simeq 2M^2 EE' \cos^2 \frac{\theta}{2} \]  
(C.55b)

The cross section becomes

\[
\frac{d\sigma}{d\Omega_{k'}} = \frac{\alpha^2}{E E' dE' d\Omega_{k'}} \frac{\alpha^2}{(-4EE' \sin^2 \frac{\theta}{2})^2} \frac{E E'}{E'} \left( 2G_M^2 \sin^2 \frac{\theta}{2} + \frac{G_E^2 + \tau G_M^2}{1 + \tau} \cos^2 \frac{\theta}{2} \right) \delta \left( \nu + \frac{q^2}{2M} \right), \]  
(C.56)

since

\[
\frac{d^2 k'}{k'^4} \simeq \frac{|k|^2 d |k| d\Omega_{k'}}{E E'} = \frac{E' dE' d\Omega_{k'}}{E'} = E' dE' d\Omega_{k'}. \]  
(C.57)

Integrating over the unobserved energy of the outgoing electron, one finally arrives to the so-called Rosenbluth formula [36] for the electron-proton elastic cross section:

\[
\frac{d\sigma}{d\Omega_{k'}} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2} \left( 1 + \frac{2G_M^2}{M^2} \sin^2 \frac{\theta}{2} \right)} \left( \frac{G_E^2 + \tau G_M^2}{1 + \tau} + 2\tau G_M^2 \tan^2 \frac{\theta}{2} \right), \]  
(C.58)

with \( (d\sigma/d\Omega)_{\text{Mott}} \) the so-called Mott cross section

\[
\frac{\left( d\sigma \right)}{d\Omega_{k'}} = \frac{\alpha^2 \cos^2 \frac{\theta}{2}}{4E^2 \sin^4 \frac{\theta}{2} \left( 1 + \frac{2G_M^2}{M^2} \sin^2 \frac{\theta}{2} \right)}, \]  
(C.59)

which describes the scattering off a structureless spinless pointlike particle.

The advantage of the Sachs form factors over the Dirac and Pauli form factors is already seen at the level of the cross section. The are no mixed terms like \( G_E G_M \) while there are terms in \( F_1 F_2 \) in

\[
\frac{d\sigma}{d\Omega_{k'}} = \left( \frac{d\sigma}{d\Omega_{k'}} \right)_{\text{Mott}} \left[ F_1^2 + \tau F_2^2 + 2\tau (F_1 + F_2)^2 \tan^2 \frac{\theta}{2} \right], \]  
(C.60)

### C.4 Definition of the electric and magnetic form factors

In the Breit frame, defined by the momenta of the initial state \( p = (E, -q/2) \) and of the final state \( p' = (E, q/2) \). In this frame there is no energy transferred from the initial to the final state:

\[ q = p' - p = (0, q). \]

The spinor of the free Dirac equation may be written as

\[ u(p, s) = \sqrt{E + M \left( \frac{\sigma \cdot p}{E + M} \phi_s \right)}. \]  
(C.61)
In the Breit frame, using the current in the form (C.31)

\[
\langle (E, q/2), S' | J^\mu | (E, -q/2), S \rangle = \bar{u} ((E, q/2), S') \left( F_1 \gamma^\mu + \frac{1}{2M} F_2 i \sigma^{\mu \nu} q_\nu \right) u ((E, -q/2), S)
\]

\[
= \delta^{00} \bar{u} ((E, q/2), S') \left( F_1 \gamma^0 + \frac{1}{2M} F_2 i \sigma^{0i} q_\nu \right) u ((E, -q/2), S)
\]

\[
+ \delta^{ik} \bar{u} ((E, q/2), S') \left( F_1 \gamma^k + \frac{1}{2M} F_2 i \sigma^{k j} q_j \right) u ((E, -q/2), S),
\]

(C.62)


together with the spinor (C.61) and the relations

\[
i \sigma^{0i} q_i = \begin{pmatrix} 0 & \sigma \cdot q \\ \sigma \cdot q & 0 \end{pmatrix}, \quad i (\Sigma \times q)^k \quad \text{with} \quad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix},
\]

one may try to express the matrix elements of the currents in terms of form factors,

\[
\langle (E, q/2), S' | J^\mu | (E, -q/2), S \rangle = \delta^{00} F_1 \left( \frac{E + M}{2M} \phi_{S'}^\dagger \phi_S - \frac{1}{2M (E + M)} \phi_{S'}^\dagger \sigma \cdot q \phi_S \right)
\]

\[
- \delta^{00} \frac{F_2}{4M^2} \phi_{S'}^\dagger \sigma \cdot q \phi_S + \delta^{0k} \frac{1}{2M} F_1 \phi_{S'}^\dagger \left[ \sigma, \sigma^k \right] \cdot q \phi_S
\]

\[
+ \delta^{0k} \frac{F_1}{2M} \phi_{S'}^\dagger \left[ i (\sigma \times q)^k + \frac{\sigma \cdot q/2}{E + M} (E + M) - \frac{q^2}{4M^2} F_2 \phi_{S'}^\dagger \phi_S \right]
\]

\[
+ \delta^{0k} \frac{F_2}{4M^2} \phi_{S'}^\dagger \left[ i (\sigma \times q)^k + \frac{q^2}{4M^2} F_2 \phi_{S'}^\dagger \phi_S \right]
\]

\[
= \delta^{00} F_1 \phi_{S'}^\dagger \phi_S \left( \frac{E + M}{2M} \phi_{S'}^\dagger \phi_S - \phi_{S'}^\dagger \phi_S \right) - \delta^{00} \frac{q^2}{4M^2} F_2 \phi_{S'}^\dagger \phi_S
\]

\[
+ \delta^{0k} \frac{F_1}{2M} \phi_{S'}^\dagger \left[ i (\sigma \times q)^k + \frac{q^2}{4M^2} F_2 \phi_{S'}^\dagger \phi_S \right]
\]

\[
+ \delta^{0k} \frac{F_2}{4M^2} \phi_{S'}^\dagger \left[ i (\sigma \times q)^k + \frac{q^2}{4M^2} F_2 \phi_{S'}^\dagger \phi_S \right]
\]

(C.64)

Using the following relations for the momenta in the Breit frame

\[
q = p' - p = (0, q), \quad q^2 = -q^2 = -4 (E^2 - M^2)
\]

(C.65a)

\[
p^2 = E^2 - \left( \frac{q}{2} \right)^2 = M^2
\]

(C.65b)

and

\[
\sigma \cdot q (\sigma \times q)^k \sigma \cdot q = \varepsilon^{mnk} \sigma^m \sigma^n q^i q^j = -q^2 (\sigma \times q)^k,
\]

(C.66)

since the Pauli matrices fulfill

\[
\sigma^i \sigma^m \sigma^j = \delta^{im} \sigma^j - \delta^{ij} \sigma^m + \delta^{mj} \sigma^i + i \varepsilon^{imj},
\]

(C.67)

one finally finds

\[
\langle (E, q/2), S' | J^\mu | (E, -q/2), S \rangle = \delta^{00} F_1 \phi_{S'}^\dagger \phi_S - \delta^{00} \frac{q^2}{4M^2} F_2 \phi_{S'}^\dagger \phi_S
\]

\[
+ \delta^{0k} \frac{1}{2M} F_1 \phi_{S'}^\dagger [i (\sigma \times q)^k + \frac{q^2}{4M^2} F_2 \phi_{S'}^\dagger \phi_S]
\]

\[
- \delta^{0k} \frac{F_2}{4M^2} (E - M) \phi_{S'}^\dagger [i (\sigma \times q)^k + \frac{q^2}{4M^2} F_2 \phi_{S'}^\dagger \phi_S]
\]

\[
= \delta^{00} \left( F_1 - \frac{q^2}{4M^2} F_2 \right) \phi_{S'}^\dagger \phi_S + \delta^{0k} \left( F_1 + F_2 \right) \frac{1}{2M} \phi_{S'}^\dagger [i (\sigma \times q)^k + \frac{q^2}{4M^2} F_2 \phi_{S'}^\dagger \phi_S]
\]

(C.68)

Looking at (C.68) one immediately recognizes the Sachs form factors, already defined in the previous section, which are found to be related to matrix elements of the components of the electromagnetic current in the Breit frame (\(Q^2 = -q^2 = q'^2\)):

\[
\langle p', S' | J^0 | p, S \rangle = \left( F_1 - \frac{q^2}{4M^2} F_2 \right) \phi_{S'}^\dagger \phi_S = (F_1 - \tau F_2) \phi_{S'}^\dagger \phi_S = G_E (Q^2) \phi_{S'}^\dagger \phi_S,
\]

(C.69a)

\[
\langle p', S' | J^k | p, S \rangle = (F_1 + F_2) \frac{1}{2M} \phi_{S'}^\dagger [i (\sigma \times q)^k + \frac{q^2}{4M^2} F_2 \phi_{S'}^\dagger \phi_S] = G_M (Q^2) \frac{1}{2M} \phi_{S'}^\dagger [i (\sigma \times q)^k + \frac{q^2}{4M^2} F_2 \phi_{S'}^\dagger \phi_S].
\]

(C.69b)
C.5 Definition of the axial-vector form factors

It is also possible to follow the same kind of steps of the previous sections in the case of the axial-vector currents $A^\alpha$. The aim of this section is again to find how a matrix element of the axial-vector current between 1/2-spin states of different momenta (for instance the octet baryons) may be decomposed in terms of form factors. Now we have $A^\alpha = \psi^\dagger 5\gamma^\alpha \gamma^5 \lambda^\chi/2\psi$.

$$
\langle B(p', s') | A^\alpha(0) | B(p, s) \rangle = \bar{u}(p', s') \Gamma_B^\alpha (p', p) u(p, s).
$$

From Section C.2, it is easy to see from (C.16) that the decomposition (C.24) based on Lorentz covariance also holds in the case of the axial current since the right-hand side of (C.16) has exactly the same form when it is multiplied by $\gamma^5$, with only the role of the tensors $A, B, C, D$ and $E$ interchanged. Using the result for the vector current (C.24)

$$
\bar{u}(p', s') \Gamma_B^\alpha (p', p) u(p, s) = \bar{u}(p', S_3') \left( G_A^{(x)}(q^2)\gamma^\alpha \gamma^5 + \frac{G_P^{(x)}(q^2)}{2M} \gamma^\alpha q^\chi - \frac{G_T^{(x)}(q^2)}{2M} P^\alpha \right) \gamma^5 u(p, S_3).
$$

where the form factors, functions of $(q^2)$, are the axial form factor $G_A$, the pseudoscalar form factor $G_P$. From factors $G_A$ and $G_P$ are of the first kind regarding $G$ parity, while $G_T$ is of the second kind. It may be shown, following the same steps of Section C.2 in proving $F_1$ and $F_2$ were real, that $G_T$ vanishes identically for all $q^2$, provided hermiticity of the axial current and time reversal hold. One may thus write

$$
\langle B(p', s') | A^\alpha(0) | B(p, s) \rangle = \bar{u}(p', S_3') \left( G_A^{(x)}(q^2)\gamma^\alpha \gamma^5 + \frac{G_P^{(x)}(q^2)}{2M} \gamma^\alpha q^\chi \right) u(p, S_3).
$$

Again the expression for the form factors in terms of the matrix elements will be worked out in the Breit frame. For the energy, in this frame and on mass-shell, form (C.65b), one has $E = \sqrt{M^2 + q^2}/4$. Inserting the free spinor in the form (C.61) one finds that for the space components (the time component vanishes in the Breit frame)

$$
\langle B(p', S_3') | A^\chi(x) | B(p, S_3) \rangle = \mathcal{N}(E + M)
$$

$$
\times \phi^\dagger(S_3') \left[ G_A^{(x)}(q^2) \left( \sigma - \frac{\sigma \cdot q}{(E + M)} \right) - \frac{G_P^{(x)}(q^2)}{2M} \frac{\sigma \cdot q}{(E + M)} \right] \phi(S_3).
$$

where $\mathcal{N}$ is the normalization constant of the spinor $u$ and $\phi$ is a two component spinor. Introducing the definitions of tranverse and longitudinal projections for $\sigma$

$$
\sigma_L = \hat{q} (\sigma \cdot \hat{q}), \quad \sigma_T = \sigma - \sigma_L
$$

one obtains with the Breit frame relation (C.5)

$$
\int d^4x \mathcal{L}^{i\chi} \mathcal{L} \langle B(S_3') | A^\chi(x) | B(S_3) \rangle = \phi^\dagger(S_3') \left[ \frac{E}{M} G_A^{(x)}(q^2) \sigma_T + \left( G_A^{(x)}(q^2) - \frac{|q|^2}{4M^2} G_P^{(x)}(q^2) \right) \sigma_L \right] \frac{\lambda^\chi}{2} \phi(S_3).
$$

since

$$
\langle B(p', S_3') | A^\chi(0) | B(p, S_3) \rangle = \int d^4x e^{-i\mathbf{q} \cdot \mathbf{x}} \langle B(S_3') | A^\chi(x) | B(S_3) \rangle.
$$

To extract the axial form factor from (C.75), one applies to both sides the external product from the left with $q$ twice and performs an average over the orientations of $q$ ($\Omega_q$). For the left-hand side one finds

$$
\frac{1}{4\pi} \int d\Omega_p \int d^4x e^{-i\mathbf{q} \cdot \mathbf{x}} \mathcal{L} \langle B(S_3') | A^\chi(x) | B(S_3) \rangle = \frac{1}{4\pi} \int d\Omega_p \int d^4x e^{-i\mathbf{q} \cdot \mathbf{x}} \left( q \cdot \langle B(S_3') | A^\chi(x) | B(S_3) \rangle - |q|^2 \langle B(S_3') | A^\chi(x) | B(S_3) \rangle \right).
$$
For the average over the orientation of $q$ writing
\[ e^{-i q \cdot x} = \sum_{lm} 4 \pi i^{l} j_{l} (|q| r) Y_{lm}^{*} (\hat{q}) Y_{lm} (\hat{r}) \] (C.78a)
\[
\hat{q} = \sqrt{\frac{4\pi}{3}} \sum_{\mu} (-1)^{\mu} Y_{1-\mu} (\theta_{q}, \phi_{q}) e_{\mu}
\] (C.78b)
the first term of (C.77) becomes
\[
\frac{1}{4 \pi} \int d\Omega_{p} \int d^{3}x e^{-i q \cdot x} q \langle q \cdot (N(S_{0}) | A^{\chi}(x) | N(S_{3})) \rangle = \frac{1}{3} |q|^{2} \int d^{3}x j_{0} (|q| r) \langle N(S_{0}) | A^{\chi}(x) | N(S_{3}) \rangle
\]
\[+ \frac{\sqrt{2}}{3} \sqrt{\frac{4\pi}{3}} |q|^{2} \int d^{3}x j_{2} (|q| r) \langle N(S_{3}) | \{ Y_{2} (\hat{x}) \otimes A_{1a}(x) \} | N(S_{3}) \rangle
\] (C.79)
and the second term of (C.77)
\[
\frac{1}{4 \pi} |q|^{2} \int d\Omega_{p} \int d^{3}x e^{-i q \cdot x} \langle B(S_{0}) | A^{\chi}(x) | B(S_{3}) \rangle = |q|^{2} \int d^{3}x j_{0} (|q| r) \langle B(S_{0}) | A^{\chi}(x) | B(S_{3}) \rangle
\] (C.80)

taking into consideration the results \(^{1}\):
\[
\frac{1}{4 \pi} \int d\Omega_{p} e^{-i q \cdot x} = j_{0} (|q| r), \quad (C.82a)
\]
\[
\frac{1}{4 \pi} \int d\Omega_{p} e^{-i q \cdot x} q^{i} q^{j} = \delta^{ij} |q|^{2} \left( \delta^{31} + \delta^{32} - \delta^{33} \right) j_{2} (|q| r) + \delta^{i3} j_{2} (|q| r). \quad (C.82b)
\]

The average over the right hand side of (C.75) yields
\[
\frac{1}{4 \pi} \int d\Omega_{p} \delta^{i \dagger} (S_{3}) \left[ \frac{E}{M} G_{A}^{(a)} (q^{2}) q \times (q \times \sigma_{T}) \right.
\]
\[+ \left. \left( G_{A}^{(a)} (q^{2}) - \frac{|q|^{2}}{4M^{2}} G_{P}^{(a)} (q^{2}) \right) q \times (q \times \sigma_{L}) \right] \phi (S_{3})
\] (C.83)
using the following results ($q \times \sigma_{L} = 0$):
\[
\frac{1}{4 \pi} \int d\Omega_{p} q \times (q \times \sigma_{T}) = - \frac{2}{3} |q|^{2} \sigma, \quad \frac{1}{4 \pi} \int d\Omega_{p} q^{i} q^{j} = \frac{1}{3} \delta^{ij} |q|^{2}. \quad (C.84)
\]
Collecting the previous results (C.80,C.81,C.83) into (C.75) allows finally to write, from the third space component of the current, the expression for the form factor
\[
G_{A}^{(a)} (q^{2}) = \frac{M}{E} \int d^{3}x (j_{0} (|q| r) \langle B(S_{3}) | A^{3\chi}(x) | B(S_{3}) \rangle
\]
\[- \sqrt{2} \pi j_{2} (|q| r) \langle B(S_{3}) | \{ Y_{2} (\hat{r}) \otimes A_{1}(x) \} | B(S_{3}) \rangle). \quad (C.85)
\]

### C.6 Expression for radii

Interpreting the form factor $G(q^{2})$ as the Fourier transform of a spherical spatial charge distribution $\rho(r)$, i.e. taking $q = |q| \hat{z}$,
\[
G(q^{2}) = \int d^{3}r \rho(r) e^{i q \cdot r} = \int dr r^{2} \rho(r) \sum_{n=0}^{\infty} |q|^{n} r^{n} \int d\Omega_{r} \cos^{n}(\theta)
\] (C.86)

\(^{1}\) $j_{2} (z) = \frac{2}{5} j_{1} (z) - j_{0} (z)$
considering \( q = |q| \hat{z} \). Using the result \( \int d\Omega, \cos^n(\theta) = 4\pi \left[ 1 - (-1)^{n+1} \right] / (n + 1) \), the form factor may be expanded at small \( |q|^2 \) and it is expressed as

\[
G(q^2) = G(0) \left( 1 - \frac{1}{6} |q|^2 \langle r^2 \rangle + \cdots \right),
\]

(C.87)

with the help of the definitions of the root mean square radius and total charge

\[
\langle r^2 \rangle = \frac{1}{G(0)} \int dr \, r^4 \rho(r) \quad \text{and} \quad G(0) = \int dr \, r^2 \rho(r).
\]

(C.88)

The mean quadratic radius of the distribution under consideration \( \langle r^2 \rangle \), usually designated as simply as radius, is given by

\[
\langle r^2 \rangle = -\frac{6}{G(0)} \frac{dG(|q|^2)}{d|q|^2} \bigg|_{|q|^2 = 0}.
\]

(C.89)

In the Breit frame \( Q^2 = -q^2 = |q|^2 \), so that

\[
\langle r^2 \rangle = -\frac{6}{G(0)} \frac{dG(Q^2)}{dQ^2} \bigg|_{Q^2 = 0},
\]

(C.90)

In the case \( G(0) = 0 \) one defines the radius from the slope, i.e. one replaces \( G(0) \) by 1 in (C.90).

### 7 Form factors in the \( N-\Delta \) transition

The matrix element (4.1) may be expressed in terms of form factors just as in the case of the elastic form factors. The difference in this case is that the \( \Delta \) is a spin 3/2 particle which must thus be described by a Rarita-Schwinger spinor instead of a Dirac one. The general expression [203,264] for the matrix element, following the same kind of reasoning of Appendix C, is

\[
\langle \Delta (p', s'_3) | J^{\mu} (0) | N (p, s_3) \rangle = \bar{u}_\beta (p', s'_3) \Gamma_\alpha \beta_{N-\Delta} u (p, s_3)
\]

\[
= \bar{u}_h (p', s'_3) \left[ G_M^M (q^2) H^M_M (q, P) + G_E^E (q^2) H^E_E (q, P) + G_C^C (q^2) H^C_C (q, P) \right] u (p, s_3)
\]

(C.91)

where \( q = p' - p \) and \( P = p' + p \) and \( M \) refers to magnetic dipole, \( E \) for electric quadrupole, and \( C \) for scalar quadrupole.

In the case of the Breit-frame and under the approximation of equal masses (which simplifies the expressions, but restricts them to the leading order in \( N_c \)) for \( N \) and \( \Delta \), the same kind of steps pursed in the case of the elastic form factors, Section C.2, would now lead [265] from (C.91) to

\[
\langle \Delta (p', s'_3) | J^{0 \mu} (0) | N (p, s_3) \rangle = \frac{3i}{4M_N} G_C^* (q^2) q^{a \mu} q^3,
\]

(C.92a)

\[
\langle \Delta (p', s'_3) | J^1 \mu (0) | N (p, s_3) \rangle = \frac{3i}{4M_N} \left[ G_E^* (q^2) - G_M^* (q^2) \right] \epsilon^{ijk} q^j \left( C^2 \delta^i_{s'_3} - s_{s_3} \right) U_{s'_3 - s_3 k},
\]

\[
+ \frac{3i}{2M_N} G_E^* (q^2) \left( \delta_{ik} - \frac{q^i q^k}{q^2} \right) a^{kl} q^l,
\]

(C.92b)

\[
a^{kl} = \sum_m C^m_{1m} U_{mk} \langle m | \sigma \rangle | s_3 \rangle,
\]

(C.92c)

where \( U \) is the matrix relating the Cartesian and the spherical basis (A.30). Relations (C.92) may be used to express the form factors in terms of the matrix elements.
D Model description of baryon states

D.1 Baryon correlation function

The correlation functions for baryonic currents of the form (2.24) are defined in Euclidian space as matrix elements in the vacuum state of these currents:

\[ C_{B^A}(T) \equiv \langle 0 | J_{B^A}(x') J_B^\dagger(x) | 0 \rangle = \]

\( \frac{1}{(N_c)^2} \Gamma_{B^A}^{\{f_1,...,f_{N_c}\}} \Gamma_B^{\{g_{1},...,g_{N_c}\}} \varepsilon_{\alpha_1...\alpha_{N_c}} \varepsilon_{\beta_1...\beta_{N_c}} \times \langle 0 | T \left\{ \psi^\dagger_{f_1 \alpha_1}(x') \cdots \psi^\dagger_{f_N \alpha_N}(x') \psi_{g_N \beta_N}(x) \cdots \psi^\dagger_{g_1 \beta_1}(x) \right\} | 0 \rangle \). \hspace{1cm} (D.1)

where it assumed that in the original correlation function the time of \( x'^4 > x^4 \) function. It is then possible to write a path integral representation in Euclidian space for the \( 2N_c \) quark fields correlation

\( \langle 0 | T \left\{ \psi^\dagger_{f_1 \alpha_1}(x') \cdots \psi^\dagger_{f_N \alpha_N}(x') \psi_{g_N \beta_N}(x) \cdots \psi^\dagger_{g_1 \beta_1}(x) \right\} | 0 \rangle = \)

\( \mathcal{N} \lim_{-\infty \rightarrow x^4, x'^4 \rightarrow \infty} \int [dU] [d\psi] \psi_{f_1 \alpha_1}(x') \cdots \psi_{f_N \alpha_N}(x') \psi^\dagger_{g_1 \beta_1}(x) \psi^\dagger_{g_N \beta_N}(x) e^{\int_{-i\pi^4}^{i\pi^4} dy dy \psi^\dagger(y) D^E(U) \psi(y)} \), \hspace{1cm} (D.2)

since in Euclidian space the Lagrangian of the model as given by (2.4) changes to

\( \mathcal{L}_{CQSM} \rightarrow \mathcal{L}^E_{CQSM} = \psi^\dagger (x) \left( -i \gamma^4 \left( i \left( -i \gamma^4 \right) \frac{\partial}{\partial (i \pi^4)} + i \gamma^4 E \frac{\partial}{\partial (i \pi^4)} - m - MU^{75} \right) \right) \psi(x) \)

\( = -\psi^\dagger (x) \left[ \partial_4 + h^E(U) \right] \psi(x) = -\psi^\dagger (x) D^E(U) \psi(x) \) \hspace{1cm} (D.3)

with the one-particle Dirac Hamiltonian

\( h^E(U) = -\gamma^4 \gamma_4 \partial^E + i \gamma^4 M U^{75} \). \hspace{1cm} (D.4)

and the integration measure \( d^4x = -id^4x^E \). \mathcal{N} is a normalization constant.

In the following the limits for \( x^4, x'^4 \) will not be displayed explicitly for notational simplicity. This limit of large time separation allows to rewrite, with the purpose of making the two Grassmann algebras \( \psi \) and \( \psi^\dagger \) independent, the statistical factor according to

\( \int d^4y \psi^\dagger (y) D^E(U) \psi(y) = \int d^4y' d^4y \psi^\dagger (y') D^E_E(U; y'; y) \psi(y) \). \hspace{1cm} (D.5)

Henceforth we drop the superscript \( E \). Allowing for the introduction of external Grassmann sources for these fermion fields and defining

\( Z[\omega^*, \omega] = \int [dU] [d\psi^\dagger] [d\psi] e^{\int d^4y' d^4y \psi^\dagger (y') D^E(U; y'; y) \psi(y) + \int d^4y \left( \psi^\dagger (y) \omega(y) + \omega^* (y) \psi(y) \right)} \)

the expression for the ordered product of quark fields can be written as

\( \langle 0 | T \left\{ \psi^\dagger_{f_1 \alpha_1}(x') \cdots \psi^\dagger_{f_N \alpha_N}(x') \psi_{g_N \beta_N}(x) \cdots \psi^\dagger_{g_1 \beta_1}(x) \right\} | 0 \rangle = \mathcal{N} \frac{\delta}{\delta \omega_{g_1 \beta_1}(x)} \cdots \frac{\delta}{\delta \omega_{g_N \beta_N}(x)} \psi_{f_N \alpha_N}(x) \cdots \frac{\delta}{\delta \omega_{f_1 \alpha_1}(x')} \psi^\dagger_{f_1 \alpha_1}(x') Z[\omega^*, \omega] \psi_{\omega=0, \omega^*=0} \) \hspace{1cm} (D.6)
The Grassmann variables in $Z[\omega^*, \omega]$ are integrated out since the action is quadratic in them. Because the variables $\psi^\dagger$ and $\psi$ are independent, the result is

$$Z[\omega^*, \omega] = \int \det Nc [D(U)] e^{-\int d^4y d^4z \omega^*_s(y) G_{ss'}(U; y, z) \omega_{s'}(z)}$$  \hspace{1cm} (D.8)$$

with $G(U)$ the Euclidean propagator in coordinate space. Now the functional derivatives at the point $x'$ can be performed

$$\frac{\delta}{\delta \omega^f_{\mu, \alpha}(x')} \cdots \frac{\delta}{\delta \omega^f_{\mu, \alpha}(x')} Z[\omega^*, \omega] = \left( -1 \right)^{\sum_{k=1}^{Nc-1} k} \int [dU] \det Nc [D(U)] \left( \int d^4z_1 G_{f_1s}(U; x', z_1) \omega^{*}_{s' \alpha}(z_1) \right)$$

$$\times \cdots \times \int d^4z_Nc G_{f_Ns'}(U; x', z_Nc) \omega^{*}_{s' \alpha}(z_Nc) e^{-\int d^4y d^4z \omega^*_s(y) G_{ss'}(U; y, z) \omega_{s'}(z)}$$  \hspace{1cm} (D.9)$$

and then the functional derivatives at the other point, with vanishing contributions coming from the action of the derivatives on the exponential. For the sum over the color indices, using

$$\epsilon_{\beta_1 \cdots \beta_{Nc}} \sum_{\theta \in S_{Nc}} (-1)^{\pi(\theta)} \delta_{\alpha_1 \beta_1} \cdots \delta_{\alpha_{Nc} \beta_{Nc}} \epsilon_{\alpha_1 \cdots \alpha_{Nc}} = (Nc)!^2$$  \hspace{1cm} (D.10)$$

where $\pi(\theta)$ is the parity of the permutation, one obtains

$$\frac{\delta}{\delta \omega^g_{\mu_1}(x')} \cdots \frac{\delta}{\delta \omega^g_{\mu_1}(x')} = \int d^4z_1 G_{f_1s'}(U; x', z_1) \omega^{*}_{s' \alpha}(z_1) \cdots = \int d^4z_Nc G_{f_Ns'}(U; x', z_Nc) \omega^{*}_{s' \alpha}(z_Nc) (-1)^{\sum_{k=1}^{Nc-1} k}$$

$$\times \sum_{\theta \in S_{Nc}} (-1)^{\pi(\theta)} G_{f_1g_1}(U; x', x) \delta_{\alpha_1 \beta_1} \cdots \delta_{\alpha_{Nc} \beta_{Nc}}$$  \hspace{1cm} (D.11)$$

and the final result in Euclidean space is

$$c_{B'B}(T) \equiv \langle 0 | J_{B'}(x') J_{B}^I(x) | 0 \rangle^E = N \Gamma_{B'}^{(f_1, \cdots, f_{Nc})} \Gamma_{B}^{(g_1, \cdots, g_{Nc})} \times \int [dU] \det Nc [D^E(U)] \prod_{j=1}^{Nc} G_{f_jg_j}^E(U; x', x).$$  \hspace{1cm} (D.13)$$

D.2 The effective action

From the last section, the functional integral over $U$ in (D.13) is the next issue. For the purposes of both applying the saddle point condition and also the semi-classical quantization of zero modes, this section is devoted to the calculation of

$$e^{Nc \text{Tr} \log [D^E(U)]} \prod_{j=1}^{Nc} G_{f_jg_j}^E(U; x', x) = e^{S_{ef}}.$$  \hspace{1cm} (D.14)$$

The calculation of this expression may be divided into that of the product of propagers and that of the functional determinant. One particularly finds that the product of propagers contributes both to the effective action and with a product of valence functions which are to be identified later as making part of the collective wave function in the context of the semi-classical quantization (2.51).

D.2.1 Product of the $Nc$ quark propagers

The starting point will be a single propagator:

$$G_{f_1g_1}(U; x', x) \equiv (T/2, x' | G_f(U) | -T/2, x)$$  \hspace{1cm} (D.15)$$

$$G_{f_1g_1}(U; x', x) \equiv (T/2, x' | G_f(U) | -T/2, x) = \langle T/2, x' | e^{i \not{P} A(t)} G_{f_1g_1}(U_c) A^\dagger(t) e^{-i \not{P}} | -T/2, x \rangle$$

$$\langle T/2, x' | e^{i \not{P} A(t)} G_{f_1g_1}(U_c) A^\dagger(t) e^{-i \not{P}} | -T/2, x \rangle$$
where the limit of time independent angular velocity was taken and it was used that
\[ g(\Omega, \delta m) \]
\[ G(\Omega, \delta m) \]
where \( P \) is the momentum operator, generator of the translations.

\[
g(\Omega^0, \delta m^0) = \sum_{n \geq 0} e^{-\varepsilon_n T} P_T \left( A \langle x' + z | n \rangle \right)_f \langle n \mid x + z \rangle A^\dagger \rangle_{g_k} P_T
\]

\[
\mathop{\longrightarrow}_{T \to \infty} e^{-\varepsilon_n T} P_T \left( A \phi_\nu(x' + z) \right)_f \langle \phi_\nu(x + z) A^\dagger \rangle_{g_k} P_T
\] (D.17)

\[
g(\Omega^1, \delta m^0) = \frac{1}{2} e^{-(\varepsilon_n + \varepsilon_m) T / 2}
\]

\[
\times \sum_{n, m \geq 0} P_T \left( A \langle x' + z | n \rangle \right)_f \langle n \mid x + z \rangle A^\dagger \rangle_{g_k} \int d\tau i\Omega_E^2(\tau) e^{-(\varepsilon_m - \varepsilon_n) \tau}
\]

\[
\mathop{\longrightarrow}_{T \to \infty} \frac{1}{2} e^{-\varepsilon_n T} P_T \left( A \phi_\nu(x' + z) \right)_f \langle \phi_\nu(x + z) A^\dagger \rangle_{g_k} P_T
\]

\[
\mathop{\longrightarrow}_{T \to \infty} \frac{1}{2} e^{-\varepsilon_n T} P_T \left( A \phi_\nu(x' + z) \right)_f \langle \phi_\nu(x + z) A^\dagger \rangle_{g_k} P_T
\] (D.18)

where the limit of time independent angular velocity was taken and it was used that \( \langle v \mid \tau^i \mid v \rangle = 0 \) and

\[
G^S_{\nu} P A^\dagger (t) \mid -T / 2, x + z \rangle \mathop{\longrightarrow}_{T \to \infty} 0
\] (D.19)

\[
g(\Omega^2, \delta m^1) = T e^{-(\varepsilon_n + \varepsilon_m) T / 2}
\]

\[
\times \sum_{n, m \geq 0} P_T \left( A \langle x' + z | n \rangle \right)_f \langle n \mid x + z \rangle A^\dagger \rangle_{g_k} P_T
\]

\[
\mathop{\longrightarrow}_{T \to \infty} T e^{-\varepsilon_n T} P_T \left( A \phi_\nu(x' + z) \right)_f \langle \phi_\nu(x + z) A^\dagger \rangle_{g_k} P_T
\]

\[
\mathop{\longrightarrow}_{T \to \infty} T e^{-\varepsilon_n T} P_T \left( A \phi_\nu(x' + z) \right)_f \langle \phi_\nu(x + z) A^\dagger \rangle_{g_k} P_T
\] (D.20)

For the second order in angular velocity, now dropping the spin-isospin indices for clarity, and remembering \( A^\dagger A(\tau_1) = \frac{1}{2} i \Omega_E^2(\tau_1) \lambda^\alpha, P_T A^\dagger A \lambda A P_T = M_1 + M_2 D^{(8)}_{88} \tau^1 + M_2 D^{(8)}_{88} / \sqrt{3} \)

\[
g(\Omega^2, \delta m^1) = \frac{1}{4} \int d\tau_1 \int d\tau_2 P_T \langle T / 2, x' + z \mid A G T \mid \tau_1 \rangle i \Omega_E^2(\tau_1) \lambda^\alpha \langle \tau_1 |
\]

\[
\times (G^T P_T \otimes G^S P_S) \mid \tau_2 \rangle i \Omega_E^2(\tau_2) \lambda^\beta \langle \tau_2 | G^T A^\dagger \mid -T / 2, x + z \rangle_{f_k g_k} P_T
\]

\[
\mathop{\longrightarrow}_{T \to \infty} \frac{1}{4} e^{-\varepsilon_n T} P_T \left( A \langle x' + z | v \rangle \right)_f \langle \phi_\nu(x + z) A^\dagger \rangle_{g_k} P_T
\]

\[
\times \left\{ \sum_{n \neq V} \frac{1}{\varepsilon_n - \varepsilon_V} \langle v \mid \tau^\alpha \mid n \rangle \langle n \mid \tau^\beta \mid v \rangle \int_{-T/2}^{T/2} d\tau i \Omega_E^2 i \Omega_E^2 - \frac{1}{6} T \int_{-T/2}^{T/2} d\tau i \Omega_E^2 i \Omega_E^2
\]

\[
+ \frac{1}{2} \sum_{n^0} \frac{1}{\varepsilon_n^0 - \varepsilon_V} \langle v | n^0 \rangle \langle n^0 | v \rangle \int_{-T/2}^{T/2} d\tau (i \Omega_E^2 i \Omega_E^2 + i \Omega_E^2 i \Omega_E^2) \right\}
\] (D.21)
The next terms to consider are of mixed order in $\delta m$ and angular velocity:

$$
g^{(\Omega, \delta m)} = \langle T/2, \mathbf{x} | e^{i\mathbf{z} \cdot \mathbf{P}} AG(U_c) A^\dagger \hat{A} G(U_c) i\gamma_4 A^\dagger \delta m AG(U_c) A^\dagger(t) e^{-i\mathbf{z} \cdot \mathbf{P}} \mathbf{|} -T/2, \mathbf{x} \rangle_{f_9 g_k}$$

For the first term

$$\langle T/2, \mathbf{x} | e^{i\mathbf{z} \cdot \mathbf{P}} AG(U_c) A^\dagger \hat{A} G(U_c) i\gamma_4 A^\dagger \delta m AG(U_c) A^\dagger(t) e^{-i\mathbf{z} \cdot \mathbf{P}} \mathbf{|} -T/2, \mathbf{x} \rangle_{f_9 g_k} =$$

$$\frac{1}{2} \int d\tau_1 \int d\tau_2 \langle T/2, \mathbf{x}' + \mathbf{z} | AG^T | \tau_1 \rangle i\Omega_E^{\tau_2} i\gamma_4 A^\dagger \delta m AG(U_c) A^\dagger(t) e^{-i\mathbf{z} \cdot \mathbf{P}} \mathbf{|} -T/2, \mathbf{x} \rangle_{f_9 g_k}$$

$$\rightarrow \frac{1}{2} e^{-\epsilon v T} \langle A (\mathbf{x}' + \mathbf{z} | \mathbf{v}) \rangle_{f_k} \left( \langle \mathbf{p} | \mathbf{x} + \mathbf{z} \rangle \right)_{g_k}$$

For the other term

$$\langle T/2, \mathbf{x} | e^{i\mathbf{z} \cdot \mathbf{P}} AG(U_c) A^\dagger \hat{A} G(U_c) i\gamma_4 A^\dagger \delta m AG(U_c) A^\dagger(t) e^{-i\mathbf{z} \cdot \mathbf{P}} \mathbf{|} -T/2, \mathbf{x} \rangle_{f_9 g_k} =$$

$$\frac{1}{2} \int d\tau_1 \int d\tau_2 \langle T/2, \mathbf{x}' + \mathbf{z} | AG^T i\gamma_4 A^\dagger \delta m A | \tau_1 \rangle \langle \tau_1 | (G^T P_T \mp G^S P_S) | \tau_2 \rangle$$

$$\times i\Omega_E^{\tau_2} i\gamma_4 A^\dagger \delta m A \langle \tau_2 | G^T P_T A^\dagger \mathbf{|} -T/2, \mathbf{x} + \mathbf{z} \rangle_{f_9 g_k}$$

$$\rightarrow \frac{1}{2} e^{-\epsilon v T} \langle A (\mathbf{x}' + \mathbf{z} | \mathbf{v}) \rangle_{f_k} \left( \langle \mathbf{p} | \mathbf{x} + \mathbf{z} \rangle \right)_{g_k}$$

Taking into account the previous results, it is clear that the product can be written as an exponential without much error. In this expression only

$$\frac{1}{2} \left( \frac{1}{2\sqrt{3}} \int_{-T/2}^{T/2} d\tau i\Omega_E^{\tau} \right)^2$$

is of second order in $T$. Writing the other terms as an exponential represents errors of order $(\Omega^4)$ for the terms quadratic in the angular velocity, errors of order $(\delta m^2)$ for the linear terms in the mass difference and errors $(\delta m^2 \Omega^2)$ for the mixed mass-angular corrections terms. The exponential reads for the product of $N_c$ propagators (hence the $N_c$ factor)

$$\prod_{k=1}^{N_c} G_{f_k g_k} (U; x', x) \rightarrow P_T \left( A(-T/2) \phi (x' + z) \right)_{f_k} \left( \phi^\dagger (x + z) A^\dagger \right)_{g_k} P_T$$

$$\times \exp \left\{ -N_c \epsilon v T - \frac{N_c}{2\sqrt{3}} \int_{-T/2}^{T/2} i\Omega_E^{\tau} - T N_c \langle \mathbf{v} | i\gamma_4 A^\dagger \delta m A | \mathbf{v} \rangle \right\}$$
D.2 The effective action

\[ + \frac{N_c}{4} \sum_{\varepsilon_n \neq \varepsilon_V} \frac{1}{\varepsilon_n - \varepsilon_V} \langle \psi | \tau^i | n \rangle \langle n | \tau^j | \psi \rangle \int_{-T/2}^{T/2} d\tau i\Omega_D^i \Omega_D^j \]

\[ + \frac{N_c}{4} \sum_{\varepsilon_n^0} \frac{1}{\varepsilon_n^0 - \varepsilon_V} \langle \psi | P_T \lambda^a P_S \lambda^b P_T | n^0 \rangle \langle n^0 | \psi \rangle \int_{-T/2}^{T/2} d\tau i\Omega_D^i \Omega_D^j \]

\[ + N_c \sum_{n \neq 0} \frac{1}{\varepsilon_n - \varepsilon_V} \left( \langle \psi | P_T \lambda^a P_T | n \rangle \langle n | i\gamma_4 P_T A^\dagger \delta m A_P T | \psi \rangle + \langle \psi | i\gamma_4 P_T A^\dagger \delta m A_P T | n \rangle \langle n | P_T \lambda^a P_T | \psi \rangle \right) \int_{-T/2}^{T/2} d\tau i\Omega_D^i \]

\[ - \frac{N_c}{2} T \left( \langle \psi | P_T \lambda^a P_T | \psi \rangle \langle \psi | i\gamma_4 P_T A^\dagger \delta m A_P T | \psi \rangle + \langle \psi | i\gamma_4 P_T A^\dagger \delta m A_P T | \psi \rangle \langle \psi | P_T \lambda^a P_T | \psi \rangle \right) \int_{-T/2}^{T/2} d\tau i\Omega_D^i \]

(D.26)

D.2.2 Fermionic determinant

**Regularized part** For the real part of the fermionic determinant one has

\[ \text{ReTr} \log |D(U)|_{\text{reg}} = - \frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{du}{u} \text{Tr} e^{-uD^\dagger D(U)D(U)} - \text{v.s.} \] (D.27)

To establish notation, we use v.s. to denote the subtraction by the term with \( D(U) = 1 \), i.e. the vacuum term. Using the Dyson expansion (G.5) together with

\[ D^\dagger D = (D_c^\dagger - A^\dagger \hat{A} + i\gamma_4 A^\dagger \delta m A) \left( D_c + A^\dagger \hat{A} + i\gamma_4 A^\dagger \delta m A \right) \]

\[ = D_c^\dagger D_c + D_c^\dagger A^\dagger \hat{A} - A^\dagger \hat{A} D_c + D_c^\dagger i\gamma_4 A^\dagger \delta m A + i\gamma_4 A^\dagger \delta m A D_c \]

\[ - A^\dagger \hat{A} A^\dagger \hat{A} - A^\dagger i\gamma_4 A^\dagger \delta m A + i\gamma_4 A^\dagger \delta m A A^\dagger \hat{A} + \cdots \] (D.28)

leads to

\[ \text{ReTr} \log |D(U)|_{\text{reg}} = - \frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{du}{u} \text{Tr} \left( e^{-uD^\dagger D_c} \right. \]

\[ - u \int_0^1 d\alpha \left. e^{-u\alpha D_Dc} \left( D_c^\dagger A^\dagger \hat{A} - A^\dagger \hat{A} D_c \right) e^{-(1-\alpha)uD_Dc} \right) \]

\[ - u \int_0^1 d\alpha \left. e^{-u\alpha D_Dc} \left( D_c^\dagger i\gamma_4 A^\dagger \delta m A + i\gamma_4 A^\dagger \delta m A D_c \right) e^{-(1-\alpha)uD_Dc} \right) \]

\[ + u \int_0^1 d\alpha \left. e^{-u\alpha D_Dc} \left( A^\dagger \hat{A} A^\dagger \hat{A} e^{-(1-\alpha)uD_Dc} \right) \right) \]

\[ + u^2 \int_0^1 d\beta \int_0^{1-\beta} d\alpha \left. e^{-u\alpha D_Dc} \left( D_c^\dagger A^\dagger \hat{A} - A^\dagger \hat{A} D_c \right) \right) \]

\[ \times e^{-(1-\alpha-\beta)uD_Dc} \left( D_c^\dagger A^\dagger \hat{A} - A^\dagger \hat{A} D_c \right) e^{-(1-\alpha-\beta)uD_Dc} + \cdots - \text{v.s.} \]

\[ = r(\Omega,\delta m^a) + r(\Omega^1,\delta m^a) + r(\Omega^2,\delta m^a) + r_A(\Omega^2,\delta m^a) + r_B(\Omega^2,\delta m^a) + \cdots \] (D.29)

These terms can now be worked out separately using the propagator to calculate the traces.

\[ r(\Omega^0,\delta m^a) = - \frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{du}{u} \int d^4 x \text{Tr} \left( e^{-uD_Dc} \right) - \text{v.s.} = - \frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{du}{2\sqrt{\pi}} \int_{-\infty}^{\infty} du e^{-u^2 + u^2} - \text{v.s.(D.30a)} \]

\[ r(\Omega^1,\delta m^a) = 0 \] (D.30b)

\[ r(\Omega^2,\delta m^a) = T \sum_n \varepsilon_n \int_{1/\Lambda^2}^{\infty} \frac{du}{u} \text{Tr} \left( e^{-u^2 + u^2} \right) \]

\[ = - \frac{1}{2} T \sum_n \langle n | i\gamma_4 P_T A^\dagger \delta m A_P T | n \rangle \int_{1/\Lambda^2}^{\infty} \frac{du}{\sqrt{u}} \varepsilon_n e^{-u^2} \] (D.30c)
\[ \begin{align*}
  \Gamma_{A}^{(\Omega_{2}^{a} \Omega_{2}^{b})} &= \frac{1}{2} \int_{1/\Lambda^{2}}^{\infty} du \text{Tr} \int_{0}^{1} d\alpha e^{-\alpha u D_{T}^{a} D_{T}^{b} A_{1}^{1} A_{1}^{1} \epsilon T_{A}} \left( e^{-u \omega_{n}^{2}} + e^{-u \omega_{m}^{2}} \right) \frac{1}{4} \int dx^{4} \Omega_{E}^{a} \Omega_{E}^{b} \\
  &= \frac{1}{2} \int_{1/\Lambda^{2}}^{\infty} du \text{Tr} \int_{0}^{1} d\alpha e^{-\alpha u D_{T}^{a} D_{T}^{b} A_{1}^{1} A_{1}^{1} \epsilon T_{A}} \left( e^{-u \omega_{n}^{2}} + e^{-u \omega_{m}^{2}} \right) \frac{1}{4} \int dx^{4} \Omega_{E}^{a} \Omega_{E}^{b} \\
  &= \frac{1}{8 \sqrt{\pi}} \int_{1/\Lambda^{2}}^{\infty} \frac{du}{u} \sum_{n, m \neq n} \langle m | P_{T} \lambda^{a} P_{T} | n \rangle \langle n | P_{T} \lambda^{b} P_{T} | m \rangle \left( e^{-u \omega_{n}^{2}} + e^{-u \omega_{m}^{2}} \right) \frac{1}{4} \int dx^{4} \Omega_{E}^{a} \Omega_{E}^{b} \\
  &= \frac{1}{12 \sqrt{\pi}} \int_{1/\Lambda^{2}}^{\infty} \frac{du}{u} \sum_{n, m \neq n} \langle m | n^{0} \rangle \langle n^{0} | m \rangle \left( e^{-u \omega_{n}^{2}} + e^{-u \omega_{m}^{2}} \right) \frac{1}{4} \int dx^{4} \Omega_{E}^{a} \Omega_{E}^{b} \\
  \end{align*} \]

\[ \begin{align*}
  \Gamma_{B}^{(\Omega_{2}^{a} \Omega_{2}^{b})} &= \frac{1}{2} \int_{1/\Lambda^{2}}^{\infty} u du \text{Tr} \int_{0}^{1} d\beta e^{-\beta u D_{T}^{a} D_{T}^{b} (D_{T}^{1} A_{1}^{1} - A_{1}^{1} \epsilon T_{A})} \left( e^{-u \omega_{n}^{2}} + e^{-u \omega_{m}^{2}} \right) \frac{1}{4} \int dx^{4} \Omega_{E}^{a} \Omega_{E}^{b} \\
  &= \frac{1}{2} \int_{1/\Lambda^{2}}^{\infty} u du \text{Tr} \int_{0}^{1} d\beta e^{-\beta u D_{T}^{a} D_{T}^{b} (D_{T}^{1} A_{1}^{1} - A_{1}^{1} \epsilon T_{A})} \left( e^{-u \omega_{n}^{2}} + e^{-u \omega_{m}^{2}} \right) \frac{1}{4} \int dx^{4} \Omega_{E}^{a} \Omega_{E}^{b} \\
  &= \frac{1}{8 \sqrt{\pi}} \int_{1/\Lambda^{2}}^{\infty} \frac{du}{u} \sum_{n, m \neq n} \langle m | n^{0} \rangle \langle n^{0} | m \rangle \left( e^{-u \omega_{n}^{2}} + e^{-u \omega_{m}^{2}} \right) \frac{1}{4} \int dx^{4} \Omega_{E}^{a} \Omega_{E}^{b} \\
  &= \frac{1}{12 \sqrt{\pi}} \int_{1/\Lambda^{2}}^{\infty} \frac{du}{u} \sum_{n, m \neq n} \langle m | n^{0} \rangle \langle n^{0} | m \rangle \left( e^{-u \omega_{n}^{2}} + e^{-u \omega_{m}^{2}} \right) \frac{1}{4} \int dx^{4} \Omega_{E}^{a} \Omega_{E}^{b} \\
  \end{align*} \]
D.2 The effective action

\[ \int_{1/\Lambda^2}^{\infty} \frac{du}{\sqrt{u}} \left( \frac{e^{-u\varepsilon_0} - e^{-u\varepsilon_m}}{\varepsilon_0 - \varepsilon_m} + 2 \frac{e^{-u\varepsilon_0} - e^{-u\varepsilon_m}}{u (\varepsilon_0^2 - \varepsilon_m^2)} \right) \]

(D.30ii)

Interchanging in \( r_{H2} \) \( m \leftrightarrow n \) in the contribution \( r_{H3}^{(\sigma^2, \delta_m^0)} \) and summing it with \( r_{H2}^{(\sigma^2, \delta_m^0)} \)

**Nonregularized part**

The imaginary part

\[ i \text{Im} \text{Tr} \log [D(U)] = \frac{1}{2} \left( \text{Tr} \log [D(U)] - \text{Tr} \log [D^\dagger(U)] \right) \]

\[ = \frac{1}{2} \text{Tr} \left( \log [D(U_c)] - \log [D^\dagger(U_c)] \right) \]

\[ + \frac{1}{2} \text{Tr} \left\{ \log \left[ 1 + G(U_c) \left( \hat{A}^\dagger \hat{A} + i\gamma_4 \hat{A}^\dagger \delta mA \right) \right] \right. \]

\[ - \left. \log \left[ 1 + G^\dagger(U_c) \left( -\hat{A}^\dagger \hat{A} + i\gamma_4 \hat{A}^\dagger \delta mA \right) \right] \right\} \]

(D.31)

\[ i \text{Im} \text{Tr} \log [D(U)] = \frac{1}{2} \text{Tr} \left( \log [D(U_c)] - \log [D^\dagger(U_c)] \right) \]

\[ + \frac{1}{2} \text{Tr} \left( G(U_c) \hat{A}^\dagger \hat{A} + G^\dagger(U_c) \hat{A}^\dagger \hat{A} \right) \]

\[ + \frac{1}{2} \text{Tr} \left( G(U_c) i\gamma_4 \hat{A}^\dagger \delta mA - G^\dagger(U_c) i\gamma_4 \hat{A}^\dagger \delta mA \right) \]

\[ - \frac{1}{4} \text{Tr} \left( G(U_c) \hat{A}^\dagger \hat{A} G(U_c) \hat{A}^\dagger \hat{A} - G^\dagger(U_c) \hat{A}^\dagger \hat{A} G^\dagger(U_c) \hat{A}^\dagger \hat{A} \right) \]

\[ - \frac{1}{4} \text{Tr} \left( G(U_c) i\gamma_4 \hat{A}^\dagger \delta mA G(U_c) \hat{A}^\dagger \hat{A} + G^\dagger(U_c) i\gamma_4 \hat{A}^\dagger \delta mA G^\dagger(U_c) \hat{A}^\dagger \hat{A} \right) \]

\[ = i(\Omega^0, \delta_m^0) + i(\Omega^1, \delta_m^0) + i(\Omega^2, \delta_m^0) + i(\Omega^1, \delta_m^0) + \ldots \]

(D.32)

These terms have the following simplified expressions:

\[ i(\Omega^0, \delta_m^0) = \frac{1}{2} \int d^4 x \left( \log [\hat{A}^\dagger + h(U)] - \log [\hat{A}^\dagger + h(1)] \right) = 0 \]

(D.33a)

\[ i(\Omega^1, \delta_m^0) = \frac{1}{2} \text{Tr} \left( G_T A^\dagger \hat{A} + G_T^\dagger A^\dagger \hat{A} \right) \]

\[ = \frac{1}{2} \sum_n \int \frac{d\omega}{2\pi} \langle n | P_T \lambda^\alpha P_T | n \rangle \left( \frac{1}{\omega + \varepsilon_n} + \frac{1}{-\omega + \varepsilon_n} \right) \frac{1}{2} \int dx^4 \Omega_E^\delta \]

\[ = \frac{1}{\sqrt{3}} \sum_n \text{sgn}(\varepsilon_n) \frac{1}{2} \int dx^4 \Omega_E^\delta \] (D.33b)

\[ i(\Omega^0, \delta_m^1) = \frac{1}{2} \text{Tr} \left( G_T i\gamma_4 A^\dagger \delta mA - G_T^\dagger i\gamma_4 A^\dagger \delta mA \right) \]

\[ = \frac{1}{2} \sum_n \int \frac{d\omega}{2\pi} \langle n | i\gamma_4 P_T A^\dagger \delta m A P_T | n \rangle \left( \frac{1}{\omega + \varepsilon_n} + \frac{1}{-\omega + \varepsilon_n} \right) \] v.s. = 0 (D.33c)

\[ i(\Omega^2, \delta_m^1) = 0 \] (D.33d)

\[ i(\Omega^1, \delta_m^1) = \sum_{n,m} \langle m | \lambda^\alpha P_T \rangle \langle n | i\gamma_4 P_T A^\dagger \delta m A P_T | n \rangle \int \frac{d\omega}{2\pi} \frac{1}{(\omega - k\varepsilon_m) (\omega - k\varepsilon_n)} \frac{1}{2} \int dx^4 \Omega_E^\delta \]

\[ = 2 \sum_{n,m} R_5(\varepsilon_m, \varepsilon_n) \langle m | \lambda^\alpha P_T \rangle \langle n | i\gamma_4 P_T A^\dagger \delta m A P_T | m \rangle \frac{1}{2} \int dx^4 \Omega_E^\delta \]

\[ + 2 \sum_{n,m} \left( \langle m | P_T \lambda^\alpha P_S \rangle \langle n^0 | i\gamma_4 P_S A^\dagger \delta m A P_T | m \rangle \right) \frac{1}{2} \int dx^4 \Omega_E^\delta \] (D.33e)
D.2.3 Effective action

Collecting the previous results one finds

\[ e^{N_c \text{Tr} \log[D^E(U)]]} \prod_{j=1}^{N_c} G_{J_{jg}(U;x',x)}^E = e^{S_{\text{at}} P_T (A \phi_V(x' + z)) f_k \left( \phi_V(x + z) A^k \right)}_{g_k} P_T \]

\[ = P_T (A \phi_V(x' + z)) f_k \left( \phi_V(x + z) A^k \right)_{g_k} P_T \]

\[ \times \exp \left\{ S_{\text{eff}}^{(\Omega_0,\delta m^0)} + S_{\text{eff}}^{(\Omega_0,\delta m^1)} + S_{\text{eff}}^{(\Omega_0,\delta m^0)} + S_{\text{eff}}^{(\Omega_0,\delta m^1)} + \cdots \right\} . \tag{D.34} \]

The individual terms are as follows: Leading term in angular velocity (\( \Omega_0 \)), order \( N_c \), and in \( \delta m \) (\( \delta m^0 \))

\[ S_{\text{eff}}^{(\Omega_0,\delta m^0)} = -T N_c \xi_V - \frac{1}{2} T N_c \sum_n \frac{1}{2 \sqrt{\pi}} \int_{1/\Lambda^2}^\infty \frac{du}{\sqrt{u}} e^{-u \xi_n^2} = -T M_c \tag{D.35} \]

with \( M_c \) the mass of the soliton; linear in the angular velocity

\[ S_{\text{eff}}^{(\Omega_0,\delta m^1)} = -\frac{N_c}{2 \sqrt{3}} \int_{-T/2}^{T/2} d\Omega_E^8 ; \tag{D.36} \]

linear in \( \delta m \),

\[ S_{\text{eff}}^{(\Omega_0,\delta m^1)} = N_c \langle \xi | \gamma_4 P_T A^k \Lambda^i 3 \delta m A P_T | \xi \rangle + \frac{1}{2} \sum_n \langle n | \gamma_4 P_T A^k \Lambda^i 3 \delta m A P_T | n \rangle \int_{1/\Lambda^2}^\infty \frac{du}{\sqrt{u}} e^{-u \xi_n^2} , \tag{D.37} \]

is the sigma term in SU(2)

\[ \sigma = \Sigma [SU(2)] \left( M_1 + \frac{1}{\sqrt{3}} M_8 D_{8s}^{(8)} (A) \right) / \overline{m} ; \tag{D.38} \]

quadratic in the angular velocity (\( \Omega^2 \)), order \( N_c^0 \), with the projector \( P_T \),

\[ S_{\text{eff:} P_T}^{(\Omega_2,\delta m^0)} = \frac{N_c}{4} \sum_{n \in V} \frac{1}{\varepsilon_n - \xi_V} \langle \xi | P_T \Lambda^k P_T | n \rangle \langle n | P_T \Lambda^k P_T | \xi \rangle \int_{-T/2}^{T/2} d\Omega_E^8 i \Omega_E^8 \]

\[ - \frac{N_c}{8 \sqrt{\pi}} \int_{1/\Lambda^2}^\infty \frac{du}{\sqrt{u}} \sum_{n,m \neq n} \langle m | P_T \Lambda^k P_T | n \rangle \langle n | P_T \Lambda^k P_T | m \rangle \frac{1}{4} \int d x^4 \Omega_E^8 i \Omega_E^8 \left( e^{-u \xi_n^2} + e^{-u \xi_m^2} \right) \]

\[ + \frac{1}{2} \sum_{n,m \neq n} \langle m | P_T \Lambda^k P_T | n \rangle \langle n | P_T \Lambda^k P_T | m \rangle \frac{1}{4} \int d x^4 \Omega_E^8 i \Omega_E^8 \]

\[ \times \frac{1}{8 \sqrt{\pi}} \int_{1/\Lambda^2}^\infty \frac{du}{\sqrt{u}} \left( \frac{\varepsilon_n e^{-u \xi_n^2} - \varepsilon_m e^{-u \xi_m^2}}{\varepsilon_n + \varepsilon_m} + 2 \frac{e^{-u \xi_n^2} - e^{-u \xi_m^2}}{u (\varepsilon_n - \varepsilon_m)} \right) \tag{D.39} \]

with \( I_1 \) the moment of inertia of the SU(2) embedded chiral field

\[ \frac{6}{N_c} I_1 = \sum_{n \in V} \langle \xi | \tau | n \rangle \cdot \langle n | \tau | \xi \rangle + \frac{1}{2} \sum_{n,m \neq n} \langle m | \tau | n \rangle \cdot \langle n | \tau | m \rangle \mathcal{R}_3 (\varepsilon_n, \varepsilon_m) \tag{D.40} \]

and the regularization function

\[ \mathcal{R}_3 (\varepsilon_n, \varepsilon_m) = \frac{1}{2 \sqrt{\pi}} \int_{1/\Lambda^2}^\infty \frac{du}{\sqrt{u}} \left( \frac{1}{u} \left( e^{-u \varepsilon_n^2} - e^{-u \varepsilon_m^2} \right) - \frac{\varepsilon_n e^{-u \varepsilon_n^2} + \varepsilon_m e^{-u \varepsilon_m^2}}{\varepsilon_n + \varepsilon_m} \right) \] ; \tag{D.41} \]

quadratic in the angular velocity, similar to the previous one, but now with the projector \( P_s \),

\[ S_{\text{eff:} P_s}^{(\Omega_2,\delta m^0)} = \frac{1}{2} I_s \int d x^4 \Omega_E^8 i \Omega_E^8 \tag{D.42} \]
and a SU(3) specific moment of inertia $I_2$ given by
\[
\frac{4}{N_c} I_2 = \sum_{n^0} \frac{1}{\varepsilon_{n^0} - \varepsilon_V} \langle \psi | n^0 \rangle \langle n^0 | \psi \rangle + \sum_{n^0,m} \langle m | n^0 \rangle \langle n^0 | m \rangle \mathcal{R}_3 (\varepsilon_{n^0}, \varepsilon_m) ;
\]  
\[\text{(D.43)}\]

linear in the angular velocity and also linear in the mass term $\delta m$, with the projector $P_T$
\[
S_{\text{eff};P_T}^{(\Omega^2, \delta m^1)} = 2N_c M_8 D_{8j}^{(8)} \left( \sum_{n \neq 0} \frac{1}{\varepsilon_n - \varepsilon_V} \langle \psi | \tau^i | n \rangle \langle n | \gamma^0 \tau^j | \psi \rangle 
+ 2N_c M_8 D_{8j}^{(8)} \sum_{n,m} \mathcal{R}_5 (\varepsilon_n, \varepsilon_m) \langle m | \tau^i | n \rangle \langle n | \gamma^0 \tau^j | m \rangle \right) \frac{1}{2} \int d^4 i \Omega_E^i ;
\]
\[\text{(D.44)}\]

which defines an anomalous moment of inertia $K_1$
\[
\frac{6}{N_c} K_1 = \sum_{n \neq 0} \frac{1}{\varepsilon_n - \varepsilon_V} \langle \psi | \tau | n \rangle \cdot \langle n | \gamma^0 \tau | \psi \rangle + \sum_{n,m} \mathcal{R}_5 (\varepsilon_n, \varepsilon_m) \langle m | \tau | n \rangle \cdot \langle n | \gamma^0 \tau | m \rangle ;
\]  
\[\text{(D.45)}\]

and, finally, a contribution linear both in the angular velocity and the mass term now with the projector $P_S$
\[
S_{\text{eff};P_S}^{(\Omega^2, \delta m^1)} = N_c M_8 D_{8a}^{(8)} \delta_{ab} \left( \sum_{n^0} \frac{1}{\varepsilon_{n^0} - \varepsilon_V} \langle \psi | n^0 \rangle \langle n^0 | \gamma^0 | \psi \rangle 
+ 2 \sum_{n^0,m} \langle m | n^0 \rangle \langle n^0 | \gamma^0 | m \rangle \mathcal{R}_5 (\varepsilon_{n^0}, \varepsilon_m) \right) \frac{1}{2} \int d^4 i \Omega_E^a ;
\]
\[\text{(D.46)}\]

defining a second anomalous moment of inertia $K_2$
\[
\frac{4}{N_c} K_2 = \sum_{n^0} \frac{1}{\varepsilon_{n^0} - \varepsilon_V} \langle \psi | n^0 \rangle \langle n^0 | \gamma^0 | \psi \rangle + 2 \sum_{n^0,m} \langle m | n^0 \rangle \langle n^0 | \gamma^0 | m \rangle \mathcal{R}_5 (\varepsilon_{n^0}, \varepsilon_m) .
\]  
\[\text{(D.47)}\]

Finally, the effective action and the Euclidian Lagrangian can be expressed up to powers quadratic in the angular velocity ($\Omega^2$), order $N_c^0$, and linear in the mass term ($\delta m^1$). The result reads
\[
S_{\text{eff}} = \int_{-T/2}^{T/2} d\tau L_E ,
\]
\[\text{(D.48a)}\]

\[
L_E = -M_c + \frac{1}{2} I_1 i \Omega_E^i i \Omega_E^i + \frac{1}{2} I_2 i \Omega_E^i i \Omega_E^i + \sigma 
+ 2M_8 D_{8i}^{(8)} K_1 i \Omega_E^i + 2M_8 D_{8a}^{(8)} K_2 i \Omega_E^a - \frac{N_c}{2\sqrt{3}} i \Omega_E^8 .
\]  
\[\text{(D.48b)}\]

**D.3 The soliton**

From equation (D.35)
\[
\frac{\delta S_{\text{eff}}}{\delta U} \bigg|_{U=U_c} = N_c \frac{\delta M}{\delta P} \bigg|_{P=P_c} 
\]
\[= N_c \frac{\delta}{\delta P} \left( \langle \psi | h (P) | \psi \rangle + \frac{1}{2} \sum_n \frac{1}{2\sqrt{n}} \int_0^\infty \frac{du}{u \sqrt{u}} e^{-u (n|h(P)|n)^2} - v.s. \right) \bigg|_{P=P_c}
\]
\[= N_c \langle \psi | r \rangle \frac{\partial}{\partial P} h (P) \bigg|_{P=P_c} \langle r | \psi \rangle
\]
\[ -N_c \sum_n \langle n | r \rangle \frac{\partial}{\partial P} h(P) \bigg|_{P=P_c} \langle r | n \rangle \left( -\langle n | h(P) | n \rangle \right) \int_{1/\Lambda^2}^{\infty} \frac{d\mu}{\sqrt{\mu}} e^{-u(n|h(P)|n)^2} - \text{v.s.} \]

\[ = N_c \langle v | r \rangle \left( \gamma^0 M \left( -\sin P(r) + i\gamma^5 \hat{n} \cdot \vec{r} \cos P(r) \right) \right) \langle r | v \rangle + N_c \sum_n \langle n | r \rangle \left( \gamma^0 M \left( -\sin P(r) + i\gamma^5 \hat{n} \cdot \vec{r} \cos P(r) \right) \right) \langle r | n \rangle R_1(\varepsilon_n(P_c)) - \text{v.s.} \]

\[ = 0 \]  \hspace{1cm} \text{(D.49)}

which, in terms of a scalar and a pseudoscalar densities, becomes

\[ S(r) \sin P_c(r) - \mathcal{P}(r) \cos P_c(r) = 0 \]  \hspace{1cm} \text{(D.50)}

\[ \frac{1}{N_c M} S(r) = -\phi_V(r) \phi_V(r) + \sum_n \phi_n(r) \phi_n(r) R_1(\varepsilon_n) - \text{v.s.} \]  \hspace{1cm} \text{(D.51a)}

\[ \frac{1}{N_c M} P(r) = -\phi_V(r) i\gamma^5 \hat{n} \cdot \vec{r} \phi_V(r) + \sum_n \phi_n(r) i\gamma^5 \hat{n} \cdot \vec{r} \phi_n(r) R_1(\varepsilon_n) - \text{v.s.} \]  \hspace{1cm} \text{(D.51b)}

with the regularization function

\[ R_1(\varepsilon_n) = -\frac{\varepsilon_n}{2\sqrt{\pi}} \int_{1/\Lambda^2}^{\infty} \frac{d\mu}{\sqrt{\mu}} \mu^{-\varepsilon_n^2} \]  \hspace{1cm} \text{(D.52)}

where the \( \phi_n \) and \( \varepsilon_n \) are the eigenfunctions and eigenenergies of \( h(P_c(r)) \)

\[ h(P_c(r)) \phi_n = \varepsilon_n \phi_n. \]  \hspace{1cm} \text{(D.53)}

### D.4 Quantization rules

This section deals with the quantum mechanical aspects of the collective coordinate quantization. It was first developed in the context of the SM in [266]. For the case of the CQSM see [75].

The Lagrangian in Euclidean space is

\[ L_E[A, \Omega] = -M_c + \sigma + \frac{1}{2} I_1 \Omega^1_E \Omega^1_E + \frac{1}{2} I_2 \Omega^2_E \Omega^2_E + 2M_8 D^{(8)}_8 (A) K_1 \Omega^1_E + 2M_8 D^{(8)}_8 (A) K_2 \Omega^2_E - \frac{N_c}{2\sqrt{3}} \Omega_M^8. \]  \hspace{1cm} \text{(D.54)}

where we used (2.44) and (2.46) and write \( \Sigma [SU(2)] (M_1 + M_8 D^{(8)}_8 (A)) \mu = \sigma \). The Lagrangian in Minkowski space (dropping afterwards the index \( M \)) is obtained by the replacement

\[ i\Omega_M^2 \rightarrow \Omega_M^2 \equiv \Omega^\alpha. \]  \hspace{1cm} \text{(D.55)}

In order to obtain the Hamiltonian, now that time-dependent solutions are specified by angular velocities, and therefore momenta, the starting point is to find the canonically conjugated momenta to the coordinates from which the velocities are defined. These coordinates are, however, not evident in the Lagrangian. Denoting by \( q^\alpha \) the coordinates of the flavor space which can written in the form of an SU(3) matrix \( A \) (one may speak of the coordinates of the matrix \( A \)), one may start by using the Maurer-Cartan formula \( (d/dt = \dot{q}^\alpha \partial_{\alpha}) \)

\[ A^\dagger(t) \dot{A}(t) = \dot{q}^\alpha A^\dagger \partial_{\alpha} A = \frac{1}{2} \dot{q}^\alpha C_\alpha^\beta \lambda_\beta \]  \hspace{1cm} \text{(D.56)}

where \( C_\alpha^\beta \) are the vielbeine and the \( q^\alpha \) the coordinates of the matrix \( A \).

The relation between coordinates and angular velocities are

\[ \Omega^\beta = \dot{q}^\alpha C_\alpha^\beta. \]  \hspace{1cm} \text{(D.57)}

For the Lagrangian in Minkowski space one finds

\[ L[q, \dot{q}] = -M_c + \sigma + \frac{1}{2} \dot{q}^a g_{a\beta} \dot{q}^\beta + Z_\alpha \dot{q}^\alpha \]  \hspace{1cm} \text{(D.58)}
with the definitions\(^2\):

\[
\begin{align*}
I_{\theta \sigma} &= I_1 \delta_{ij} + I_2 \delta_{ab} + \varepsilon \delta_{88}, \\
K_i &= 2M_8 K_1 D^{(8)}_{88}, i = 1, 2, 3, \\
K_\alpha &= 2M_8 K_2 D^{(8)}_{88}, \alpha = 4, 5, 6, 7, \\
K_8 &= -\frac{N_c}{2\sqrt{3}}, \\
g_{\alpha \beta} &= C_\alpha^\beta I_{\theta \sigma} C_\sigma^\beta, \\
(g_{\alpha \beta})^{-1} &= g^{\beta \alpha} = C_\beta^\alpha T_{\theta \sigma}^{-1} C_\sigma^\alpha. 
\end{align*}
\]

The canonically conjugate momenta are

\[
\pi_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} = g_{\alpha \beta} \dot{q}^\beta + Z_\alpha. 
\]

Now the velocities can be expressed in terms of the momenta

\[
\dot{q}^\beta = (g_{\alpha \beta})^{-1} (\pi_\alpha - Z_\alpha) = g^{\beta \alpha} (\pi_\alpha - Z_\alpha) 
\]

For the Hamiltonian one finds

\[
H[q, \pi] = q^\alpha \pi_\alpha - L = M_c - \sigma + \frac{1}{2} (\pi_\alpha - Z_\alpha) g^{\beta \alpha} (\pi_\beta - Z_\beta). 
\]

It is better now to define right generators \( R_\alpha \) as

\[
R_\alpha = -\pi_\beta C_\beta^\alpha 
\]

for the Hamiltonian will gain a simple form

\[
H[q, \pi] = M_c - \sigma + \frac{1}{2} \frac{K_1}{I_1} R_i R_i + \frac{1}{2} \frac{K_2}{I_2} R_\alpha R_\alpha + \frac{1}{2\varepsilon} \left( R_8 - \frac{N_c}{2\sqrt{3}} \right)^2 \]

\[
+ 2M_8 \frac{K_1}{I_1} D^{(8)}_{88} R_i + 2M_8 \frac{K_2}{I_2} D^{(8)}_{8a} R_\alpha + O(M_8^2) 
\]

\[
= M_c + H^{coll}_{\text{coll}} + H^{ab}_{\text{coll}}. 
\]

The right generators may be related back to the angular velocities

\[
-R_\alpha = I_{\alpha \sigma} \Omega^\sigma + K_\alpha. 
\]

In a quantum mechanical treatment, one defines the momentum operators in a symmetrized form

\[
\pi_\alpha = \frac{1}{2} \{ q_\beta, g_{\beta \alpha} \} + Z_\alpha 
\]

and then imposes canonical commutation relations between collective coordinates \( q^\alpha \) and their conjugated momenta \( \pi_\alpha \)

\[
[q^\alpha, \pi_\beta] = i\delta^\alpha_\beta, \quad [\pi^\alpha, \pi_\beta] = [q^\alpha, q_\beta] = 0. 
\]

The quantum right generators as operators are also defined by means of a symmetrized expression

\[
R_\alpha = -\frac{1}{2} \{ \pi_\beta, C_\beta^\alpha \} 
\]

which is explicitly hermitian. Since the vielbeine are functions of the coordinates, one has

\[
[\pi_\alpha, C_\alpha^\beta] = -i \partial_\alpha C_\alpha^\beta. 
\]

Using (D.65) the right generators may be written in terms of the angular velocities according to

\[
-R_\alpha = \begin{cases} 
I_1 \Omega_\alpha + 2M_8 K_1 D^{(8)}_{88}, & \alpha = 1, 2, 3 \\
I_2 \Omega_\alpha + 2M_8 K_2 D^{(8)}_{8a}, & \alpha = 4, 5, 6, 7 \\
-N_c/2\sqrt{3}, & \alpha = 8. 
\end{cases} 
\]

It is possible to interpret [55, 74, 267] the operators \( J_i = -R_i, i = 1, 2, 3, \) as the angular momentum operators, \( J_a \) being a constraint \( (J_a = -R_\alpha, \alpha = 4, \ldots, 7 \) do not have any connection with rotations in configuration space). These relations may be used in order to replace the classical angular

\(^2\)The vielbeine and their inverses are distinguished by the indices.
velocities $\Omega_{\alpha}$ with generalized quantum mechanical angular momentum operators $J_{\alpha}$, once the expansion in $\Omega_{\alpha}$ is done. This is the usually intended meaning of quantization in this work (2.58).

The commutation relations among the right generators, from the definitions (D.68), are given by
\begin{equation}
[R_{\alpha}, R_{\eta}] = -i f_{\alpha \mu \eta} R_{\mu},
\end{equation}
(Thus fulfilling an SU(3) algebra,) using the Maurer-Cartan identity
\begin{equation}
C^\alpha_{\beta} \partial_{\alpha} C^\eta_{\eta} - C^\beta_{\beta} \partial_{\beta} C^\epsilon_{\epsilon} = -f_{\epsilon \mu \eta} C^\mu_{\epsilon}. 
\end{equation}
Using the definitions of the right generators (D.68) and the vielbeine (D.56) one may deduce the action of the right generators upon the matrices $A$ with the result
\begin{equation}
[R_{\alpha}, A] = -\frac{1}{2} A \lambda_{\alpha},
\end{equation}
and, afterwards, their action upon the matrices $D^{(8)}(A)$ defined in the regular representation by (2.45):
\begin{equation}
[R_{\alpha}, D^{(8)}_{\chi \gamma} (A)] = -i f_{\alpha \beta \gamma} D^{(8)}_{\chi \beta} (A).
\end{equation}

Defining similarly left generators $L_{\alpha}$ by
\begin{equation}
L_{\alpha} = -\frac{1}{2} \{ \pi_{\beta}, E^\beta_{\alpha} \}
\end{equation}
with the vielbeine now defined from
\begin{equation}
(\partial_{\alpha} A) A^\dagger = i \frac{1}{2} E^\beta_{\alpha} \lambda_{\beta}
\end{equation}
and using the Maurer-Cartan identity for these vielbeine
\begin{equation}
E^\alpha_{\beta} \partial_{\alpha} E^\epsilon_{\eta} - E^\beta_{\beta} \partial_{\beta} E^\epsilon_{\epsilon} = f_{\epsilon \mu \eta} E^\epsilon_{\mu}.
\end{equation}
on one obtains the corresponding results:
\begin{equation}
[L_{\alpha}, L_{\eta}] = i f_{\alpha \mu \eta} L_{\mu},
\end{equation}
\begin{equation}
[L_{\alpha}, A] = -\frac{1}{2} \lambda_{\alpha} A,
\end{equation}
\begin{equation}
[L_{\alpha}, D^{(8)}_{\beta \chi} (A)] = i f_{\alpha \beta \gamma} D^{(8)}_{\chi \beta} (A).
\end{equation}
Equation (D.74) and (D.78c) show that the right and left generators act on the right and left indices of the matrices $D$, respectively.

The relation between right and left generators may be found from (2.46)
\begin{equation}
\lambda_{\alpha} A = D^{(8)}_{\alpha \beta} (A) A \lambda_{\beta}
\end{equation}
using the relations (D.78b) and (D.73) together with
\begin{equation}
D^{(8)}_{\alpha \beta} (A) D^{(8)}_{\chi \gamma} (A) = \delta_{\chi \gamma}
\end{equation}
with the result that
\begin{equation}
L_{\alpha} = D^{(8)}_{\alpha \beta} R_{\beta}.
\end{equation}
A consequence of this result is that
\begin{equation}
L_{\alpha} L_{\alpha} = R_{\alpha} R_{\alpha},
\end{equation}
which is the SU(3) version of the SU(2) hedgehog result $J^2 = T^2$.

The main interest in deriving (2.69) is its application to the Gell-Mann–Nishijima relation (3.31).

D.5 About the symmetry conserving quantization

Following [91] and restricting the expression to the sea contribution for simplicity, one may user (E.136) to compute the sea contribution to the generalized angular momentum operator $J^\lambda$. To this end one must realize that the angular momentum operators are defined in the laboratory reference frame, i.e. the rotating frame, and thus do not need to be rotated to the frame. It enough to replace
in (E.136), written in terms of the angular velocities \((J_\alpha/2I \to \Omega_\alpha)\), the matrices \(D_{\chi\beta} (\Lambda)\) by \(-\delta_{\chi\beta}\) (making all the troublesome commutators vanish) and \(\Gamma^\mu\) by 1/2. Neglecting the mass correction terms, which play no role in this discussion, the result, using (A.15a), (A.15b), is \((\eta = -1)\)

\[
J^X_S = \frac{-N_c}{2} \delta_{\chi 8} \sum_n \langle n | P_T \lambda^X P_T | n \rangle \left( -\frac{1}{2} \text{sgn}(\varepsilon_n) \right)
+ \frac{N_c}{2} \Omega_i \delta_{\chi i} \sum_{n,m} \langle m | P_T \lambda^X P_T | n \rangle \langle n | P_T \lambda^X P_T | m \rangle \mathcal{R}_3(\varepsilon_n, \varepsilon_m)
+ \frac{N_c}{2} \Omega_\alpha \delta_{\chi \alpha} \sum_{n,m^0} \langle m^0 | P_S \lambda^X P_T | n \rangle \langle n | P_T \lambda^\alpha P_S | m^0 \rangle \mathcal{R}_3(\varepsilon_n, \varepsilon_{m^0})
+ \frac{N_c}{2} \Omega_\alpha \delta_{\chi \alpha} \sum_{n^0,m} \langle m | P_T \lambda^\beta P_S | n^0 \rangle \langle n^0 | P_S \lambda^\alpha P_T | m \rangle \mathcal{R}_3(\varepsilon_{n^0}, \varepsilon_m)
\]

\[= \frac{-N_c}{2\sqrt{3}} B_8 \delta_{\chi 8} - I_{1S} \sum_{i=1}^3 \delta_{\chi i} \Omega_i - I_{2S} \sum_{\alpha=4}^7 \delta_{\chi \alpha} \Omega_\alpha \]  \hspace{1cm} (D.83)

where \(B_8, I_{1S}, I_{2S}\) are the sea contributions to the baryon number and the moments of inertia. The same steps may be applied to the valence contribution. The sum may be written as

\[
J^X = \frac{-N_c}{2\sqrt{3}} \delta_{\chi 8} - I_1 \sum_{i=1}^3 \delta_{\chi i} \Omega_i - I_2 \sum_{\alpha=4}^7 \delta_{\chi \alpha} \Omega_\alpha \]  \hspace{1cm} (D.84)

On the contrary, the flavor generators are identified with the intrinsic, soliton fixed, frame of reference and thus have to be rotated to the laboratory frame. This means that we may use (E.136), again written in terms of the angular velocities, just replacing \(\Gamma^\mu\) by 1/2 \((\eta = -1)\).

\[
T^X_S = \frac{N_c}{2} D_{\chi\beta}^{(8)} \sum_n \langle n | P_T \lambda^\beta P_T | n \rangle \left( -\frac{1}{2} \text{sgn}(\varepsilon_n) \right)
- \frac{N_c}{4} \left\{ \Omega_i, D_{\chi\beta}^{(8)} \right\} \sum_{n,m} \langle m | P_T \lambda^\beta P_T | n \rangle \langle n | P_T \lambda^\beta P_T | m \rangle \mathcal{R}_3(\varepsilon_n, \varepsilon_m)
+ \frac{N_c}{4} \left[ \Omega_i, D_{\chi\beta}^{(8)} \right] \sum_{n,m^0} \langle m^0 | P_S \lambda^\beta P_T | n \rangle \langle n | P_T \lambda^\alpha P_S | m^0 \rangle \mathcal{R}_6(\varepsilon_n, \varepsilon_{m^0})
- \frac{N_c}{4} \left\{ \Omega_\alpha, D_{\chi\beta}^{(8)} \right\} \left( \sum_{n,m^0} \langle m^0 | P_S \lambda^\beta P_T | n \rangle \langle n | P_T \lambda^\alpha P_S | m^0 \rangle \mathcal{R}_3(\varepsilon_n, \varepsilon_{m^0}) \right)
+ \sum_{n^0,m} \langle m | P_T \lambda^\beta P_S | n^0 \rangle \langle n^0 | P_S \lambda^\alpha P_T | m \rangle \mathcal{R}_3(\varepsilon_{n^0}, \varepsilon_m)\]
+ \frac{N_c}{4} \left[ \Omega_\alpha, D_{\chi\beta}^{(8)} \right] \left( \sum_{n,m^0} \langle m^0 | P_S \lambda^\beta P_T | n \rangle \langle n | P_T \lambda^\alpha P_S | m^0 \rangle \mathcal{R}_6(\varepsilon_n, \varepsilon_{m^0}) \right)
+ \sum_{n^0,m} \langle m | P_T \lambda^\beta P_S | n^0 \rangle \langle n^0 | P_S \lambda^\alpha P_T | m \rangle \mathcal{R}_6(\varepsilon_{n^0}, \varepsilon_m)\]
\[= \frac{N_c}{2\sqrt{3}} B_8 D_{\chi\beta}^{(8)} + I_{1S} \sum_{i=1}^3 \delta_{\beta i} \left\{ \Omega_i, D_{\chi\beta}^{(8)} \right\} + I_{2S} \sum_{\alpha=4}^7 \delta_{\beta \alpha} \left\{ \Omega_\alpha, D_{\chi\beta}^{(8)} \right\}
+ \frac{N_c}{4} \left[ \Omega_\alpha, D_{\chi\beta}^{(8)} \right] \frac{2i}{\sqrt{3}} f_{\alpha\beta} \sum_{n,m^0} \langle m^0 | n \rangle \langle n | m^0 \rangle \mathcal{R}_6(\varepsilon_n, \varepsilon_{m^0}) \]  \hspace{1cm} (D.85)
Defining the following quantity, similar to the moment of inertia $I_2$,
\[
I'_{2s} = \frac{N_c}{4} \sum_{n,m^0} \langle m^0 | n \rangle \langle n | m^0 \rangle \mathcal{R}_6(\varepsilon_n, \varepsilon_{m^0})
\]
(D.86)

the generalized flavor generators are, adding the valence contributions,
\[
T^\chi = \frac{N_c}{2\sqrt{3}} D^{(8)}_{\chi 8} + I_1 \sum_{i=1}^3 \delta_{\beta i} \{ \Omega_i, D^{(8)}_{\chi \beta} \} + I_2 \sum_{a=4}^7 \delta_{\beta a} \{ \Omega_a, D^{(8)}_{\chi \beta} \} + \frac{2i}{\sqrt{3}} f_{a\beta 8} \left[ \Omega_a, D^{(8)}_{\chi \beta} \right] I'_2
\]
(D.87)

Finally, the relation between the generalized angular momentum and flavor operators is, from (D.84) and (D.87),
\[
T^\chi = -D^{(8)}_{\chi \alpha} J_\alpha - \sqrt{3} D^{(8)}_{\chi 8} I'_2.
\]
(D.88)
E Matrix elements of observables

In this appendix, one uses the results for the correlation function ((E.8),(E.9)) together with the quantization rules of Section D.4 to derive expressions for matrix elements of observables up to order $1/N_c$ in the rotational corrections and linear in the strange quark mass $m_s$.

The SU(3) part of an operator acting on the flavor space may either be one of the $\lambda$ matrices, or it may be just the unit matrix in the flavor space. In this last case one speaks of singlet observables, a designation also applicable to other objects, for instance currents. Since the behavior of the $\lambda$ matrices is different from that of the unit matrix under rotations, one is lead to think that the results for the observables should be quite different. They indeed are. However, there is a simple prescription to obtain the singlet case from the general one, as explained in the following section.

E.1 Correlation function for Lorentz-vector currents

The correlation function in the limit of large time separation $T$, $C_{B'B}^\mu(T)$, for a Lorentz vector-type operator $O^{\mu\chi} = \Gamma^\mu \lambda^\chi$ ($\Gamma$ is the Dirac part of the operator and $\lambda$ the flavor part) is written in Minkowski space as

$$C_{B'B}^{\mu(M)}(T) = \langle 0 | J_{B'}(T/2, x') \psi^\dagger(y) O^{\mu\chi}(y) \psi(y) J_B^\dagger(-T/2, x) | 0 \rangle =$$

\[
\frac{1}{Z} \int [d\psi^\dagger] [d\psi] [dU] \ J_{B'}(T/2, x') \psi^\dagger(y) O^{\mu\chi}(y) \psi(y) J_{B}^\dagger(-T/2, x) \ e^{\int d^4x \frac{\tau}{\tau_2} dt \ L_{CQM}} \tag{E.1}
\]

and

\[
Z = \int [d\psi^\dagger] [d\psi] [dU] \ J_{B'}(T/2, x') J_{B}^\dagger(-T/2, x) \ e^{\int d^4x \frac{\tau}{\tau_2} dt \ L_{CQM}}. \tag{E.2}
\]

In order to calculate ((E.1)), an external vector current $s^\mu$ can be associated with the operator $O^{\mu\chi}$ according to

$$C_{CQM} \to L_{CQM} = \psi^\dagger (\gamma^\mu \bar{\psi} - s_{\mu} O^{\mu\chi} - \delta^\mu \Gamma^\chi M U_{\gamma}) \psi,$$

which is the simple minimal substitution prescription $\partial_\mu \to \partial_\mu + is_{\mu}$ with the charge included in the current $s_{\mu}$.

In going to the Euclidian space the statistical factor of the functional integral includes now the coupling to the external current $s^\mu$ according to

\[
i \int d^4x \ L_{CQM} \to -\int d^4x E \ \psi^\dagger [\partial_\mu + h^E(U) - s^\mu E] \psi \tag{E.4}
\]

where $h^E(U)$ is the one-particle Dirac operator (D.4). The correlation function has now the form

\[
C_{B'B}^{\mu(M)}(T) = \frac{1}{Z_E} \int [d\psi^\dagger] [d\psi] [dU] \ \Gamma_{B'}^{(g)} \Gamma_{B}^{(g)} \ \epsilon_{\alpha_1 \cdots \alpha_N \epsilon_{\beta_1 \cdots \beta_N}} \ \frac{1}{(N_c!)^2} \psi_{g_1 \alpha_1} \psi_{g_2, \cdots, \psi_{g_N, \alpha_N}}(x') \times \psi_{g_1 \alpha_1}(y) O^{\mu\chi}(y) \psi_{g_2, \cdots, \psi_{g_N, \alpha_N}}(x) \ e^{-\int d^4x E \ \psi^\dagger D(U,s) \psi} \big|_{s^\mu=0} \tag{E.5}
\]

with the operator $D(U,s)$ defined by

\[
D(U,s) = \partial_\mu + h^E(U) - s^\mu E \ L_{CQM} \tag{E.6}
\]

Substitution of the fermion fields by functional derivatives is now possible leading to

\[
C_{B'B}^{\mu(M)}(T) = \Gamma_{B'}^{(g)} \Gamma_{B}^{(g)} \ \frac{1}{(N_c!)^2} \epsilon_{\alpha_1 \cdots \alpha_N \epsilon_{\beta_1 \cdots \beta_N}} \ \frac{1}{Z_E[s=0]}(-1)^{\sum_{k=1}^{2Nc-1} k} \times \int [d\psi^\dagger] [d\psi] [dU] \epsilon_{\frac{\delta}{\delta\omega_{g_1, \beta_1}(x)}} \cdots \epsilon_{\frac{\delta}{\delta\omega_{g_N, \beta_N}(x)}} \tag{E.5}
\]
In the second case, one obtains the so-called sea contribution $s$, given by
\[ \delta \frac{\delta}{\delta s^\mu (y)} \frac{\delta}{\delta \omega^*_{f_{k,\alpha_1}(x')}} \cdots \frac{\delta}{\delta \omega^*_{f_{k,\alpha_1}(x')}} e^{-fd^4ud^4v \psi^\dagger (u) D_u(U, s; u, v) \psi (v) + fd^4u \psi^\dagger (u) \omega (u) + \omega^* (u) \psi (u)} \bigg|_{\omega = 0, \omega^* = 0, s^\mu = 0}. \]  
(E.7)

The main difference relative to (D.9) is now the functional derivative with respect to the external current. It is clear that the main differences come from this derivative. It can act on the propagators which result from the functional derivatives with respect to the $\omega^a$s or it can operate on the exponential. In the first case, similar to (D.9), one obtains the so-called valence contribution denoted by $v$ which reads
\[ \langle 0 | J_B^\alpha (x') \psi^\dagger \rangle \mathcal{O}^\mu x \psi J_B^\dagger (x) | 0 \rangle_v = \frac{N_c}{Z} \int [dU] \Gamma^{(f)}_{B'} \Gamma^{(g)}_B | \prod_{k=2}^{N_c} \langle T/2, x' | G_{f_k g_k} (U) | -T/2, x \rangle \times \langle T/2, x' | G_{f_i d_i} (U) | 0, 0 \rangle \mathcal{O}^\mu x \Gamma (E.11a) \langle 0, 0 \rangle \mathcal{O}^\mu x \rho (E.11b) \times \mathcal{O}^\mu x \psi \rangle | -T/2, x \rangle e^{N_c \text{Tr} \log D(U)}. \]  
(E.8)

In the second case, one obtains the so-called sea contribution $s$, given by
\[ \langle 0 | J_B^\alpha (x') \psi^\dagger \mathcal{O}^\mu x \psi J_B^\dagger (x) | 0 \rangle_s = \frac{N_c}{Z} \Gamma^{(f)}_{B'} \Gamma^{(g)}_B | \prod_{k=1}^{N_c} \langle T/2, x' | G_{f_k g_k} (U) | -T/2, x \rangle \times \mathcal{O}^\mu x \psi \rangle | -T/2, x \rangle e^{N_c \text{Tr} \log D(U)}. \]  
(E.9)

### E.2 Valence and sea contributions

#### E.2.1 How to obtain expressions for $\chi = 0$

Taking the projections onto the nonstrange and strange subspaces of the expression for $A^\dagger \lambda^\dagger A$ for $\chi \neq 0$ one obtains:
\[ P_T A^\dagger \lambda^\dagger A P_T = \sum_{\alpha} D^{8}_{\alpha} A P_T \lambda^\dagger P_T = \sum_i D^{8}_{\alpha} A P_T \lambda^\dagger P_T + D^{8}_{\lambda^\dagger}, \]  
(E.10a)
\[ P_T A^\dagger \lambda^\dagger A P_S = \sum_{\alpha} D^{8}_{\alpha} A P_T \lambda^\dagger P_S = \sum_a D^{8}_{\alpha} A P_T \lambda^\dagger P_S, \]  
(E.10b)

The related expressions for $\chi = 0$ are in turn
\[ P_T A^\dagger \lambda^0 A P_T = 1, \]  
(E.11a)
\[ P_T A^\dagger \lambda^0 A P_S = 0. \]  
(E.11b)

The case $\chi = 0$ can be then recovered from the case $\chi \neq 0$ by the replacements
\[ \chi \to 0, \]  
\[ D^{(8)}_{0\beta} \to \delta_{8\beta} \sqrt{3}. \]  
(E.12a)

#### E.2.2 The valence contribution

With the restriction to the zero modes (E.8) becomes
\[ \langle 0 | J_B^\alpha (x') \psi^\dagger \mathcal{O}^\mu x \psi J_B^\dagger (x) | 0 \rangle_v = \frac{N_c}{Z} \Gamma^{(f)}_{B'} \Gamma^{(g)}_B | \prod_{k=2}^{N_c} \langle T/2, x' | G_{f_k g_k} (U) | -T/2, x \rangle \times \langle T/2, x' | G_{f_i d_i} (U) | 0, 0 \rangle \mathcal{O}^\mu x \rho (E.13) \times \mathcal{O}^\mu x \psi \rangle | -T/2, x \rangle e^{N_c \text{Tr} \log D(U)}, \]  
with the propagators $G(U)$ to be treated in the way of expression (2.41). The main point to consider in the expression above is the expansion of the propagators: the product of propagators not attached to the operator will yield the valence one-particle wave functions which will enter the construction of the collective wave functions and a contribution which will be exponentiated to contribute to the effective action, in a similar way to what is described in the quantization of the
Finally, in the limit $T \to \infty$

\[ N_c \langle T/2, x' | G_{f_1d}(U) | 0, 0 \rangle \mathcal{O}_{dd}^{\mu\chi} \langle 0, 0 | G_{d'g_1}(U) | -T/2, x \rangle = e^{-v_T} \left( A\phi_V(x' + z) \right)_{f_1} \mathcal{F}_V^{\mu\chi}(z) \left( \phi_V(x + z) A \right)_{g_1} \quad \text{(E.17)} \]
with
\[
\frac{1}{N_c} \mathcal{F}^\Omega_V(z) = \phi_V^i(z) P_T A^i \mathcal{O}^\Omega A P_T \phi_V(z)
\]
\[
- \phi_V^i(z) P_T A^i \mathcal{O}^\Omega \int d\tau e^{-v\tau} \langle 0, z | AG \rangle \left( A^i \hat{A} + \gamma_4 A^i \delta m A \right) |v\rangle P_T
\]
\[
- \int d\tau e^{-v\tau} \langle v | A^i \hat{A} + \gamma_4 A^i \delta m A \rangle \langle \tau | G A^i | z, 0 \rangle P_T \mathcal{O}^\Omega P_T A \phi_V(z)
\] (E.18)
dropping the flavor/spin indices. The quantized version of this function is obtained by means of the substitutions (2.58).

Using the above result for the expansion of the propagators in (E.13), it can rewritten as
\[
\langle 0 | J_B(x') \psi^\dagger \mathcal{O}^\Omega \psi J_B(x) | 0 \rangle_V = \frac{1}{Z} \Gamma_{B'}^{(f)} \Gamma_B^{(g)*}
\]
\[
\times \int d^3z \int [dA] \prod_{k=1}^{N_c} (A \phi_V(x' + z)) f_k \left( \phi_V^i(x + z) A^i \right) g_k \mathcal{F}^\Omega_V(z) e^{-S^\Omega_{sc}(AUA')}.
\] (E.19)

omitting the explicit expansion of the second order terms of the propagators and the expansion of the effective action which give together the semiclassical effective action $S^\Omega_{sc}(AUA')$, as explained in Section D.4. The integration over $x, x'$ gives
\[
\int d^3x d^3x' e^{i p \cdot x - i p' \cdot x'} \langle 0 | J_B(x') \psi^\dagger \mathcal{O}^\Omega \psi J_B(x) | 0 \rangle_V =
\]
\[
\frac{1}{Z} \int d^3z \int [dA] \left[ \int d^3x e^{-i p' \cdot x'} \Gamma_{B'}^{(f)} \prod_{k=1}^{N_c} (A \phi_V(x' + z)) f_k \right]
\times \mathcal{F}^\Omega_V(z) \left[ \int d^3x e^{i p \cdot x} \Gamma_B^{(g)*} \prod_{k=1}^{N_c} (\phi_V^i(x + z) A^i) g_k \right] e^{-S^\Omega_{sc}(AUA')}.
\]
\[
= \frac{1}{Z} \int d^3z \int [dA] e^{i p' \cdot z} \left[ \int d^3y e^{-i p' \cdot y'} \Gamma_{B'}^{(f)} \prod_{k=1}^{N_c} (A \phi_V(y')) f_k \right]
\times \mathcal{F}^\Omega_V(z) e^{-i p \cdot z} \left[ \int d^3y e^{i p \cdot y} \Gamma_B^{(g)*} \prod_{k=1}^{N_c} (\phi_V^i(y) A^i) g_k \right] e^{-S^\Omega_{sc}(AUA')}.
\] (E.20)

with the change of variables $x' = y' - z$, $x = y - z$ and the definitions of the collective have functions (2.51)
\[
\int d^3y' e^{-i p' \cdot y'} \Gamma_{B'}^{(f)} \prod_{k=1}^{N_c} (A \phi_V(y')) f_k \rightarrow \Psi^*_{B'}(A),
\] (E.21)
\[
\int d^3y e^{i p \cdot y} \Gamma_B^{(g)*} \prod_{k=1}^{N_c} (\phi_V^i(y) A^i) g_k \rightarrow \Psi_B(A).
\] (E.22)

Now the integration over the collective orientation $A$ can be performed using
\[
\int [dA] \Psi^*_{B'}(A) \mathcal{O} \Psi_B(A) e^{-T H_{coll}} = \langle B' | \mathcal{O} | B \rangle = \int [dA] \Psi^*_{B'}(A) \mathcal{O} \Psi_B(A).
\] (E.23)

The final result for the valence contribution to $\langle B'(S'_3, p') | \psi^\dagger \mathcal{O}^\Omega \psi | B(S_3, p) \rangle_V$ is, substituting the normalization constants,
\[
\langle B'(S'_3, p') | \psi^\dagger \mathcal{O}^\Omega \psi | B(S_3, p) \rangle_V = \int d^3z \ e^{i q \cdot z} \langle B' | \mathcal{F}^\Omega_V(z) | B \rangle
\] (E.24)
with $q = p' - p$ (E.25).
the momentum transfer. The function $F^\mu_3(z)$ defined by (E.18) is in a next step written in terms of the one-particle Dirac eigenvalues and functions taking into account the spectral representation of the remaining propagator and the effects of the collective rotation operators on the operators. Such a calculation is involved but straightforward and so left to the Section E.3.

E.2 Valence and see contributions

We start with the sea contribution from (E.9) and again making a restriction to the zero modes (2.38)

$$\langle 0 | J_B(x') \psi^\dagger \mathcal{O}^{\mu \chi} \psi J_B(x) | 0 \rangle_s =$$

$$\begin{aligned}
&\frac{N_c}{2} \Gamma_f \Gamma_f \int d^3 z \left[ N_c \prod_{k=1}^{N_c} \langle T/2, x' | G_{f_k g_k} (AU_c A^\dagger) | -T/2, x \rangle \\
&\times \frac{\delta}{\delta s^\mu(0, z)} \text{Tr} \left[ \log A^\dagger D(U, s) A \right]_{s^\mu = 0} e^{N_c \text{Tr} \log D(U_c A^\dagger)}.
\end{aligned}
$$

(2.26)

with

$$D(U, s) = D(U) - s^\mu A^\dagger \mathcal{O}^{\mu \chi} A.$$  

(2.27)

The product of propagators is treated in the same way as in the valence case (E.14)

$$\begin{aligned}
&\prod_{k=1}^{N_c} \langle T/2, x' | G_{f_k g_k} (AU_c A^\dagger) | -T/2, x \rangle \\
&\times \frac{\delta}{\delta s^\mu(0, z)} \text{Tr} \left[ \log A^\dagger D(U, s) A \right]_{s^\mu = 0}.
\end{aligned}
$$

(2.28)

where the last factor will be exponentiated. Thus

$$\begin{aligned}
&\langle 0 | J_B(x') \psi^\dagger \mathcal{O}^{\mu \chi} \psi J_B(x) | 0 \rangle_s = \\
&\frac{1}{2} \Gamma_f \Gamma_f \int dA (-T/2) \int dA (T/2) \\
&\times \int d^3 z \left[ N_c \prod_{k=1}^{N_c} (A (T/2) \phi_{V_k} (x' + z)) f_k \left( \phi_{V_k}^\dagger (x + z) A^\dagger \right)_{g_k} (1 + \cdots),
\end{aligned}
$$

(2.29)

with

$$\begin{aligned}
&\mathcal{F}^{\mu \chi}_s(z) = N_c \frac{\delta}{\delta s^\mu(0, z)} \text{Tr} \left[ \log A^\dagger D(U, s) A \right]_{s^\mu = 0}.
\end{aligned}
$$

(2.30)

Proceeding in the same way as in the valence case for the integration over $x, x'$ (E.20), making the change of variables $x' = y' - z, x = y - z$, using the collective wave function definitions (E.21),(E.22) it is possible to arrive at

$$\begin{aligned}
&\int d^3 x d^3 x' e^{i P \cdot x} e^{-i P' \cdot x'} \langle 0 | J_B(x') \psi^\dagger \mathcal{O}^{\mu \chi} \psi J_B(x) | 0 \rangle_s = \\
&= \frac{1}{2} \int d^3 z e^{i (P' - P) \cdot z} \int dA \Psi^\dagger_B(A) \mathcal{F}^{\mu \chi}_s(z) \Psi_B(A)
\end{aligned}
$$

(2.31)

and finally at

$$\langle B'(S_{3}', p') | \psi^\dagger \mathcal{O}^{\mu \chi} \psi | B(S_3, p) \rangle_s = \int d^3 z e^{i q \cdot z} \langle B' | \mathcal{F}^{\mu \chi}_s(z) | B \rangle.$$

(2.32)

As for the functional trace, it is convenient as this point to separate the contributions coming from the real and from the imaginary part of the effective action, since the real part will have to be regularized contrary to the contribution coming from the imaginary part which is finite and does not need regularization. The trace splits thus into two terms

$$\begin{aligned}
&\text{Tr} \left[ \log A^\dagger D(U, s) A \right] = \frac{1}{2} \text{Tr} \left[ \log D^\dagger(U, s) D(U, s) \right] + \frac{1}{2} \text{Tr} \left[ \log D(U, s) \right]
\end{aligned}
$$

(2.33)

the first being divergent and the second not. It better to calculate them separately according to

$$\langle 0 | J_B(x') \psi^\dagger \mathcal{O}^{\mu \chi} \psi J_B(x) | 0 \rangle_s = \int d^3 z e^{i q \cdot z} \langle B' | \mathcal{F}^{\mu \chi}_{s, R}(z) + \mathcal{F}^{\mu \chi}_{s, I}(z) | B \rangle.$$  

(2.34)
defining

\[ F_{S,R}^{\mu_N}(z) = N_c \frac{\delta}{8 \pi^2} \delta s^\mu(0, z) \text{Tr} \left[ \log D^\dagger(U, s) D(U, s) \right] \rightarrow F_{S,R}^{\mu_N}(z) \tag{E.35} \]

with

\[ F_{S,R}^{\mu_N}(z) = -N_c \frac{\delta}{8 \pi^2} \delta s^\mu(0, z) \text{Tr} \left[ \log D^\dagger(U, s) D(U, s) \right] \bigg|_{\mu^\mu=0} \tag{E.36} \]

\[ F_{S,I}^{\mu_N}(z) = N_c \frac{\delta}{8 \pi^2} \delta s^\mu(0, z) \text{Tr} \left[ \log D^\dagger(U, s) \right] \bigg|_{\mu^\mu=0} \tag{E.37} \]

The proper-time regularization and its associated cut-off \( \Lambda \) have been introduced for \( F_{S,R}^{\mu_N}(z) \) (the subscript 'regul' will be dropped henceforth). The final expressions for \( F_{S,R}^{\mu_N}(z) \) allow for the calculation of flavor components of the form factors.

### E.3 Expansions in the angular velocity and \( \delta m \)

Here we deal with the technicalities like introducing the quantization (2.58) and subsequent determination of the results up to orders of \( 1/N_c \) and \( m_5^1 \), with particular emphasis given to the time ordering of the collective operators. As a remainder in the following the summation over repeated indices is implicit, unless otherwise stated: Latin indices \( i, j, k \ldots \) run from 1 to 3, Latin indices \( a, b, c, \ldots \) run from 4 to 7, and Greek indices run from 0 to 8.

The operator \( O^{\mu_N} \) acting on the flavor space may either be one of the \( \lambda \) matrices \( \lambda_\chi \), or it may be just the unit matrix in the flavor space. In this last case one speaks of singlet observables, a designation also applicable to other objects, for instance currents. Since the behavior of the \( \lambda \) matrices is different from that of the unit matrix under rotations, one is lead to think that the results for the observables should be quite different. They indeed are. However, there is a simple prescription to obtain the singlet case from the general one, as explained in the following section.

#### E.3.1 Valence

The general expression for \( F_{\chi}(z) \) in the case of a general Dirac structure \( O^{\mu_N} = O^{\mu} \lambda_\chi \) is given by (E.18). The case \( \chi = 0 \) does not need to be explicitly evaluated as explained in Section E.2.1.

With \( A^\dagger \lambda^\chi A = D^{(8)}_5 \lambda^\chi \)

\[
\frac{1}{N_c} F_{V}^{\mu_N}(z) = \langle v | z \rangle \Gamma^\mu P_T A^\dagger \lambda^\chi A P_T \langle z | v \rangle \tag{E.38a}
\]

\[- \langle v | z \rangle \Gamma^\mu P_T A^\dagger \lambda^\chi \int_{-T/2}^{T/2} d\tau \ e^{-\epsilon \nu \tau} \langle 0, z | G | \tau \rangle A^\dagger \hat{A} | v \rangle P_T \tag{E.38b}\]

\[- \langle v | z \rangle \Gamma^\mu P_T A^\dagger \lambda^\chi \int_{-T/2}^{T/2} d\tau \ e^{-\epsilon \nu \tau} \langle 0, z | G | \tau \rangle i\gamma_A A^\dagger \delta m A | v \rangle P_T \tag{E.38c}\]

\[- \int_{-T/2}^{T/2} d\tau \ e^{\nu \tau} \langle v | P_T A^\dagger \hat{A} \langle \tau | G | z, 0 \rangle \Gamma^\mu A^\dagger \lambda^\chi A P_T \langle z | v \rangle \tag{E.38d}\]

\[ \int_{-T/2}^{T/2} d\tau \ e^{\nu \tau} \langle v | i\gamma_A P_T A^\dagger \delta m A \langle \tau | G | z, 0 \rangle \Gamma^\mu A^\dagger \lambda^\chi A P_T \langle z | v \rangle \tag{E.38e}\]

\[ = \sum_{i=1}^{5} f_{iV}^{\mu_N}(z). \]

The first of these terms is independent either of the angular velocity and mass corrections; the second and fourth depend linearly on the angular velocity and the remaining ones are linear on the mass correction \( \delta m \). The next step consists in the projection of the quark propagator on strange \( P_S \) and nonstrange \( P_T \) flavor subspaces, \( G = G_T P_T + G_S P_S \), after which its spectral decomposition
In the nonstrange contribution collective operators, \( A \) performing the integrals over the intermediate time and taking the limit of large time separation.

\[ E.3 \text{ Expansions in the angular velocity and } \delta m \]

is taken into account.

This gives for the first term (E.38a)

\[
\begin{align*}
\int_{-T/2}^{T/2} d\tau & e^{-i\omega \tau} \langle 0, z | G | \tau \rangle \; A^\dagger A P_T | v \rangle = \int_{-T/2}^{T/2} d\tau e^{-i\omega \tau} \langle 0, z | G | \tau \rangle \; A^\dagger A P_T | v \rangle
\end{align*}
\]

(E.39)

The next term (E.38b) splits into strange and nonstrange contributions

\[ f_{2V}^{\mu X} = - \langle v | z \rangle \Gamma^\mu P_T A^\dagger \lambda^X A P_T | z | v \rangle + \frac{1}{\sqrt{3}} D^{(8)}_{\chi_8} \langle v | z \rangle \Gamma^\mu \langle z | v \rangle \]

(E.40)

In the nonstrange contribution \( f_{2V}^{\mu X} \) it is necessary to take into account the time ordering of the collective operators, \( A^\dagger \lambda^X A \) and \( A^\dagger A \).

\[ f_{2V}^{\mu X} = - \langle v | z \rangle \Gamma^\mu P_T A^\dagger \lambda^X A P_T \int_{-T/2}^{T/2} d\tau e^{-i\omega \tau} \langle 0, z | G | \tau \rangle \; A^\dagger A P_T | v \rangle = f_{2V}^{\mu T} + f_{2V}^{\mu S} 
\]

(E.41)

Performing the integrals over the intermediate time and taking the limit of large time separation. For the forth term, also linear in the angular velocity, is treated similarly:

\[ f_{4V}^{\mu X} = - \int_{-T/2}^{T/2} d\tau \; \langle v | z | v \rangle \; P_T A^\dagger A (\tau | G | z, 0) A^\dagger A \lambda^X A P_T | z | v \rangle = f_{4V}^{\mu X, T} + f_{4V}^{\mu X, S} 
\]

(E.43)

(E.44a)

(E.44b)

For the terms which involve the symmetry breaking term \( \delta m \) :
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\[ f_{\chi,i}^{\mu,S,T} = - (\langle z | P_T \Gamma^\mu A^i \lambda^\chi A P_T \int_{-T/2}^0 dt e^{-\epsilon v t} \langle \chi | \sum_{n \geq 0} e^{\epsilon n t} | n \rangle \langle n | i \gamma_4 P_T A^i \delta m A P_T | v \rangle + \langle v | P_T \Gamma^\mu A^i \lambda^\chi A P_T \int_{0}^{T/2} dt e^{-\epsilon v t} \langle \chi | \sum_{n < 0} e^{\epsilon n t} | n \rangle \langle n | i \gamma_4 P_T A^i \delta m A P_T | v \rangle = \left( \frac{1}{\sqrt{3}} M_1 D^{(8)}_{\chi^8} + \frac{1}{\sqrt{3}} M_8 D^{(8)}_{\chi^8} \right) \sum_{n \neq 0} \frac{1}{\epsilon_v - \epsilon_n} \langle \chi | \sum_{n \neq 0} e^{-\epsilon n t} | n \rangle \langle n | i \gamma_4 | v \rangle + \left( M_1 D^{(8)}_{\chi^8} + \frac{1}{\sqrt{3}} M_8 D^{(8)}_{\chi^8} \right) \sum_{n \neq 0} \frac{1}{\epsilon_v - \epsilon_n} \langle \chi | \sum_{n \neq 0} e^{-\epsilon n t} | n \rangle \langle n | i \gamma_4 | v \rangle + \frac{1}{\sqrt{3}} M_8 D^{(8)}_{\chi^8} \sum_{n \neq 0} \frac{1}{\epsilon_v - \epsilon_n} \langle \chi | \sum_{n \neq 0} e^{-\epsilon n t} | n \rangle \langle n | i \gamma_4 | v \rangle + M_8 D^{(8)}_{\chi^8} \sum_{n \neq 0} \frac{1}{\epsilon_v - \epsilon_n} \langle \chi | \sum_{n \neq 0} e^{-\epsilon n t} | n \rangle \langle n | i \gamma_4 | v \rangle \right) \]

\[ f_{\Delta}^{\mu,S,T} = \frac{M_8 D^{(8)}_{\chi^8} \sum_{n \neq 0} \frac{1}{\epsilon_v - \epsilon_n} \langle \chi | \sum_{n \neq 0} e^{-\epsilon n t} | n \rangle \langle n | i \gamma_4 | v \rangle \right) \]

usings (A.15a) and (A.15b).

For the remaining term, also linear in \( \delta m \),

\[ f_{S,F}^{\mu,S} = - \frac{1}{T/2} \int_{-T/2} dx \langle v | i \gamma_4 A^i \delta m A \langle \tau | G | z, 0 \rangle \Gamma^\mu P_T A^i \lambda^\chi A P_T \langle z | v \rangle = f_{S,F}^{\mu,S} + f_{S,F}^{\mu,T} \]

\[ f_{S,F}^{\mu,S} = M_8 D^{(8)}_{\chi^8} \sum_{n \neq 0} \frac{1}{\epsilon_v - \epsilon_n} \langle \chi | \sum_{n \neq 0} e^{-\epsilon n t} | n \rangle \langle n | i \gamma_4 | v \rangle \right) \]

These contributions can now be joined together. For example, and further making use of the relations

\[ \frac{J_i}{2I_1} D^{(8)}_{\chi^8} = \frac{1}{2} \left\{ J_i, D^{(8)}_{\chi^8} \right\} + \frac{1}{2} \left[ J_i, D^{(8)}_{\chi^8} \right] \]

\[ \frac{D^{(8)}_{\chi^8}}{2I_1} \frac{J_i}{2I_1} = \frac{1}{2} \left\{ J_i, D^{(8)}_{\chi^8} \right\} - \frac{1}{2} \left[ J_i, D^{(8)}_{\chi^8} \right] \]

one obtains for the nonstrange part

\[ f_{2,4,F}^{\mu,S,T} + f_{4,4,F}^{\mu,S,T} = - \frac{K_1}{I_1} m_8 D^{(8)}_{\chi^8} \sum_{n \neq 0} \frac{1}{\epsilon_v - \epsilon_n} \left( \langle v | P_T \lambda^\alpha P_T \langle n | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | v \rangle \right) + \langle v | \Gamma^\mu P_T \lambda^\beta P_T \langle z | n \rangle \langle n | P_T \lambda^\chi P_T | v \rangle \right) \]
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\[ \frac{1}{2} \left\{ J_a \right\}_{\chi^{(d)}} \sum_{n \neq 0} \frac{1}{\varepsilon_n} \left( \langle v | P_T \lambda^i P_T | n \rangle \langle n | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | v \rangle \right) \]

\[ + \frac{1}{2} \left\{ J_a \right\}_{\chi^{(d)}} \sum_{n > 0} \frac{1}{\varepsilon_n} \left( \langle v | P_T \lambda^i P_T | n \rangle \langle n | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | v \rangle \right) \]

\[ - \frac{1}{2} \left\{ J_a \right\}_{\chi^{(d)}} \sum_{n < 0} \frac{1}{\varepsilon_n} \left( \langle v | P_T \lambda^i P_T | n \rangle \langle n | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | v \rangle \right) \]

and for the other contribution

\[ f_2^{\mu|S} + f_4^{\mu|S} = - \frac{K_2}{I_2} m s D_8^{(8)} D_8^{(8)} \chi^{(d)} \sum_{n \neq 0} \frac{1}{\varepsilon_n} \left( \langle v | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | n \rangle \langle n | v \rangle \right) \]

\[ + \frac{1}{2} \left\{ J_a \right\}_{\chi^{(d)}} \sum_{n > 0} \frac{1}{\varepsilon_n} \left( \langle v | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | n \rangle \langle n | v \rangle \right) \]

\[ + \frac{1}{2} \left\{ J_a \right\}_{\chi^{(d)}} \sum_{n < 0} \frac{1}{\varepsilon_n} \left( \langle v | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | n \rangle \langle n | v \rangle \right) \]

\[ - \frac{1}{2} \left\{ J_a \right\}_{\chi^{(d)}} \sum_{n < 0} \frac{1}{\varepsilon_n} \left( \langle v | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | n \rangle \langle n | v \rangle \right) \]

\[ - \langle v | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | n \rangle \langle n | v \rangle \]

Taking the same kind of procedure for terms (E.38c) and (E.38c) leads to the following result for the valence part ($P_T \lambda^i P_T = \tau^i$):

\[ \frac{1}{N_c} \mathcal{F}^{\mu|j}(z) = D_8^{(8)} \langle v | z \rangle \Gamma^\mu \tau^j \langle z | v \rangle + \frac{1}{\sqrt{3}} D_8^{(8)} \langle v | z \rangle \Gamma^\mu \langle z | v \rangle \]

\[ + \frac{1}{2} \left\{ J_a \right\}_{\chi^{(d)}} \sum_{n \neq 0} \frac{1}{\varepsilon_n} \left( \langle v | \tau^i | n \rangle \langle n | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | v \rangle \right) \]

\[ \times \left( \langle v | \tau^i | n \rangle \langle n | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | v \rangle \right) \]

\[ + \frac{1}{2} \left\{ J_a \right\}_{\chi^{(d)}} \sum_{n > 0} \frac{1}{\varepsilon_n} \left( \langle v | \tau^i | n \rangle \langle n | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | v \rangle \right) \]

\[ \times \left( \langle v | \tau^i | n \rangle \langle n | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | v \rangle \right) \]

\[ + \frac{1}{2} \left\{ J_a \right\}_{\chi^{(d)}} \sum_{n > 0} \frac{1}{\varepsilon_n} \left( \langle v | \tau^i | n \rangle \langle n | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | v \rangle \right) \]

\[ - \langle v | \tau^i | n \rangle \langle n | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | v \rangle \]

\[ - \langle v | \tau^i | n \rangle \langle n | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | v \rangle \]

\[ - \langle v | \tau^i | n \rangle \langle n | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | v \rangle \]
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\[ \times \left( \langle v | n^0 \rangle \langle n^0 | z \rangle \Gamma^\mu P_T \lambda^\alpha P_S \lambda^\beta P_T \langle z | v \rangle - \langle v | z \rangle \Gamma^\mu P_T \lambda^\beta P_S \lambda^\alpha P_T \langle z | n^0 \rangle \langle n^0 | v \rangle \right) \]

\[ - \frac{K_1}{I_1} M_8 D_{8\delta}^{(8)} D_{\chi^\delta}^{(8)} \sum_{n \neq 0} \frac{1}{\varepsilon_n - \varepsilon_n} \]

\[ \times \left( \langle v | z \rangle \Gamma^\mu P_T \lambda^\beta P_S \lambda^\alpha P_T \langle z | n^0 \rangle \langle n^0 | v \rangle + \langle v | n^0 \rangle \langle n^0 | z \rangle \Gamma^\mu P_T \lambda^\alpha P_S \lambda^\beta P_T \langle z | v \rangle \right) \]

\[ + \left( \frac{1}{\sqrt{3}} M_1 D_{\chi^8}^{(8)} + \frac{1}{3} M_8 D_{8\delta}^{(8)} D_{\chi^8}^{(8)} \right) \sum_{n \neq 0} \frac{1}{\varepsilon_n - \varepsilon_n} \]

\[ \times \left( \langle v | z \rangle \Gamma^\mu \langle z | n \rangle \langle n | \gamma^0 | v \rangle + \langle n | z \rangle \Gamma^\mu \langle z | v \rangle \langle v | \gamma^0 | n \rangle \right) \]

\[ + \left( \frac{1}{\sqrt{3}} M_1 D_{\chi^8}^{(8)} + \frac{1}{3} M_8 D_{8\delta}^{(8)} D_{\chi^8}^{(8)} \right) \sum_{n \neq 0} \frac{1}{\varepsilon_n - \varepsilon_n} \]

\[ \times \left( \langle v | z \rangle \Gamma^\mu \tau^j \langle z | n \rangle \langle n | \gamma^0 \tau^k | v \rangle + \langle n | z \rangle \Gamma^\mu \tau^j \langle z | v \rangle \langle v | \gamma^0 \tau^k | n \rangle \right) \]

\[ + \frac{1}{3} M_2 D_{8\delta}^{(8)} D_{\chi^8}^{(8)} \sum_{n \neq 0} \frac{1}{\varepsilon_n - \varepsilon_n} \left( \langle v | z \rangle \Gamma^\mu \tau^j \langle z | n \rangle \langle n | \gamma^0 \tau^k | v \rangle + \langle n | z \rangle \Gamma^\mu \tau^j \langle z | v \rangle \langle v | \gamma^0 \tau^k | n \rangle \right) \]

\[ + \left( \frac{1}{\sqrt{3}} M_2 D_{8\delta}^{(8)} \right) \sum_{n \neq 0} \frac{1}{\varepsilon_n - \varepsilon_n} \left( \langle v | z \rangle \Gamma^\mu P_T \lambda^\beta P_S \lambda^\alpha P_T \langle z | n^0 \rangle \langle n^0 | \gamma^0 | v \rangle \right) \]

\[ + \langle n^0 | z \rangle \Gamma^\mu P_T \lambda^\alpha P_S \lambda^\beta P_T \langle z | v \rangle \langle v | \gamma^0 | n^0 \rangle \) \quad (E.52) \]

which can only be simplified with further knowledge of \( \Gamma^\mu \).

E.3.2 Sea contribution

For this quantity the real (regularized) and imaginary (finite) parts will be calculated separately.

Regularized real part

This section is concerned with the quantization and calculation of (E.37) The first step concerns the calculation of the functional derivative:

\[ \frac{\delta}{\delta s^\mu(0, z)} \text{Tr} e^{-uD^1(U, s)D(U, s)} \bigg|_{s^\mu=0} = \frac{\delta}{\delta s^\mu(0, z)} \int d^4x \langle x | e^{-uD^1(U, s)D(U, s)} | x \rangle \bigg|_{s^\mu=0} \]

\[ = u \text{tr} \langle 0, z | e^{-uD^1D} \left( D^1(A^\dagger O^\mu A) + (A^\dagger O^\mu A) D \right) | 0, z \rangle. \] \quad (E.53)

Now the integration over \( u \) can be made:

\[ \int_{1/\Lambda^2}^\infty du e^{-uD^1D} = e^{-D^1D/\Lambda^2} \left( D^1D \right)^{-1}. \] \quad (E.54)

The result of these steps is

\[ \mathcal{F}_{S, s}^{\mu}(z) = -N_c \frac{1}{2} \text{tr} \langle 0, z | e^{-D^1D/\Lambda^2} \left( D^1D \right)^{-1} (A^\dagger O^\mu A) + (A^\dagger O^\mu A) D \right) | 0, z \rangle - \text{v.s.} \]

\[ = -N_c \frac{1}{2} \text{tr} \langle 0, z | e^{-D^1D/\Lambda^2} D^{-1} | 0, z \rangle A^\dagger O^\mu A \]

\[ - N_c \frac{1}{2} \text{tr} \langle 0, z | e^{-D^1D/\Lambda^2} D^{-1} | 0, z \rangle A^\dagger O^\mu A - \text{v.s.} \) \quad (E.55) \]

where, in the last term, the cyclic property of the trace was used together with the fact that \( D \) commutes with the exponential. In this expression v.s. stands for the vacuum subtraction, the
second term in (E.37)). The expression above can be simplified further since
\[ \langle 0, z | e^{-D^2 D/\Lambda^2} D^{-1} | 0, z \rangle = \langle 0, z | e^{-D^2 D/\Lambda^2} D^{-1} | 0, z \rangle^* , \]
i.e.
\[ F_{\mu \nu}^{\mu_X}(z) = -N_c \frac{1}{2} \text{tr} \left( \langle 0, z | e^{-D^2 D/\Lambda^2} D^{-1} | 0, z \rangle \right. \]
\[ \left. + \eta \langle 0, z | e^{-D^2 D/\Lambda^2} D^{-1} | 0, z \rangle^* \right) A^\dagger O^{\mu_X} A - \text{v.s.} \]
defining
\[ O^{\mu_X} = \eta O^{\mu_X} . \]
It is now clear that the next step is to calculate \( \langle 0, z | e^{-D^2 D/\Lambda^2} G | 0, z \rangle \). It is calculated in the limit of large \( N_c \) and of the expansion in \( \delta m \). Since
\[ D^\dagger D = \left( D_c^\dagger - A^\dagger \dot{A} + i \gamma_4 A^\dagger \delta m A \right) \left( D_c + A^\dagger \dot{A} + i \gamma_4 A^\dagger \delta m A \right) \]
\[ = D^\dagger D_c + D^\dagger_i A^\dagger \dot{A} - A^\dagger \dot{A} D_c + D^\dagger_i i \gamma_4 A^\dagger \delta m A + i \gamma_4 A^\dagger \delta m AD_c \]
one can use the operator relation
\[ e^{A+B} = e^A + \int_0^1 da \ e^{A} B e^{(1-a)A} + \ldots \]
to obtain
\[ e^{-D^2 D/\Lambda^2} = e^{-D^\dagger D_c/\Lambda^2} - \frac{1}{N_c} \int_0^1 da \ e^{-aD^\dagger D_c/\Lambda^2} \]
\[ \times \left( D^\dagger_i A^\dagger \dot{A} - A^\dagger \dot{A} D_c + D^\dagger_i i \gamma_4 A^\dagger \delta m A + i \gamma_4 A^\dagger \delta m AD_c \right) e^{-(1-a)D^\dagger D_c/\Lambda^2} + \ldots \]
For \( G = D^{-1} \) we use as before
\[ G = D^{-1} = \left( D_c + A^\dagger \dot{A} + i \gamma_4 A^\dagger \delta m A \right)^{-1} = G_c - G_c \left( A^\dagger \dot{A} + i \gamma_4 A^\dagger \delta m A \right) G_c + \ldots \]
Keeping terms up to the first order in the angular velocity \( A^\dagger \dot{A} \) and in the symmetry breaking term \( \delta m \) the product \( \exp(-D^2 D/\Lambda^2)G \) becomes
\[ e^{-D^2 D/\Lambda^2} G + \frac{1}{N_c} \int_0^1 da \ e^{-aD^\dagger D_c/\Lambda^2} \left( D^\dagger_i A^\dagger \dot{A} - A^\dagger \dot{A} D_c + D^\dagger_i i \gamma_4 A^\dagger \delta m A + i \gamma_4 A^\dagger \delta m AD_c \right) \]
\[ = e^{-D^\dagger D_c/\Lambda^2} G_c - e^{-D^\dagger D_c/\Lambda^2} G_c A^\dagger \dot{A} G_c - e^{-D^\dagger D_c/\Lambda^2} G_c i \gamma_4 A^\dagger \delta m AG_c \]
\[ - \frac{1}{N_c} \int_0^1 da \ e^{-aD^\dagger D_c/\Lambda^2} \left( D^\dagger_i A^\dagger \dot{A} G_c - A^\dagger \dot{A} \right) e^{-(1-a)D^\dagger D_c/\Lambda^2} \]
\[ - \frac{1}{N_c} \int_0^1 da \ e^{-aD^\dagger D_c/\Lambda^2} \left( D^\dagger_i i \gamma_4 A^\dagger \delta m G_c + i \gamma_4 A^\dagger \delta m A \right) e^{-(1-a)D^\dagger D_c/\Lambda^2} + \ldots \]
which allows to rewrite (E.57) as a sum of five terms denoted, in the above order, by
\[ \frac{1}{N_c} F_{\mu \nu}^{\mu_X}(z) = \sum_{i=1}^{5} f_{1,i}^{\mu_X}(z) . \]
Again from the projections into nonstrange and strange subspaces one has
\[ e^{-D^\dagger D_c/\Lambda^2} = e^{-\left(D^\dagger_T P_T + D^\dagger_S P_S\right)/\Lambda^2} = e^{-D^\dagger_T D_T/\Lambda^2 P_T} + e^{-D^\dagger_S D_S/\Lambda^2 P_S} \]
with \( G_c = G_T P_T + G_S P_S \).
The first term coming from (E.63) turns thus into
\[ f_{1,1}^{\mu_X} = -\frac{1}{2} D^{(8)}_{\chi \beta} \text{tr} \left( \langle 0, z | e^{-D^\dagger D_c/\Lambda^2} G_c | 0, z \rangle + \eta \langle 0, z | e^{-D^\dagger D_c/\Lambda^2} G_c | 0, z \rangle^* \right) \lambda^\beta \Gamma^\mu - \text{v.s.} \]
\[ = -\frac{1}{2} D^{(8)}_{\chi \beta} \sum_n \langle n | z \rangle P_T \lambda^\beta \Gamma^\mu \langle z | n \rangle \frac{1+n}{2} \int \frac{d\omega}{2\pi} e^{-i\omega z_n} \frac{2\zeta n}{\omega^2 + \zeta^2} \]
\[ = D^{(8)}_{\chi \beta} \sum_n \langle n | z \rangle P_T \lambda^\beta \Gamma^\mu P_T \langle z | n \rangle \mathcal{R}_1(\varepsilon_n, \eta) \]
with the regularization function defined by
\[ R_1(\varepsilon_n, \eta) = -\frac{1 + \eta}{2} \varepsilon_n \int_{-2\pi}^{2\pi} \frac{d\omega}{2\pi} e^{-\omega^2 + \varepsilon_n^2} \delta_{\eta 1} R_1(\varepsilon_n) \] (E.67)
with \( R_1(\varepsilon_n) \) defined in (E.70).

Making \( \alpha = 1/\Lambda^2 \) in the regularization function (E.67) and using
\[ \frac{d}{d\omega} \int_{-2\pi}^{2\pi} \frac{d\omega}{2\pi} e^{-\omega^2 + \varepsilon_n^2} = \int_{-2\pi}^{2\pi} \frac{d\omega}{2\pi} e^{-\omega^2 + \varepsilon_n^2} = \frac{1}{2\sqrt{\pi}} e^{-\varepsilon_n^2} \] (E.68)
one can easily rewrite the function \( R_1(\varepsilon_n, \eta) \) in the form
\[ R_1(\varepsilon_n, \eta) = -\delta_{\eta 1} \frac{1}{2\sqrt{\pi}} \varepsilon_n \int_{1/\Lambda^2}^{\infty} \frac{du}{\sqrt{u}} e^{-\varepsilon_n^2 u} = \delta_{\eta 1} \frac{1}{2\sqrt{\pi}} \varepsilon_n \int_{1/\Lambda^2}^{\infty} \frac{du}{\sqrt{u}} e^{-\varepsilon_n^2 u} = \delta_{\eta 1} R_1(\varepsilon_n) \] (E.69)
since \( \frac{1}{2\sqrt{\pi}} = \delta_{\eta 1} \) because \( \eta \) can only take the values 1 or –1 and with
\[ R_1(\varepsilon_n) = -\frac{1}{2\sqrt{\pi}} \varepsilon_n \int_{1/\Lambda^2}^{\infty} \frac{du}{\sqrt{u}} e^{-\varepsilon_n^2 u}. \] (E.70)

An equivalent way to write the same function relies upon the change of variables
\[ u = \frac{\varepsilon_n^2}{\varepsilon_n} \] (E.71)
leading to
\[ \frac{1}{2\sqrt{\pi}} \int_{1/\Lambda^2}^{\infty} \frac{du}{\sqrt{u}} e^{-\varepsilon_n^2 u} = \frac{1}{\sqrt{\pi}} \int_{|\varepsilon_n|/\Lambda}^{\infty} \frac{dy}{\sqrt{y}} e^{-y^2} = \frac{1}{2|\varepsilon_n|} \text{erfc} \left( \frac{|\varepsilon_n|}{\Lambda} \right) \] (E.72)
which allows to rewrite \( R_1 \) in a form making it easy to find the behavior of this function for in the limit of large value for the cut-off:
\[ R_1(\varepsilon_n, \eta) = -\frac{1}{2\sqrt{\pi}} \varepsilon_n \int_{1/\Lambda^2}^{\infty} \frac{du}{\sqrt{u}} e^{-\varepsilon_n^2 u}. \] (E.73)

The second term coming from (E.63)
\[ f^{PS}_{2S} = \frac{1}{2} D^{(8)}_{\chi \beta} \text{tr} \left( \begin{array}{c} \langle 0, z | e^{-D^1_{\chi \beta}} G_c A^1 \hat{A} G_c | 0, z \rangle \\ + \eta \langle 0, z | e^{-D^1_{\chi \beta}} G_c A^1 \hat{A} G_c | 0, z \rangle^* \end{array} \right) \lambda^\beta \Gamma^\mu - \text{v.s.} \] (E.74)
needs time ordering at some point, since the time associated with the angular velocity operator may be before or after the time associated with \( D^{(8)}_{\chi \beta} \).

\[ \frac{1}{2} D^{(8)}_{\chi \beta} \text{tr} \langle 0, z | e^{-D^1_{\chi \beta}} G_c A^1 \hat{A} G_c | 0, z \rangle \lambda^\beta \Gamma^\mu = \] (E.75)
\[ \frac{1}{2} D^{(8)}_{\chi \beta} \text{tr} \langle 0, z | e^{-D^1_{\chi \beta}} P^T G_T P^T A^1 \hat{A} G_c | 0, z \rangle \lambda^\beta \Gamma^\mu = \]
\[ + \frac{1}{2} D^{(8)}_{\chi \beta} \text{tr} \langle 0, z | e^{-D^1_{\chi \beta}} P^T G_T P^T A^1 \hat{A} G_c | 0, z \rangle \lambda^\beta \Gamma^\mu + \langle P_T \leftrightarrow P_S \rangle \]
from which, using the quantization relations
\[ \frac{1}{2} D^{(8)}_{\chi \beta} \text{tr} \langle 0, z | e^{-D^1_{\chi \beta}} P^T G_T P^T A^1 \hat{A} G_c | 0, z \rangle \lambda^\beta \Gamma^\mu = \] (E.76)
where the time ordering is now clear in the limits of the integrations over \( v^4 \). Using
\[ \int_{-\infty}^{\infty} dv^4 \langle \omega | v^4 \rangle \langle v^4 | \omega' \rangle = \int_{-\infty}^{\infty} dv^4 e^{-i(\omega - \omega')v^4} = \lim_{\zeta \to 0^+} \frac{1}{-i(\omega' - \omega) + \zeta} \] (E.77a)
\[ \int_{-\infty}^{\infty} dv^4 \langle \omega | v^4 \rangle \langle v^4 | \omega' \rangle = \int_{-\infty}^{\infty} dv^4 e^{-i(\omega - \omega')v^4} = \lim_{\zeta \to 0^+} \frac{1}{i(\omega' - \omega) + \zeta} \] (E.77b)
one can write
\[
\frac{1}{2} D^{(8)}_{\chi \beta} \text{tr} \langle 0, z | e^{-D_T^1 D_T / \Lambda^2} P_T G_T P_T A^1 \hat{A} G_T P_T | 0, z \rangle \lambda^\beta \Gamma^\mu =
\]
\[
\frac{J_1}{2 I_1} D^{(8)}_{\chi \beta} \sum_{n, m} \langle m | z \rangle P_T \lambda^\beta P_T \Gamma^\mu \langle z | n \rangle \langle n | P_T \lambda^\beta P_T | m \rangle Q^{(-)}(\varepsilon_n, \varepsilon_m)
\]
\[
+ D^{(8)}_{\chi \beta} \frac{J_1}{2 I_1} \sum_{n, m} \langle m | z \rangle P_T \lambda^\beta P_T \Gamma^\mu \langle z | n \rangle \langle n | P_T \lambda^\beta P_T | m \rangle Q^{(+)}(\varepsilon_n, \varepsilon_m)
\]
\[
- \frac{K_1}{I_1} M_S D^{(8)}_{8\iota} \sum_{n, m} \langle m | z \rangle P_T \lambda^\beta P_T \Gamma^\mu \langle z | n \rangle \langle n | P_T \lambda^\beta P_T | m \rangle Q(\varepsilon_n, \varepsilon_m)
\]
\[
(\text{E.78})
\]

defining the functions:
\[
Q^{(\pm)}(\varepsilon_n, \varepsilon_m) = \frac{1}{2} \int \frac{d\omega}{2\pi} \frac{1}{2\pi} \frac{e^{-(\omega^2 + \varepsilon_n^2) / \Lambda^2}}{(i\omega + \varepsilon_n) (i\omega + \varepsilon_m)}
\]
\[
(\text{E.79a})
\]
\[
Q(\varepsilon_n, \varepsilon_m) = \frac{1}{2} \int \frac{d\omega}{2\pi} \frac{e^{-(\omega^2 + \varepsilon_n^2) / \Lambda^2}}{(i\omega + \varepsilon_n) (i\omega + \varepsilon_m)}
\]
\[
(\text{E.79b})
\]

For the other projections, one is
\[
\frac{1}{2} D^{(8)}_{\chi \beta} \text{tr} \langle 0, z | e^{-D_T^1 D_T / \Lambda^2} P_T G_T P_T A^1 \hat{A} G_T P_T | 0, z \rangle \lambda^\beta \Gamma^\mu =
\]
\[
\frac{J_a}{2 I_2} D^{(8)}_{\chi \beta} \sum_{n, m^b} \langle m^0 | z \rangle P_S \lambda^\beta P_T \Gamma^\mu \langle z | n^0 \rangle \langle n^0 | P_T \lambda^\beta P_S | m^0 \rangle Q^{(-)}(\varepsilon_n, \varepsilon_m^0)
\]
\[
+ D^{(8)}_{\chi \beta} \frac{J_a}{2 I_2} \sum_{n, m^b} \langle m^0 | z \rangle P_S \lambda^\beta P_T \Gamma^\mu \langle z | n^0 \rangle \langle n^0 | P_T \lambda^\beta P_S | m^0 \rangle Q^{(+)}(\varepsilon_n, \varepsilon_m^0)
\]
\[
- \frac{K_2}{I_2} M_S D^{(8)}_{8\iota} \sum_{n, m^b} \langle m^0 | z \rangle P_S \lambda^\beta P_T \Gamma^\mu \langle z | n^0 \rangle \langle n^0 | P_T \lambda^\beta P_S | m^0 \rangle Q(\varepsilon_n, \varepsilon_m^0)
\]
\[
(\text{E.80})
\]

and the other is
\[
\frac{1}{2} D^{(8)}_{\chi \beta} \text{tr} \langle 0, z | e^{-D_S^1 D_S / \Lambda^2} P_S G_S P_S A^1 \hat{A} G_T P_T | 0, z \rangle \lambda^\beta \Gamma^\mu =
\]
\[
\frac{J_a}{2 I_2} D^{(8)}_{\chi \beta} \sum_{n^a, m} \langle m | z \rangle P_T \lambda^\beta P_S \Gamma^\mu \langle z | n^0 \rangle \langle n^0 | P_S \lambda^\beta P_T | m \rangle Q^{(-)}(\varepsilon_n^a, \varepsilon_m)
\]
\[
+ D^{(8)}_{\chi \beta} \frac{J_a}{2 I_2} \sum_{n^a, m} \langle m | z \rangle P_T \lambda^\beta P_S \Gamma^\mu \langle z | n^0 \rangle \langle n^0 | P_S \lambda^\beta P_T | m \rangle Q^{(+)}(\varepsilon_n^a, \varepsilon_m)
\]
\[
- \frac{K_2}{I_2} M_S D^{(8)}_{8\iota} \sum_{n^a, m} \langle m | z \rangle P_T \lambda^\beta P_S \Gamma^\mu \langle z | n^0 \rangle \langle n^0 | P_S \lambda^\beta P_T | m \rangle Q(\varepsilon_n^a, \varepsilon_m).
\]
\[
(\text{E.81})
\]

Now we can write, with the pure strange part taken away by the vacuum subtraction and defining
\[
L^{(\pm)}(\varepsilon_n, \varepsilon_m; \eta) = Q^{(\pm)}(\varepsilon_n, \varepsilon_m) - \eta Q^{(\pm)}(\varepsilon_m, \varepsilon_n)
\]
\[
L(\varepsilon_n, \varepsilon_m; \eta) = Q(\varepsilon_n, \varepsilon_m) - \eta Q(\varepsilon_m, \varepsilon_n)
\]
\[
(\text{E.82a})
\]
\[
(\text{E.82b})
\]

Finally
\[
f^{\mu\nu}_{2,8} = \left( \frac{J_1}{2 I_1} D^{(8)}_{\chi \beta} L^{(-)}(\varepsilon_n, \varepsilon_m; \eta) + D^{(8)}_{\chi \beta} \frac{J_1}{2 I_1} L^{(+)}(\varepsilon_n, \varepsilon_m; \eta) - \frac{K_1}{I_1} M_S D^{(8)}_{8\iota} L(\varepsilon_n, \varepsilon_m; \eta) \right)
\]
\[
\times \sum_{n, m} \langle m | z \rangle P_T \lambda^\beta P_T \Gamma^\mu \langle z | n \rangle \langle n | P_T \lambda^\beta P_T | m \rangle
\]
\[
+ \left( \frac{J_a}{2 I_2} D^{(8)}_{\chi \beta} L^{(-)}(\varepsilon_n, \varepsilon_m^a; \eta) + D^{(8)}_{\chi \beta} \frac{J_a}{2 I_2} L^{(+)}(\varepsilon_n, \varepsilon_m^a; \eta) - \frac{K_2}{I_2} M_S D^{(8)}_{8\iota} L(\varepsilon_n, \varepsilon_m^a; \eta) \right)
\]
\[
\times \sum_{n, m^b} \langle m^0 | z \rangle P_S \lambda^\beta P_T \Gamma^\mu \langle z | n^0 \rangle \langle n^0 | P_T \lambda^\beta P_S | m^0 \rangle
\]
\[
(\text{E.83})
\]
\( + \left( \frac{J_a}{2I_2} D^{(8)}_{\chi\beta} \mathcal{L}^{-}(\varepsilon_{m\theta}; \varepsilon_{m\eta}) + D^{(8)}_{\chi\beta} \frac{J_a}{2I_2} \mathcal{L}^{(8)}_{\theta}(\varepsilon_{m\theta}; \varepsilon_{m\eta}) - \frac{K_2}{I_2} M_8 D^{(8)}_{8i} \mathcal{L}(\varepsilon_{m\theta}; \varepsilon_{m\eta}) \right) \times \sum_{n, m} \langle m \mid z \rangle P_T \lambda^{\mu} P_S \Gamma^\mu \left( \frac{z}{n^0} \right) \langle n^0 \mid P_S \lambda^{\eta} P_T \mid m \rangle \) \hspace{1cm} (E.83)

Regarding the minus sign in \( \mathcal{L}^{(8)} \) one must remember that the matrix element \( \langle n \mid A^\dagger \hat{A} \mid m \rangle \) has to be considered before the quantization, since the operator \( A^\dagger \hat{A} \) is anti-hermitian in Euclidian space: this causes the \((-)\) sign in \( \mathcal{L}^{(8)} \), and also on \( \mathcal{P}^{(8)}(\varepsilon_n, \varepsilon_m; \eta) \) below appearing in \( f^{(8)}_{4,8} \).

The third term
\[ f^{(8)}_{3,8} = \frac{1}{2} D^{(8)}_{\chi\beta} \text{tr} \left( \langle 0, z \mid e^{-D^{(8)}_{\chi\beta}} | 0, z \rangle \lambda^\beta \Gamma^\mu \langle z | n^0 \rangle \langle n^0 | P_T \lambda^{\eta} P_T | m \rangle \right) \lambda^\beta \Gamma^\mu - \text{v.s.} \] \hspace{1cm} (E.84)
does not need time ordering. Repeating the same steps as before
\[ f^{(8)}_{3,8} = \left[ \left( M_1 + \frac{1}{\sqrt{3}} M_8 D^{(8)}_{88} \right) D^{(8)}_{\chi\beta} \sum_{n, m} \langle m | z \rangle P_T \lambda^\beta \Gamma^\mu \langle z | n \rangle \langle n | i \gamma_4 | m \rangle \right. \]
\[ + M_8 D^{(8)}_{8i} D^{(8)}_{\chi\beta} \sum_{n, m} \langle m | z \rangle P_T \lambda^\beta \Gamma^\mu P_T \langle z | n \rangle \langle n | i \gamma_4 | m \rangle \left. \mathcal{M}(\varepsilon_n, \varepsilon_m; \eta) \right] \]
\[ + M_8 D^{(8)}_{sa} D^{(8)}_{\chi\beta} \sum_{n, m} \langle m | z \rangle P_T \lambda^\beta \Gamma^\mu P_T \langle z | n \rangle \langle n | i \gamma_4 | m \rangle \mathcal{M}(\varepsilon_n, \varepsilon_m; \eta) \]
\[ + M_8 D^{(8)}_{sa} D^{(8)}_{\chi\beta} \sum_{n, m} \langle m | z \rangle P_T \lambda^\beta \Gamma^\mu P_T \langle z | n \rangle \langle n | i \gamma_4 | m \rangle \mathcal{M}(\varepsilon_n, \varepsilon_m; \eta) \] \hspace{1cm} (E.85)

with
\[ \mathcal{M}(\varepsilon_n, \varepsilon_m; \eta) = \mathcal{Q}(\varepsilon_n, \varepsilon_m) + \eta \mathcal{Q}(\varepsilon_n, \varepsilon_m) \] \hspace{1cm} (E.86)

The fourth term
\[ f^{(8)}_{4,8} = \frac{1}{2} \frac{1}{\Lambda^2} D^{(8)}_{\chi\beta} \text{tr} \left( \langle 0, z | \int_0^1 d\alpha e^{-D^{(8)}_{\chi\beta}} \left( D^{(8)}_{c} A^\dagger \hat{A} \right) \left( \int_0^1 d\alpha e^{-D^{(8)}_{\chi\beta}} \left( D^{(8)}_{c} A^\dagger \hat{A} \right) \right) \right. \]
\[ \times e^{-(1-\alpha)D^{(8)}_{\chi\beta} \Lambda^2} \left( D^{(8)}_{c} A^\dagger \hat{A} \right) \left( \int_0^1 d\alpha e^{-D^{(8)}_{\chi\beta}} \left( D^{(8)}_{c} A^\dagger \hat{A} \right) \right) \]
\[ \times e^{-(1-\alpha)D^{(8)}_{\chi\beta} \Lambda^2} | 0, z \rangle \lambda^\beta \Gamma^\mu \left. + \text{v.s.} \right] \] \hspace{1cm} (E.87)

again needs time ordering. The projections over strange and nonstrange flavor spaces lead for the trace to
\[ \text{tr} \langle 0, z | \int_0^1 d\alpha e^{-D^{(8)}_{\chi\beta} \Lambda^2} \left( D^{(8)}_{T} A^\dagger \hat{A} \right) e^{-(1-\alpha)D^{(8)}_{\chi\beta} \Lambda^2} | 0, z \rangle \lambda^\beta \Gamma^\mu = \]
\[ \text{tr} \langle 0, z | \int_0^1 d\alpha e^{-D^{(8)}_{\chi\beta} \Lambda^2} P_T \left( D^{(8)}_{\chi\beta} A^\dagger \hat{A} \right) P_T e^{-(1-\alpha)D^{(8)}_{\chi\beta} \Lambda^2} | 0, z \rangle \lambda^\beta \Gamma^\mu = \]
\[ + \text{tr} \langle 0, z | \int_0^1 d\alpha e^{-D^{(8)}_{\chi\beta} \Lambda^2} P_T \left( D^{(8)}_{\chi\beta} A^\dagger \hat{A} \right) P_T e^{-(1-\alpha)D^{(8)}_{\chi\beta} \Lambda^2} | 0, z \rangle \lambda^\beta \Gamma^\mu = \] \hspace{1cm} (E.88)

The nonstrange part needs time ordering:
\[ D^{(8)}_{\chi\beta} \frac{1}{2} \frac{1}{\Lambda^2} \int_0^1 d\alpha \langle 0, z | e^{-D^{(8)}_{\chi\beta} \Lambda^2} P_T \left( D^{(8)}_{\chi\beta} A^\dagger \hat{A} \right) P_T e^{-(1-\alpha)D^{(8)}_{\chi\beta} \Lambda^2} | 0, z \rangle \lambda^\beta \Gamma^\mu = \]
\[ = \sum_{n, m} \langle m | z \rangle P_T \lambda^\beta P_T \langle z | n \rangle \langle n | P_T \lambda^{\eta} P_T | m \rangle \]
E.3 Expansions in the angular velocity and $\delta m$

\[
\left( \frac{J_1}{2I_1} D^{(8)}_{\chi \beta} \mathcal{P}(-)(\varepsilon_n, \varepsilon_m) + \frac{J_1}{2I_1} D^{(8)}_{\chi \beta} \mathcal{P}(+)(\varepsilon_n, \varepsilon_m) - \frac{K_1}{I_1} M_8 D^{(8)}_{8i} \mathcal{P}(\varepsilon_n, \varepsilon_m) \right)
\]  

(E.89)

with the definitions

\[
\mathcal{P}^{(\pm)}(\varepsilon_n, \varepsilon_m) = \frac{1}{2 \Lambda^2} \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} \int_{-\pi}^{\pi} \frac{d\omega'}{2\pi} \int_0^1 d\alpha \frac{e^{-\alpha(\omega^2 + \varepsilon_n^2/\Lambda^2 - (1-\alpha)(\omega^2 + \varepsilon_m^2/\Lambda^2)}}{\pm i(\omega' - \omega) + \zeta} \left( -i\omega + \varepsilon_n + 1 \right)
\]  

(E.90a)

\[
\mathcal{P}(\varepsilon_n, \varepsilon_m) = \frac{1}{2 \Lambda^2} \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} \int_{-\pi}^{\pi} \frac{d\omega'}{2\pi} \int_0^1 d\alpha \frac{e^{-\alpha(\omega^2 + \varepsilon_n^2/\Lambda^2 - (1-\alpha)(\omega^2 + \varepsilon_m^2/\Lambda^2)}}{i(\omega' + \varepsilon_m - 1)}
\]  

(E.90b)

The $(-)$ sign in $\mathcal{P}^{(\pm)}(\varepsilon_n, \varepsilon_m; \eta)$ comes again from $\langle n| A^f A^\dagger | m \rangle^*$. In this way, the fourth term is given by

\[
f_{4,5}^{\mu} = \left( \frac{J_1}{2I_1} D^{(8)}_{\chi \beta} \mathcal{P}(-)(\varepsilon_n, \varepsilon_m; \eta) + \frac{J_1}{2I_1} D^{(8)}_{\chi \beta} \mathcal{P}(+)(\varepsilon_n, \varepsilon_m; \eta) - \frac{K_1}{I_1} M_8 D^{(8)}_{8i} \mathcal{P}(\varepsilon_n, \varepsilon_m; \eta) \right)
\]

\[
\times \sum_{n,m} \langle m| z \rangle P_T \lambda^\beta \Gamma^\mu P_T \langle z| n \rangle \langle n| P_T \lambda^\delta P_T | m \rangle
\]

\[
+ \left( \frac{J_2}{2I_2} D^{(8)}_{\chi \beta} \mathcal{P}(-)(\varepsilon_n, \varepsilon_m^0; \eta) + \frac{J_2}{2I_2} D^{(8)}_{\chi \beta} \mathcal{P}(+)(\varepsilon_n, \varepsilon_m^0; \eta) - \frac{K_2}{I_2} M_8 D^{(8)}_{8i} \mathcal{P}(\varepsilon_n, \varepsilon_m^0; \eta) \right)
\]

\[
\times \sum_{n,m^0} \langle m^0| z \rangle P_S \lambda^\beta \Gamma^\mu P_T \langle z| n^0 \rangle \langle n^0| P_T \lambda^\delta P_S | m^0 \rangle
\]

\[
+ \left( \frac{J_2}{2I_2} D^{(8)}_{\chi \beta} \mathcal{P}(-)(\varepsilon_n^0, \varepsilon_m; \eta) + \frac{J_2}{2I_2} D^{(8)}_{\chi \beta} \mathcal{P}(+)(\varepsilon_n^0, \varepsilon_m; \eta) - \frac{K_2}{I_2} M_8 D^{(8)}_{8i} \mathcal{P}(\varepsilon_n^0, \varepsilon_m; \eta) \right)
\]

\[
\times \sum_{n^0,m} \langle m| z \rangle P_T \lambda^\beta \Gamma^\mu P_S \langle z| n^0 \rangle \langle n^0| P_S \lambda^\delta P_T | m \rangle
\]

(E.91)

with the definitions

\[
\mathcal{P}^{(\pm)}(\varepsilon_n, \varepsilon_m; \eta) = \mathcal{P}^{(\pm)}(\varepsilon_n, \varepsilon_m) - \eta \mathcal{P}^{(\pm)}(\varepsilon_n, \varepsilon_m),
\]

(E.92a)

\[
\mathcal{P}(\varepsilon_n, \varepsilon_m; \eta) = \mathcal{P}(\varepsilon_n, \varepsilon_m) - \eta \mathcal{P}(\varepsilon_n, \varepsilon_m).
\]

(E.92b)

Finally, the same steps can be applied to the fifth term, linear in $\delta m$,

\[
f_{5,6}^{\mu} = D^{(8)}_{\chi \beta} \frac{1}{2 \Lambda^2} \text{tr} \left[ (0, z) \int_0^1 d\alpha e^{-\alpha D^\dagger_{\chi} D_{\chi}/\Lambda^2} \left( D_{\chi}^\dagger i\gamma_4 A^l \delta m \Lambda_{\chi} e + i\gamma_4 A^l \delta m A \right) \right.
\]

\[
\times e^{-\alpha D^\dagger_{\chi} D_{\chi}/\Lambda^2} |0, z\rangle \langle 0, z| \lambda^\beta \Gamma^\mu + \eta \langle 0, z| \int_0^1 d\alpha e^{-\alpha D^\dagger_{\chi} D_{\chi}/\Lambda^2} \left( D_{\chi}^\dagger i\gamma_4 A^l \delta m \Lambda_{\chi} e + i\gamma_4 A^l \delta m A \right)
\]

\[
\times e^{-\alpha D^\dagger_{\chi} D_{\chi}/\Lambda^2} |0, z\rangle \langle 0, z| \lambda^\beta \Gamma^\mu \lambda^\beta \Gamma^\mu \right] - v.s.
\]

(E.93)

with the result

\[
f_{5,6}^{\mu} = \left[ \left( M_1 + \frac{1}{\sqrt{3}} M_8 D^{(8)}_{88} \right) D^{(8)}_{\chi \beta} \sum_{n,m} \langle m| z \rangle \Gamma^\mu \langle z| n \rangle \langle n| i\gamma_4 | m \rangle \right]
\]

\[
+ M_8 D^{(8)}_{8j} D^{(8)}_{\chi \beta} \sum_{n,m} \langle m| z \rangle \Gamma^\mu \langle z| n \rangle \langle n| i\gamma_4 \tau^j | m \rangle \right] \mathcal{N}(\varepsilon_n, \varepsilon_m; \eta)
\]

\[
+ M_8 D^{(8)}_{8a} D^{(8)}_{\chi \beta} \sum_{n,m^0} \langle m^0| z \rangle P_S \lambda^\beta \Gamma^\mu P_T \langle z| n \rangle \langle n| i\gamma_4 P_T \lambda^\delta P_S | m^0 \rangle \mathcal{N}(\varepsilon_n, \varepsilon_m^0; \eta)
\]

\[
+ M_8 D^{(8)}_{8a} D^{(8)}_{\chi \beta} \sum_{n^0,m} \langle m| z \rangle P_T \lambda^\beta \Gamma^\mu \langle z| n^0 \rangle \langle n^0| i\gamma_4 P_S \lambda^\delta P_T | m \rangle \mathcal{N}(\varepsilon_n^0, \varepsilon_m; \eta)
\]

(E.94)

with the definitions:

\[
\mathcal{N}(\varepsilon_n, \varepsilon_m; \eta) = \mathcal{N}(\varepsilon_n, \varepsilon_n) + \eta \mathcal{N}(\varepsilon_n, \varepsilon_n),
\]

(E.95a)

\[
\mathcal{N}(\varepsilon_n, \varepsilon_m) = \frac{1}{2 \Lambda^2} \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} \int_0^1 d\alpha e^{-\alpha(\omega^2 + \varepsilon_n^2/\Lambda^2 - (1-\alpha)(\omega^2 + \varepsilon_m^2/\Lambda^2)}} \left( -i\omega + \varepsilon_n + 1 \right)
\]

(E.95b)
In order to combine the previous results it is useful to introduce the regularization functions in the form

\[
\mathcal{R}(\pm)(\varepsilon_n, \varepsilon_m; \eta) = \mathcal{L}(\pm)(\varepsilon_n, \varepsilon_m; \eta) + \mathcal{P}(\pm)(\varepsilon_n, \varepsilon_m; \eta) \tag{E.96a}
\]

\[
\mathcal{R}(\varepsilon_n, \varepsilon_m; \eta) = \mathcal{L}(\varepsilon_n, \varepsilon_m; \eta) + \mathcal{P}(\varepsilon_n, \varepsilon_m; \eta) \tag{E.96b}
\]

to be used together with the relations ((E.49)), ((E.49)). In this way one finds that

\[
f_{2S}^{\pm} + f_{4S}^{\pm} = \sum_{n,m} \langle m| z \rangle P_T \lambda^\beta P_T \Gamma^\mu \langle z| n \rangle \langle n| P_T \lambda^\alpha P_T | m \rangle
\]

\[
\times \left( \frac{1}{2} \left\{ \frac{J_1}{2I_1}, D^{(8)}_{\chi \delta} \right\} \mathcal{R}_A(\varepsilon_n, \varepsilon_m; \eta) + \frac{1}{2} \left\{ \frac{J_1}{2I_1}, D^{(8)}_{\chi \delta} \right\} \mathcal{R}_B(\varepsilon_n, \varepsilon_m; \eta) \right)
\]

\[
-f_{1} \frac{M_s D^{(8)}_{8i}}{I_1} \mathcal{R}(\varepsilon_n, \varepsilon_m; \eta)
\]

\[
+ \sum_{n,m^0} \langle m^0| z \rangle P_S \lambda^\beta P_T \Gamma^\mu \langle z| n \rangle \langle n| P_T \lambda^\alpha P_S | m^0 \rangle
\]

\[
\times \left( \frac{1}{2} \left\{ \frac{J_0}{2I_2}, D^{(8)}_{\chi \delta} \right\} \mathcal{R}_A(\varepsilon_n, \varepsilon_m^0; \eta) + \frac{1}{2} \left\{ \frac{J_0}{2I_2}, D^{(8)}_{\chi \delta} \right\} \mathcal{R}_B(\varepsilon_n, \varepsilon_m^0; \eta) \right)
\]

\[
-f_{2} \frac{M_s D^{(8)}_{8i}}{I_2} \mathcal{R}(\varepsilon_n, \varepsilon_m^0; \eta)
\]

\[
+ \sum_{n^0,m} \langle m| z \rangle \Gamma^\mu P_T \lambda^\beta P_S \langle z| n^0 \rangle \langle n^0| P_S \lambda^\alpha P_T | m \rangle
\]

\[
\times \left( \frac{1}{2} \left\{ \frac{J_0}{2I_2}, D^{(8)}_{\chi \delta} \right\} \mathcal{R}_A(\varepsilon_n^0, \varepsilon_m; \eta) + \frac{1}{2} \left\{ \frac{J_0}{2I_2}, D^{(8)}_{\chi \delta} \right\} \mathcal{R}_B(\varepsilon_n^0, \varepsilon_m; \eta) \right)
\]

\[
-f_{2} \frac{M_s D^{(8)}_{8i}}{I_2} \mathcal{R}(\varepsilon_n^0, \varepsilon_m; \eta)
\]

where

\[
\mathcal{R}_A(\varepsilon_n, \varepsilon_m; \eta) = \mathcal{R}_A^{-}(\varepsilon_n, \varepsilon_m; \eta) + \mathcal{R}_A^{+}(\varepsilon_n, \varepsilon_m; \eta) \tag{E.98a}
\]

\[
\mathcal{R}_B(\varepsilon_n, \varepsilon_m; \eta) = \mathcal{R}_B^{-}(\varepsilon_n, \varepsilon_m; \eta) - \mathcal{R}_B^{+}(\varepsilon_n, \varepsilon_m; \eta) \tag{E.98b}
\]

The regularization functions related to the terms with time ordering \( \mathcal{R}_A \) and \( \mathcal{R}_B \) are defined in terms of

\[
\mathcal{R}(\pm)(\varepsilon_n, \varepsilon_m; \eta) = \mathcal{L}(\pm)(\varepsilon_n, \varepsilon_m; \eta) + \mathcal{P}(\pm)(\varepsilon_n, \varepsilon_m; \eta)
\]

\[
= Q(\pm)(\varepsilon_n, \varepsilon_m) + P(\pm)(\varepsilon_n, \varepsilon_m) - \eta \left( Q(\pm)(\varepsilon_m, \varepsilon_n) + P(\pm)(\varepsilon_n, \varepsilon_m) \right)
\]

\[
= \frac{1}{2} \int \frac{d\omega}{2 \pi} \int \frac{d\omega'}{2 \pi} \pm i(\omega' - \omega) + \zeta \left[ e^{-(\omega^2 + \varepsilon_n^2)/\Lambda^2} - e^{-(\omega^2 + \varepsilon_m^2)/\Lambda^2} \right]
\]

\[
- \eta \left( e^{-(\omega'\omega + \varepsilon_n^2)/\Lambda^2} - e^{-(\omega'\omega + \varepsilon_m^2)/\Lambda^2} \right)
\]

\[
+ \frac{1}{\Lambda^2} \int_0^1 \frac{d\alpha}{\Lambda^2} e^{-\alpha \omega^2 - \varepsilon_n^2}/\Lambda^2 - (1-\alpha)(\omega^2 + \varepsilon_m^2)/\Lambda^2 \left( \frac{-\omega + \varepsilon_n}{\omega' + \varepsilon_m} - 1 \right)
\]

\[
- \eta \left( \frac{-\omega + \varepsilon_m}{\omega' + \varepsilon_n} - 1 \right) \right] \tag{E.99}
\]

by (E.98a) and (E.98b). It is easy to find, on account of simple changes of the integration variables, that \( \mathcal{R}_A(\varepsilon_n, \varepsilon_m; 1) = 0 \). Function \( \mathcal{R}_A \) is related to the delta function from \( 1/(\pm i(\omega' - \omega) + \zeta) \) while \( \mathcal{R}_B \) is related to the Cauchy principal value. For \( \mathcal{R}_A(\varepsilon_n, \varepsilon_m; \eta) \) one has then

\[
\mathcal{R}_A(\varepsilon_n, \varepsilon_m; \eta) = -\delta_{-1\eta} \mathcal{R}_3(\varepsilon_n, \varepsilon_m) \tag{E.100}
\]

where the regularization function \( \mathcal{R}_3(\varepsilon_n, \varepsilon_m) \), defined as

\[
\mathcal{R}_3(\varepsilon_n, \varepsilon_m) = \frac{1}{2 \sqrt{\pi}} \int_{1/\Lambda^2}^{\infty} \frac{du}{\sqrt{u}} \left( \frac{1}{u} \frac{e^{-\varepsilon_n^2 u} - e^{-\varepsilon_m^2 u}}{\varepsilon_n^2 - \varepsilon_m^2} - \frac{\varepsilon_m e^{-\varepsilon_n^2 u} + \varepsilon_n e^{-\varepsilon_m^2 u}}{\varepsilon_n + \varepsilon_m} \right), \tag{E.101}
\]
was already encountered in the effective action (D.41).

\[ R_A(\epsilon_n, \epsilon_m; \eta) = \delta_{\epsilon_n \epsilon_m} R^{(-)}(\epsilon_n, \epsilon_m; -1) + R^{(+)}(\epsilon_n, \epsilon_m; -1) \]

\[ = \frac{1}{2} \int \frac{d\omega}{2\pi} \left[ e^{-(\omega^2 + \epsilon_n^2)/\Lambda^2} - e^{-(\omega^2 + \epsilon_m^2)/\Lambda^2} \right] \frac{1}{\Lambda^2} \int_0^1 da e^{-a(\omega^2 + \epsilon_n^2)/\Lambda^2 - (1-a)(\omega^2 + \epsilon_m^2)/\Lambda^2} \left( 2 + \frac{i\omega - \epsilon_n}{i\omega + \epsilon_n} + \frac{i\omega - \epsilon_m}{i\omega + \epsilon_m} \right) \]  

(E.102)

At this point it is easy to see that the function is real and that it is even regarding the interchange of \( \epsilon_n, \epsilon_m \). Integrating over \( \alpha \) and following (E.68) \( (\alpha = 1/\Lambda^2) \) to find

\[ \int \frac{d\omega}{2\pi} (e^{-(\omega^2 + \epsilon_n^2)/\Lambda^2} - \eta e^{-(\omega^2 + \epsilon_n^2)/\Lambda^2}) = \int \frac{d\omega}{\sqrt{\pi}} \left( e^{-\epsilon_n^2/\Lambda^2} - \eta e^{-\epsilon_n^2/\Lambda^2} \right) \]

(E.103)

we arrive at (E.101)

\[ \int_{|\epsilon_m|/\Lambda^2}^{\infty} \frac{dy}{y} e^{-y^2} = \sqrt{\pi} \text{erfc} \left( \frac{|\epsilon_m|}{\Lambda} \right) \]  

(E.104)

which allow to easily find the large cut-off limit

\[ \mathcal{R}_3(\epsilon_n, \epsilon_m) = \frac{1}{\sqrt{\pi}} \left[ \frac{1}{\epsilon_n - \epsilon_m} \left( \epsilon_m - \epsilon_n \right) \int_{|\epsilon_m|/\Lambda^2}^{\infty} \frac{dy}{y^2} e^{-y^2} - \int_{|\epsilon_n|/\Lambda^2}^{\infty} \frac{dy}{y^2} e^{-y^2} \right] \]

\[ + \frac{1}{\sqrt{\pi}} \frac{\epsilon_n + \epsilon_m}{2} \left( \text{sgn}(\epsilon_n) \text{erfc} \left( \frac{|\epsilon_n|}{\Lambda} \right) + \text{sgn}(\epsilon_n) \text{erfc} \left( \frac{|\epsilon_n|}{\Lambda} \right) \right) \]

\[ = \frac{\Lambda}{\sqrt{\pi}} \frac{\epsilon_n^2 - \epsilon_m^2}{2(\epsilon_n - \epsilon_m)} = \mathcal{R}_3(\epsilon_n, \epsilon_m). \]  

(E.105)

The function \( \mathcal{R} \) (E.96b) is equal to \( \mathcal{R}_A \) and therefore is related to \( \mathcal{R}_3(\epsilon_n, \epsilon_m) \)

\[ \mathcal{R}(\epsilon_n, \epsilon_m; \eta) = -\delta_{\epsilon_n \epsilon_m} \mathcal{R}_3(\epsilon_n, \epsilon_m). \]  

(E.106)

As to the regularization function \( \mathcal{R}_B \), starting with \( \eta = 1 \) (it is easy to see on the same grounds that \( \mathcal{R}_B(\epsilon_n, \epsilon_m; -1) = 0 \)),

\[ \mathcal{R}_B(\epsilon_n, \epsilon_m; 1) = \mathcal{R}^{(-)}(\epsilon_n, \epsilon_m; 1) - \mathcal{R}^{(+)}(\epsilon_n, \epsilon_m; 1) = \]

\[ i \text{ P.V.} \int_{2\pi} \frac{d\omega}{2\pi} \omega' - \omega \left[ e^{-(\omega^2 + \epsilon_n^2)/\Lambda^2} - e^{-(\omega^2 + \epsilon_m^2)/\Lambda^2} \right] \frac{1}{\Lambda^2} \int_0^1 da e^{-a(\omega^2 + \epsilon_n^2)/\Lambda^2 - (1-a)(\omega^2 + \epsilon_m^2)/\Lambda^2} \left( \frac{-i\omega + \epsilon_n}{i\omega' + \epsilon_n} - 1 \right) \]

\[ - \frac{1}{\Lambda^2} \int_0^1 da e^{-a(\omega^2 + \epsilon_n^2)/\Lambda^2 - (1-a)(\omega^2 + \epsilon_m^2)/\Lambda^2} \left( \frac{-i\omega' + \epsilon_m}{i\omega + \epsilon_m} - 1 \right) \]  

(E.107)

The replacements \( \omega' \rightarrow -\omega' \), \( \omega \rightarrow -\omega \) show that the function is real.
$\mathcal{R}_B(\varepsilon_n, \varepsilon_m; 1) = \mathcal{R}(-)(\varepsilon_n, \varepsilon_m; 1) - \mathcal{R}(+)(\varepsilon_n, \varepsilon_m; 1) =$

$$P.V. \int \frac{du\, dw}{2\pi^2} \frac{1}{w - \omega} \left[ (\omega \varepsilon_m + \omega' \varepsilon_n) \int_0^\infty dv \int_{1/\Lambda^2}^{\infty} du' e^{-(\omega^2 + \omega'^2)v - (\omega + \omega')u} ight. \\
+ \frac{1}{\sqrt{2}} \int_0^1 da \int_0^\infty dw \left( \omega \varepsilon_m + \omega' \varepsilon_n \right) e^{-\alpha(\omega^2 + \omega'^2)v - \alpha(\omega + \omega')u} \\
- \frac{1}{\sqrt{2}} \int_0^1 da \int_0^\infty dw \left( \omega \varepsilon_m + \omega' \varepsilon_n \right) e^{-[(1 - \alpha)/\Lambda^2 + \alpha](\omega^2 + \omega'^2)v - \alpha(\omega + \omega')u} \right]$$

(E.108)

Making the change of variables

$$u = \frac{\alpha}{\Lambda^2}, \quad v = \frac{1 - \alpha}{\Lambda^2} + w$$

with $w > 0, 0 < u < \frac{1}{\Lambda^2}, u + v > 1/\Lambda^2$, one finds

$$\mathcal{R}_B(\varepsilon_n, \varepsilon_m; 1) = \int_{u,v>0; u+v>1/\Lambda^2} du dv \left( e^{-\frac{\alpha}{\Lambda^2}v^2 - \frac{1}{\Lambda^2}v} \frac{u \varepsilon_n - v \varepsilon_m}{\sqrt{uv}(u + v)} - e^{-\frac{1}{\Lambda^2}v^2 - \frac{1}{\Lambda^2}v} \frac{u \varepsilon_m - v \varepsilon_n}{\sqrt{uv}(u + v)} \right)$$

(E.110)

The further use of the relation

$$P.V. \int \frac{dw\, dw'}{2\pi^2} \frac{\omega' + \omega'}{w' - \omega} e^{-\alpha(\omega^2 + \omega'^2)v - \alpha(\omega + \omega')u} = \frac{\epsilon^2 - \epsilon d}{4\pi\sqrt{cd}(c + d)}$$

(E.111)

and of another change of variables

$$u = (1 - \beta), \quad v = \beta w,$$

where $w > 1/\Lambda^2$ leads finally to

$$\mathcal{R}_B(\varepsilon_n, \varepsilon_m; \eta) = \frac{1}{2\pi} \int_{u,v>0; u+v>1/\Lambda^2} du dv \left( \frac{u \varepsilon_n - v \varepsilon_m}{\sqrt{uv}(u + v)} - e^{-\frac{1}{\Lambda^2}v^2 - \frac{1}{\Lambda^2}v} \right)$$

(E.113)

The regularization function $\mathcal{R}_4(\varepsilon_n, \varepsilon_m)$ is odd under the interchange $\varepsilon_n \leftrightarrow \varepsilon_m$ and is given by

$$\mathcal{R}_4(\varepsilon_n, \varepsilon_m) = \frac{1}{2\pi} \int_{1/\Lambda^2}^{\infty} dw \int_0^1 d\beta \frac{(1 - \beta) \varepsilon_n - \beta \varepsilon_m}{\sqrt{(1 - \beta) \beta}} e^{-\frac{1}{\Lambda^2}(1 - \beta) + \frac{\varepsilon^2_m}{\beta}}$$

(E.114)

The asymptotic behavior of this function is found to be

$$\mathcal{R}_4(\varepsilon_n, \varepsilon_m) \xrightarrow{\Lambda \to \infty} \frac{1}{2} \frac{1 - sgn(\varepsilon_n)sgn(\varepsilon_m)}{\varepsilon_n - \varepsilon_m}$$

(E.115)

where $\mathcal{R}_6(\varepsilon_n, \varepsilon_m)$ is not a regularization function

$$\mathcal{R}_6(\varepsilon_n, \varepsilon_m) \xrightarrow{\Lambda \to \infty} \frac{1}{2} \frac{1 - sgn(\varepsilon_n)sgn(\varepsilon_m)}{\varepsilon_n - \varepsilon_m}$$

(E.116)

Similarly for the two remaining terms one finds

$$f_{3, S}^{P_X} + f_{3, S}^{P_X} = \left( M_1 + \frac{1}{\sqrt{3}} M_6 D_8^{(8)} \right) D_8^{(8)} \sum_{n,m} \langle m | z \rangle \Gamma^\mu P_T^\lambda \lambda^\beta P_T \langle z | n \rangle \langle \gamma_4 | m \rangle S(\varepsilon_n, \varepsilon_m; \eta)$$

(E.117)

$$S(\varepsilon_n, \varepsilon_m; \eta) = M(\varepsilon_n, \varepsilon_m; \eta) + N(\varepsilon_n, \varepsilon_m; \eta).$$

(E.118)
The function $S(\varepsilon_n, \varepsilon_m; \eta)$ explicitly reads from (E.118)

$$S(\varepsilon_n, \varepsilon_m; \eta) = M(\varepsilon_n, \varepsilon_m; \eta) + N(\varepsilon_n, \varepsilon_m; \eta)$$

$$= \mathcal{Q}(\varepsilon_n, \varepsilon_m) + N(\varepsilon_n, \varepsilon_m) + \eta \left( N(\varepsilon_m, \varepsilon_n) + \mathcal{Q}(\varepsilon_m, \varepsilon_n) \right)$$

$$= \frac{1}{2} \int \frac{d\omega}{2\pi} \left\{ \frac{e^{-(\omega^2 + \varepsilon_m^2)/\Lambda^2}}{(i\omega + \varepsilon_n)(i\omega + \varepsilon_m)} + \frac{\eta}{(i\omega + \varepsilon_n)(i\omega + \varepsilon_m)} - \frac{\eta e^{-(\omega^2 + \varepsilon_n^2)/\Lambda^2}}{(i\omega + \varepsilon_n)(i\omega + \varepsilon_m)} \right\}$$

$$+ \frac{1}{\Lambda^2} \int_0^1 d\alpha e^{-\alpha(\omega^2 + \varepsilon_n^2)/\Lambda^2} \left( -\frac{i\omega + \varepsilon_n}{i\omega + \varepsilon_m} + 1 \right) \left( \frac{i\omega + \varepsilon_m}{i\omega + \varepsilon_n} + 1 \right).$$

(E.119)

Performing the integration over $\alpha$, it is easy to see that this function vanishes when $\eta = -1$ with the final result reading

$$S(\varepsilon_n, \varepsilon_m; \eta) = \delta_{\eta 1} R_2(\varepsilon_n, \varepsilon_m)$$

(E.120)

\begin{align}
R_2(\varepsilon_n, \varepsilon_m) &= \frac{1}{\varepsilon_n - \varepsilon_m} \int \frac{d\omega}{2\pi} \left( \frac{e^{-(\omega^2 + \varepsilon_m^2)/\Lambda^2} - \varepsilon_n e^{-(\omega^2 + \varepsilon_n^2)/\Lambda^2}}{(\omega^2 + \varepsilon_m^2)/(\omega^2 + \varepsilon_n^2)} \right) \\
&= \frac{1}{2\sqrt{\pi}} \int_1^{\infty} \frac{du}{\sqrt{u}} e^{-\frac{\varepsilon_m^2}{u} - \varepsilon_n e^{-\frac{\varepsilon_n^2}{u}}} \\
&= \frac{1}{2} \frac{\text{sgn}(\varepsilon_m) \text{erfc}(|\varepsilon_m|/\Lambda) - \text{sgn}(\varepsilon_n) \text{erfc}(|\varepsilon_n|/\Lambda)}{\varepsilon_n - \varepsilon_m}
\end{align}

(E.121a, b, c)

using the change of variables ((E.68)) and the result ((E.72)) in the last step.

$$S(\varepsilon_n, \varepsilon_m; \eta) \xrightarrow{\Lambda \to \infty} \delta_{\eta 1} \frac{1}{2} \frac{\text{sgn}(\varepsilon_m) - \text{sgn}(\varepsilon_n)}{\varepsilon_n - \varepsilon_m} = -\delta_{\eta 1} R_5(\varepsilon_n, \varepsilon_m)$$

(E.122)

$$R_5(\varepsilon_n, \varepsilon_m) = \frac{1}{2} \frac{\text{sgn}(\varepsilon_m) - \text{sgn}(\varepsilon_n)}{\varepsilon_n - \varepsilon_m}$$

(E.123)

being $R_5$ not, because of the limit, a regularization function and will thus also appear in the expression for the unregularized part of the next section.

**Unregularized part**

The unregularized contribution is connected with the imaginary part of the action which is finite. This contribution is

$$\mathcal{F}_{\varepsilon_n, \varepsilon_m}^{\mu \chi}(z) = N_c \frac{1}{2} \frac{\delta}{\delta s^\mu(0, z)} \text{Tr} \left[ \log \left( \frac{D(U, s)}{D(1/U, s)} \right) \right]_{s^\mu = 0} - \text{v.s.}$$

(E.124)

One has, performing the functional derivative and remembering (E.58) $O^{\mu \chi} = \eta O^{\mu \chi}$

$$\mathcal{F}_{\varepsilon_n, \varepsilon_m}^{\mu \chi}(z) = N_c \frac{1}{2} \frac{\delta}{\delta s^\mu(0, z)} \text{Tr} \left[ \log \left( \frac{D(U, s)}{D(1/U, s)} \right) \right]_{s^\mu = 0} - \text{v.s.}$$

$$= -N_c \frac{1}{2} \text{tr} \left[ \left( (0, z) G(U) |0, z\rangle - \eta \langle 0, z | G(U) |0, z\rangle^* \right) A^\dagger O^{\mu \chi} A \right] - \text{v.s.}$$

(E.125)

The expansion in angular velocity and strange quark mass is as before, up to linear term in the angular velocity and strange mass corrections,

$$\text{tr} \langle 0, z | G(U) |0, z\rangle A^\dagger O^{\mu \chi} A = \text{tr} \langle 0, z | (D_c + A^\dagger \dot{A} + i\gamma_4 A^\dagger \delta m A)^{\dagger} |0, z\rangle A^\dagger O^{\mu \chi} A$$

$$= \text{tr} \langle 0, z | G_c |0, z\rangle A^\dagger O^{\mu \chi} A$$

(E.126a)

$$- \text{tr} \langle 0, z | G_c A^\dagger \dot{A} G_c |0, z\rangle A^\dagger O^{\mu \chi} A$$

(E.126b)

$$- \text{tr} \langle 0, z | G_c i\gamma_4 A^\dagger \delta m A G_c |0, z\rangle A^\dagger O^{\mu \chi} A$$

(E.126c)
The trace can now be split according to the strange and nonstrange projections

$$f_{1,S,I}^p(z) = \sum_{n} \text{sgn}(\varepsilon_n) \langle z | n \rangle \left( \frac{1}{\sqrt{3}} D^{(8)}_{\chi^8} - D^{(8)}_{\chi^8} \tau^i \right) \Gamma^\mu \langle n | z \rangle$$  \hspace{1cm} (E.127)

The second term (E.126b) is linear in the angular velocity:

$$f_{2,S,I}^p(z) = \frac{1}{2} \left[ \left( \langle 0, z | G_c A^\dagger \bar{A} G_c | 0, z \rangle - \eta \langle 0, z | G_c A^\dagger \bar{A} G_c | 0, z \rangle^* \right) A^\dagger \mathcal{O}^{\mu \nu} A \right] - \text{v.s.}$$  \hspace{1cm} (E.128)

The trace can now be split according to the strange and nonstrange projections

$$\text{tr} \left[ \langle 0, z | (G_T P_T + G_S P_S) A^\dagger \bar{A} G_T P_T + G_S P_S | 0, z \rangle A^\dagger \mathcal{O}^{\mu \nu} A \right] =$$

$$- \int_0^1 dv^4 \text{tr} \left[ \langle 0, z | G_T P_T | v^4 \rangle A^\dagger \bar{A} \langle v^4 | G_T P_T | 0, z \rangle A^\dagger \mathcal{O}^{\mu \nu} A \right]$$

$$- \int_{-\infty}^0 dv^4 \text{tr} \left[ A^\dagger \mathcal{O}^{\mu \nu} A \langle 0, z | \left( \sum_{n<0} e^{\varepsilon_n v^4} | n \rangle \langle n | \right) | v^4 \rangle | 0, z \rangle P_T \right]$$

$$- \int_0^\infty dv^4 \text{tr} \left[ \langle 0, z | \left( \sum_{n<0} e^{\varepsilon_n v^4} | n \rangle \langle n | \right) | v^4 \rangle | 0, z \rangle P_T \right]$$

$$- \frac{K_1}{T_1} M_8 D^{(8)}_{\chi^8} \int \frac{d\omega}{2\pi} \text{tr} \langle z | \frac{1}{i\omega + h} \tau^i \frac{1}{i\omega + h} | 0, z \rangle A^\dagger \mathcal{O}^{\mu \nu}$$

$$= - \frac{1}{2} \left\{ \frac{J_i}{2T_1}, D^{(8)}_{\chi^8} \right\} \left( \sum_{n>0, m<0} \frac{1}{\varepsilon_n - \varepsilon_m} - \sum_{n<0, m>0} \frac{1}{\varepsilon_n - \varepsilon_m} \right) \langle m | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | n \rangle \langle n | \tau^i | m \rangle$$

$$+ \frac{1}{2} \left[ \frac{J_i}{2T_1}, D^{(8)}_{\chi^8} \right] \left( \sum_{n>0, m<0} \frac{1}{\varepsilon_n - \varepsilon_m} + \sum_{n<0, m>0} \frac{1}{\varepsilon_n - \varepsilon_m} \right) \langle m | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | n \rangle \langle n | \lambda^i | m \rangle$$

$$+ \frac{K_1}{T_1} M_8 D^{(8)}_{\chi^8} \sum_{n,m} \mathcal{R}_5 (\varepsilon_n, \varepsilon_m) \langle m | z \rangle \Gamma^\mu P_T \lambda^\beta P_T \langle z | n \rangle \langle n | \lambda^i | m \rangle$$  \hspace{1cm} (E.130)

where the sums can be simplified using the functions \(\mathcal{R}_5(\varepsilon_n, \varepsilon_m)\) (E.123) and \(\mathcal{R}_6(\varepsilon_n, \varepsilon_m)\) (E.116)

$$\left( \sum_{n>0, m<0} \frac{1}{\varepsilon_n - \varepsilon_m} - \sum_{n<0, m>0} \frac{1}{\varepsilon_n - \varepsilon_m} \right) A_{nm} = \sum_{n,m} \mathcal{R}_5(\varepsilon_n, \varepsilon_m) A_{nm},$$  \hspace{1cm} (E.131a)

$$\left( \sum_{n>0, m<0} \frac{1}{\varepsilon_n - \varepsilon_m} + \sum_{n<0, m>0} \frac{1}{\varepsilon_n - \varepsilon_m} \right) A_{nm} = \sum_{n,m} \mathcal{R}_6(\varepsilon_n, \varepsilon_m) A_{nm}.$$  \hspace{1cm} (E.131b)
E.3 Expansions in the angular velocity and $\delta m$

As for the second parcel of eq. (E.128), with the complex conjugation, the main difference is that now it is necessary to take into account the spectral representation of the conjugated propagator $G^\dagger$:

$$G^\dagger(x_4, x'_4) = \langle x_4 | \frac{1}{-\partial_x + i} | x'_4 \rangle$$

$$= \theta(x'_4 - x_4) \sum_{n>0} e^{-\varepsilon_n(x'_4 - x_4)} | n \rangle \langle n | - \theta(x_4 - x'_4) \sum_{n<0} e^{-\varepsilon_n(x_4 - x'_4)} | n \rangle \langle n |$$

(E.132)

Carrying out similar steps for the remaining terms, with strange quark contributions, and adding everything together, leads to the following form for the second term:

$$f_{2,8,l}^\mu(z) =$$

$$-\delta_{n1} \left[ \frac{1}{2} \left\{ \frac{J_1}{2I_1} D^{(8)}_{\chi\beta} \right\} - \frac{K_1}{I_1} M_S D^{(8)}_{8i} D^{(8)}_{\chi\beta} \right] \times \sum_{n,m} R_5(\varepsilon_n, \varepsilon_m) \langle m|z\rangle \Gamma^\mu P_T \lambda^\beta P_T (z|n) \langle n | P_T \lambda^\alpha P_T | m \rangle$$

$$-\delta_{n1} \left[ \frac{1}{2} \left\{ \frac{J_2}{2I_2} D^{(8)}_{\chi\beta} \right\} - \frac{K_2}{I_2} M_S D^{(8)}_{8a} D^{(8)}_{\chi\beta} \right] \times \sum_{n,m^0} R_5(\varepsilon_n, \varepsilon_{m^0}) \langle m^0|z\rangle P_S \lambda^\beta P_T \Gamma^\mu (z|n) \langle n | P_T \lambda^\alpha P_S | m^0 \rangle$$

(E.133)

For the third term (E.126c) linear in $\delta m$

$$f_{3,8,l}^\mu(z) = \frac{1}{2} \text{tr} \left[ 0, z | G_c i \gamma_4 A^1 \delta m A G_c | 0, z \rangle A^1 \mathcal{O}^{\mu\chi} A \right.$$

$$- \eta \langle 0, z | G_c i \gamma_4 A^1 \delta m A G_c | 0, z \rangle^* A^1 \mathcal{O}^{\mu\chi} A \right] - v.s$$

(E.134)

completely analogous steps lead to

$$f_{3,8,l}^\mu(z) = -\delta_{n1} \left[ \left( M_1 D^{(8)}_{\chi\beta} + \frac{1}{\sqrt{3}} M_S D^{(8)}_{8i} D^{(8)}_{\chi\beta} \right) \times \sum_{n,m} R_5(\varepsilon_n, \varepsilon_m) \langle m|z\rangle \Gamma^\mu P_T \lambda^\beta P_T (z|n) \langle n | P_T i \gamma_4 P_T | m \rangle$$

$$+ M_S D^{(8)}_{\chi\beta} D^{(8)}_{8j} \sum_{n,m} R_5(\varepsilon_n, \varepsilon_m) \langle m|z\rangle \Gamma^\mu P_T \lambda^\beta P_T (z|n) \langle n | i \gamma_4 \tau^j | m \rangle$$

$$+ M_S D^{(8)}_{\beta\chi} D^{(8)}_{8a} \sum_{n,m^0} R_5(\varepsilon_n, \varepsilon_{m^0}) \langle m^0|z\rangle P_S \lambda^\beta P_T \Gamma^\mu (z|n) \langle n | i \gamma_4 P_T \lambda^\alpha P_S | m^0 \rangle$$

$$+ M_S D^{(8)}_{\beta\chi} D^{(8)}_{8a} \sum_{n^0,m} R_5(\varepsilon_{n^0}, \varepsilon_m) \langle m|z\rangle \Gamma^\mu P_T \lambda^\beta P_S (z|n^0) \langle n^0 | P_S \lambda^\alpha P_T i \gamma_4 | m \rangle$$

(E.135)
Result for the sea

The sea contribution is now simply the sum of the terms of (E.63) and (E.126):

\[
\frac{1}{N_c} F_{\delta \chi}^{(0)}(z) = \frac{1}{N_c} F_{\delta \chi}^{(0)}(z) + \frac{1}{N_c} F_{\delta \chi}^{(0)}(z)
\]

\[
\begin{align*}
= & D_{\chi \delta}^{(8)} \sum_n \langle m | z \rangle P_T \lambda^\beta P_T \Gamma^\mu (z | n) \left( \delta_{\eta_1} R_1(\varepsilon_n) - \delta_{\eta_1 - 1} \right) \\
+ & \sum_{n,m} \langle m | z \rangle P_T \lambda^\beta P_T \Gamma^\mu (z | n) \langle n | P_T \lambda^\alpha P_S | m \rangle \\
& \quad \times \left( -\frac{1}{2} \left[ \frac{J_1}{2 T_1} D_{\chi \beta}^{(8)} \right] \left[ \delta_{\eta_1 - 1} R_3(\varepsilon_n, \varepsilon_m) + \delta_{\eta_1} R_5(\varepsilon_n, \varepsilon_m) \right] \\
& \quad + \frac{1}{2} \left[ \frac{J_2}{2 T_2} D_{\chi \beta}^{(8)} \right] \left[ \delta_{\eta_1} R_4(\varepsilon_n, \varepsilon_m) + \delta_{\eta_1 - 1} R_6(\varepsilon_n, \varepsilon_m) \right] \\
& \quad + \frac{K_1}{T_1} M_8 D_{8 a}^{(8)} D_{\chi \beta}^{(8)} \left[ \delta_{\eta_1 - 1} R_3(\varepsilon_n, \varepsilon_m) + \delta_{\eta_1} R_5(\varepsilon_n, \varepsilon_m) \right] \right) \\
+ & \sum_{n,m^0} \langle m | z \rangle P_T \Gamma^\mu P_T \lambda^\beta P_S (z | n) \langle n | P_S \lambda^\alpha P_T | m \rangle \\
& \quad \times \left( \frac{1}{2} \left[ \frac{J_2}{2 T_2} D_{\chi \beta}^{(8)} \right] \left[ \delta_{\eta_1 - 1} R_3(\varepsilon_n, \varepsilon_m) + \delta_{\eta_1} R_5(\varepsilon_n, \varepsilon_m) \right] \\
& \quad + \frac{1}{2} \left[ \frac{J_2}{2 T_2} D_{\chi \beta}^{(8)} \right] \left[ \delta_{\eta_1} R_4(\varepsilon_n, \varepsilon_m) + \delta_{\eta_1 - 1} R_6(\varepsilon_n, \varepsilon_m) \right] \\
& \quad + \frac{K_2}{T_2} M_8 D_{8 a}^{(8)} D_{\chi \beta}^{(8)} \left[ \delta_{\eta_1 - 1} R_3(\varepsilon_n, \varepsilon_m) + \delta_{\eta_1} R_5(\varepsilon_n, \varepsilon_m) \right] \right) \\
+ & \left( M_1 + \frac{1}{\sqrt{3}} M_8 D_{8 a}^{(8)} D_{\chi \beta}^{(8)} \sum_{n,m} \langle m | z \rangle \Gamma^\mu P_T \lambda^\beta P_T (z | n) \langle n | i \gamma_4 | m \rangle \\
& \quad \times \left( \delta_{\eta_1} R_2(\varepsilon_n, \varepsilon_m) - \delta_{\eta_1 - 1} R_5(\varepsilon_n, \varepsilon_m) \right) \\
& \quad + M_8 D_{8 a}^{(8)} D_{\chi \beta}^{(8)} \sum_{n,m} \langle m | z \rangle \Gamma^\mu P_T \lambda^\beta P_T (z | n) \langle n | i \gamma_4 | m \rangle \\
& \quad \times \left( \delta_{\eta_1} R_2(\varepsilon_n, \varepsilon_m) - \delta_{\eta_1 - 1} R_5(\varepsilon_n, \varepsilon_m) \right) \\
& \quad + M_8 D_{8 a}^{(8)} D_{\chi \beta}^{(8)} \sum_{n,m^0} \langle m_0 | z \rangle P_S \lambda^\beta P_T \Gamma^\mu (z | n) \langle n | i \gamma_4 | P_T | m \rangle \\
& \quad \times \left( \delta_{\eta_1} R_2(\varepsilon_n, \varepsilon_m) - \delta_{\eta_1 - 1} R_5(\varepsilon_n, \varepsilon_m) \right) \right) \quad (E.136)
\end{align*}
\]
with the regularization functions: \( R_1 \) (E.70), \( R_2 \) (E.121), \( R_3 \) (E.101), \( R_4 \) (E.114). The other functions are \( R_5 \) (E.123), \( R_6 \) (E.116).

In order to be further simplify these expressions it is necessary to know which case \( \eta = 0 \) or \( \eta \neq 0 \) is fulfilled by the operator \( \mathcal{O}^{\mu \nu} \). This knowledge, apart from the obvious replacement of the Kronecker \( \delta \) functions in the expressions above, determines the behaviour of the one-particle matrix elements under the \( G_5 \)-parity transformation.

### E.4 Intermediate results for form factors

This section relates the previous results (E.52) and (E.136) to the various form factors considered in this work.

#### E.4.1 Electric

The model result can be constructed from (E.52,E.136) with just some minor simplifications concerning the operator for this case. In particular it is possible to go back to Minkowski notation for the gamma matrices, since they appear in the same form in both sides of the relations (E.52,E.136).

The Lorentz part of the operator is \( \Gamma^\mu = 1 \) for which \( \eta = -1 \). The \( G_5 \) relations are also used to simplify the expressions. For \( F^{\mu \nu}_E(z) \) we have:

\[
\frac{1}{N_\epsilon} F^{\mu \nu}_E(z) = \frac{1}{\sqrt{3}} D^{(8)}_{\chi^8} \left( \langle \psi | z | \psi \rangle - \frac{1}{2} \sum_n \text{sgn}(\epsilon_n) \langle z | n \rangle \langle n | z \rangle \right) \\
+ \left\{ \frac{J_1}{2 I_1} D^{(8)}_{\chi^8} \right\} \left( \sum_{n \neq 0, \pm n} \frac{1}{\bar{\epsilon}_n - \epsilon_n} \langle \psi | z \rangle \tau^j \langle z | n \rangle \langle n | \tau^j | \psi \rangle \right. \\
- \frac{1}{2} \sum_{n, m} \mathcal{R}_3(\epsilon_n, \epsilon_m) \langle m | z \rangle \tau^j \langle z | n \rangle \langle n | \tau^j | m \rangle \right) \\
+ \left( \frac{J_n}{2 I_2}, D^{(8)}_{\chi^8} \right) \left\{ \frac{2}{3} + \frac{J_n}{2 I_2}, D^{(8)}_{\chi^8} \right\} d_{ab} \left( \sum_{n^0 > 0} \frac{1}{\bar{\epsilon}_n - \epsilon_n} [\langle \psi | z \rangle \langle z | n^0 \rangle \langle n^0 | \psi \rangle] \\
- \sum_{n, m^0} \mathcal{R}_3(\epsilon_n, \epsilon_m^0) \langle m^0 | z \rangle \langle z | n \rangle \langle n | m^0 \rangle \right) \\
+ \left( \frac{1}{\sqrt{3}} \left[ \frac{J_1}{2 I_2}, D^{(8)}_{\chi^8} \right] \right) i D^{(8)}_{\chi^8} \left( \sum_{n \neq 0, \pm n} \frac{1}{\bar{\epsilon}_n - \epsilon_n} \langle \psi | z \rangle \tau^j \langle z | n \rangle \langle n | \tau^j | \psi \rangle \right. \\
+ \sum_{n, m^0} \mathcal{R}_3(\epsilon_n, \epsilon_m^0) \langle m^0 | z \rangle \langle z | n \rangle \langle n | m^0 \rangle \right) \\
+ \frac{K_1}{T_1} M_s D^{(8)}_{\chi^8} D^{(8)}_{\chi^8} \left( \sum_{n \neq 0, \pm n} \frac{1}{\bar{\epsilon}_n - \epsilon_n} \langle \psi | z \rangle \tau^j \langle z | n \rangle \langle n | \tau^j | \psi \rangle \right. \\
- \frac{1}{2} \sum_{n, m} \mathcal{R}_3(\epsilon_n, \epsilon_m) \langle m | z \rangle \tau^j \langle z | n \rangle \langle n | \tau^j | m \rangle \right) \\
+ \frac{2 K_2}{T_2} M_s \left( \frac{2}{3} D^{(8)}_{\chi^8} D^{(8)}_{\chi^8} \right) \left( \sum_{n \neq 0, \pm n} \frac{1}{\bar{\epsilon}_n - \epsilon_n} \langle \psi | z \rangle \langle z | n^0 \rangle \langle n^0 | \psi \rangle \\
- \sum_{n, m^0} \mathcal{R}_3(\epsilon_n, \epsilon_m^0) \langle m^0 | z \rangle \langle z | n \rangle \langle n | m^0 \rangle \right) \\
+ 2 \left( \frac{1}{\sqrt{3}} M_1 D^{(8)}_{\chi^8} + \frac{1}{3} M_s D^{(8)}_{\chi^8} D^{(8)}_{\chi^8} \right) \left( \sum_{n \neq 0, \pm n} \frac{1}{\bar{\epsilon}_n - \epsilon_n} \langle \psi | z \rangle \langle z | n \rangle \gamma^0 | \psi \rangle \\
- \frac{1}{2} \sum_{n, m} \mathcal{R}_3(\epsilon_n, \epsilon_m) \langle m | z \rangle \langle z | n \rangle \gamma^0 | m \rangle \right) \\
+ 2 \left( \frac{1}{\sqrt{3}} M_1 D^{(8)}_{\chi^8} + \frac{1}{3} M_s D^{(8)}_{\chi^8} D^{(8)}_{\chi^8} \right) \left( \sum_{n \neq 0, \pm n} \frac{1}{\bar{\epsilon}_n - \epsilon_n} \langle \psi | z \rangle \tau^j \langle z | n \rangle \langle n | \tau^j | \psi \rangle \right.
\]

\[
+ 2 M_s D^{(8)}_{\chi^8} D^{(8)}_{\chi^8} \left( \sum_{n \neq 0, \pm n} \frac{1}{\bar{\epsilon}_n - \epsilon_n} \langle \psi | z \rangle \tau^j \langle z | n \rangle \langle n | \gamma^0 \tau^j | \psi \rangle \right)
\]
\[ -\frac{1}{2} \sum_{n,m} \mathcal{R}_3(\varepsilon_n, \varepsilon_m) \langle m|z\rangle \tau^k \langle z|n\rangle \gamma^0 \tau^l |m\rangle \]  
\tag{E.137g}

\[ + 2M_b \left( \frac{2}{3} D^{(8)}_{\delta a} D^{(8)}_{\chi a} + \frac{1}{\sqrt{3}} D^{(8)}_{\delta a} D^{(8)}_{\chi b} d_{abs} \right) \left( \sum_{n^0} \frac{1}{\varepsilon_V - \varepsilon_{n^0}} \langle v|z\rangle \langle z|n^0\rangle \gamma^0 |v\rangle \right) \]
\[ - \sum_{n,m^0} \mathcal{R}_3(\varepsilon_n, \varepsilon_{m^0}) \langle m^0|z\rangle \langle z|n\rangle \gamma^0 |m^0\rangle \]  
\tag{E.137h}

Using the fact that the SU(3) structure constants fulfill
\[ \sum_{\alpha,\beta=1}^8 f_{\alpha\beta\gamma} f_{\alpha\beta\sigma} = 3\delta_{\gamma\sigma}, \quad \sum_{a,b=4}^7 f_{abs} f_{abs} = 0, \quad \sum_{a,b=4}^7 f_{abs} f_{abs} = 3; \]
\tag{E.138}

the commutators and anticommutators of collective operators \( (J \text{ and } D^{(8)}) \) are easily simplified
\[ \left\{ \frac{J_i}{2I_1}, D^{(8)}_{\chi^a} \right\} = \frac{1}{I_2} D^{(8)}_{\chi a} \frac{1}{I_2} J_a D^{(8)} \]  
\tag{E.139a}

\[ \left\{ \frac{J_a}{2I_2}, D^{(8)}_{\chi^a} \right\} = \frac{1}{I_2} D^{(8)}_{\chi a} \frac{1}{I_2} J_a D^{(8)} \]  
\tag{E.139b}

where the notation for the matrix element of the collective operator \( \mathcal{O} \) (\( J \), \( D^{(8)} \), or products of them) \( \langle B(S_3)| \mathcal{O} |B(S_3)\rangle = \langle \mathcal{O} \rangle \) is used.

### E.4.2 Magnetic

The expressions (E.52) and (E.136) developed in Section E.3 contain the magnetic density in the case \( \mu = k \) and \( \eta = 1 \). Further simplifications of these expressions use the properties studied in Section G.2.1. As an example, using (G.36b)
\[ \langle m|z\rangle \gamma^0 \gamma^k \langle z|m\rangle = - \langle n|z\rangle \gamma^0 \gamma^k \langle z|m\rangle \]
\tag{E.140a}

\[ \langle m|z\rangle \gamma^0 \gamma^k \tau^l \langle z|n\rangle = \langle n|z\rangle \gamma^0 \gamma^k \tau^l \langle z|m\rangle. \]
\tag{E.140b}

These properties allow for simplifications like, here for (E.52),
\[ \langle v|z\rangle \gamma^0 \gamma^k \langle z|n\rangle \langle n| \tau^l |v\rangle + \langle n|z\rangle \gamma^0 \gamma^k \langle z|v\rangle \langle v| \tau^l |n\rangle = 2 \langle v|z\rangle \gamma^0 \gamma^k \langle z|n\rangle \langle n| \tau^l |v\rangle. \]
\tag{E.141}

For \( \chi = 3, 8 \) the magnetic density is
\[ \frac{1}{N_e}\mathcal{R}^{(8)}_M(z) = D^{(8)}_{\chi^a} \left( \langle v|z\rangle \gamma^0 \gamma^k \tau^l \phi_\nu(z) + \sum_n \langle n|z\rangle \gamma^0 \gamma^k \tau^l \langle z|n\rangle \mathcal{R}_1(\varepsilon_n) \right) \]
\[ + \frac{1}{\sqrt{3}} \left\{ \frac{J_i}{2I_1}, D^{(8)}_{\chi^a} \right\} \left( \sum_{n^0} \frac{1}{\varepsilon_V - \varepsilon_{n^0}} \langle v|z\rangle \gamma^0 \gamma^k \langle z|n^0\rangle \langle n^0| \tau^l |v\rangle \right) \]
\[ - \frac{1}{2} \sum_{n,m} \mathcal{R}_3(\varepsilon_n, \varepsilon_m) \langle m|z\rangle \gamma^0 \gamma^k \langle z|n\rangle \langle n| \tau^l |m\rangle \]  
\tag{E.142a}

\[ + \left\{ \frac{J_a}{2I_2}, D^{(8)}_{\chi^a} \right\} d_{abs} \left( \sum_{n^0} \frac{1}{\varepsilon_V - \varepsilon_{n^0}} \langle v|z\rangle \gamma^0 \gamma^k \tau^l \langle z|n^0\rangle \langle n| \tau^l |v\rangle \right) \]
\[ - \sum_{m^0} \mathcal{R}_3(\varepsilon_{m^0}, \varepsilon_{n^0}) \langle m^0|z\rangle \gamma^0 \gamma^k \tau^l \langle z|n^0\rangle \langle n^0| \tau^l |m^0\rangle \]  
\tag{E.142b}

\[ + \left[ \frac{J_i}{2I_1}, D^{(8)}_{\chi^a} \right] d_{abs} \left( \sum_{n^0} \frac{\text{sgn}(\varepsilon_n)}{\varepsilon_n - \varepsilon_V} \langle v|z\rangle \gamma^0 \gamma^k \tau^l \langle z|n^0\rangle \langle n| \tau^l |v\rangle \right) \]
\[ + \frac{1}{2} \sum_{n,m^0} \mathcal{R}_4(\varepsilon_n, \varepsilon_{m^0}) \langle m|z\rangle \gamma^0 \gamma^k \tau^l \langle z|n^0\rangle \langle n| \tau^l |m^0\rangle \]  
\tag{E.142c}

\[ + \left[ \frac{J_a}{2I_2}, D^{(8)}_{\chi^a} \right] i f_{abs} \left( \sum_{n^0} \frac{\text{sgn}(\varepsilon_{n^0})}{\varepsilon_{n^0} - \varepsilon_V} \langle v|z\rangle \gamma^0 \gamma^k \tau^l \langle z|n^0\rangle \langle n| \tau^l |v\rangle \right) \]
\[ + \sum_{m^0} \mathcal{R}_4(\varepsilon_n, \varepsilon_{m^0}) \langle n|z\rangle \gamma^0 \gamma^k \tau^l \langle z|m^0\rangle \langle m^0|n\rangle \]  
\tag{E.142d}
E.4 Intermediate results for form factors

\[ -\frac{2}{\sqrt{3}} \frac{K_1}{I_1} \sum_{n \neq 0} \frac{1}{\varepsilon - \varepsilon_n} \langle v|z \rangle \gamma^0 \gamma^k \langle z|n \rangle \langle n| \tau^i |v \rangle \]

\[ -\frac{1}{\sqrt{3}} \sum_{n,m} R_5(\varepsilon_n, \varepsilon_m) \langle m|z \rangle \gamma^0 \gamma^k \langle z|n \rangle \langle n| \tau^i |m \rangle \]  

(E.142f)

\[ -\frac{K_2}{I_2} \sum_{n \neq 0} \frac{1}{\varepsilon - \varepsilon_n} \langle v|z \rangle \gamma^0 \gamma^k \tau^j \langle z|n^0 \rangle \langle n^0|v \rangle \]

\[ -\frac{1}{\sqrt{3}} \sum_{n,m} R_5(\varepsilon_n, \varepsilon_m) \langle n|z \rangle \gamma^0 \gamma^k \tau^j \langle z|m^0 \rangle \langle m^0|n \rangle \]  

(E.142g)

\[ +2 \left( M_1 D_{\chi}^{(8)} + \frac{1}{\sqrt{3}} M_s D_{\chi}^{(8)} D_{\chi}^{(8)} \right) \left( \sum_{n \neq 0} \frac{1}{\varepsilon - \varepsilon_n} \langle v|z \rangle \gamma^0 \gamma^k \tau^j \langle z|n \rangle \langle n| \gamma^0 |v \rangle \right) \]

\[ + \frac{1}{\sqrt{3}} \sum_{n,m} R_2(\varepsilon_n, \varepsilon_m) \langle m|z \rangle \gamma^0 \gamma^k \tau^j \langle z|n \rangle \langle n| \gamma^0 |m \rangle \]  

(E.142h)

\[ +2 \frac{1}{\sqrt{3}} M_s D_{\chi}^{(8)} D_{\chi}^{(8)} \sum_{n \neq 0} \frac{1}{\varepsilon - \varepsilon_n} \langle v|z \rangle \gamma^0 \gamma^k \tau^j \langle z|n \rangle \langle n| \gamma^0 |v \rangle \]

\[ + \frac{1}{\sqrt{3}} \sum_{n,m} R_2(\varepsilon_n, \varepsilon_m) \langle n|z \rangle \gamma^0 \gamma^k \tau^j \langle z|m^0 \rangle \langle m^0|n \rangle \]  

(E.142i)

\[ +2 M_s D_{\chi}^{(8)} \sum_{n \neq 0} \frac{1}{\varepsilon - \varepsilon_n} \langle v|z \rangle \gamma^0 \gamma^k \tau^j \langle z|n \rangle \langle n| \gamma^0 |v \rangle \]

\[ + \sum_{n,m} R_2(\varepsilon_n, \varepsilon_m) \langle n|z \rangle i \gamma_3 \gamma^k \tau^j P_T \langle \tau|n \rangle \langle n| \gamma^0 |m \rangle \]  

(E.142j)

E.4.3 Axial

The most relevant results from the $G_5$ transformations are:

\[ \langle m|z \rangle \gamma^0 \gamma^3 \gamma^5 \tau^j \langle z|n \rangle = \langle n|z \rangle \gamma^0 \gamma^3 \gamma^5 \tau^j \langle z|m \rangle \]

(E.143a)

\[ \langle m|z \rangle \gamma^0 \gamma^3 \gamma^5 \langle z|n \rangle = - \langle n|z \rangle \gamma^0 \gamma^3 \gamma^5 \langle z|m \rangle \]

(E.143b)

\[ \langle m| \gamma^0 |n \rangle = \langle n| \gamma^0 |m \rangle \]

(E.143c)

\[ \langle m| \gamma^0 \tau^i |n \rangle = - \langle n| \gamma^0 \tau^i |m \rangle \]

(E.143d)

Using these results ($\gamma^0 \gamma^3 \gamma^5 = \Sigma^5$)

\[ \frac{1}{N_c} A_{\gamma^k}(z) = D^{(8)}_{\chi} \langle v|z \rangle \sigma^2 \tau^j \langle \tau|v \rangle + D^{(8)}_{\chi} \sum_{n} \langle n|z \rangle \sigma^2 \tau^j \langle \tau|n \rangle R_1(\varepsilon_n) \]

(E.144a)

\[ + \frac{1}{\sqrt{3}} \frac{J_1}{2 I_1} D^{(8)}_{\chi} \left( \sum_{n \neq 0} \frac{1}{\varepsilon - \varepsilon_n} \langle n|z \rangle \sigma^3 \langle z|v \rangle \langle v| \tau^i |n \rangle \right) \]

\[ - \frac{1}{\sqrt{3}} \sum_{n,m} R_5(\varepsilon_n, \varepsilon_m) \langle m|z \rangle \sigma^3 \langle z|n \rangle \langle n| \tau^i |m \rangle \]  

(E.144b)

\[ + \left( \frac{J_1}{2 I_1} D^{(8)}_{\chi} \right) d_{ab} \langle m^0|z \rangle \sigma^3 \tau^j \langle z|0 \rangle \langle 0| \tau^i |m^0 \rangle \]

\[ - \sum_{n,m} R_5(\varepsilon_n, \varepsilon_m) \langle n|z \rangle \sigma^3 \tau^j \langle z|m_0 \rangle \langle m_0|n \rangle \]  

(E.144c)

\[ + \left( \frac{J_1}{2 I_1} D^{(8)}_{\chi} \right) \left( \sum_{n \neq 0} \frac{\text{sgn}(\varepsilon_n)}{\varepsilon - \varepsilon_n} \langle n|z \rangle \sigma^3 \tau^j \langle z|0 \rangle \langle 0| \tau^i |n \rangle \right) \]

\[ + \frac{1}{\sqrt{3}} \sum_{n,m} R_4(\varepsilon_n, \varepsilon_m) \langle m|z \rangle \sigma^3 \tau^j \langle z|n \rangle \langle n| \tau^i |m \rangle \]  

(E.144d)

\[ + \left( \frac{J_1}{2 I_1} D^{(8)}_{\chi} \right) i f_{ab} \left( \sum_{m \neq 0} \frac{\text{sgn}(\varepsilon_m)}{\varepsilon - \varepsilon_m} \langle m^0|z \rangle \sigma^3 \tau^j \langle z|0 \rangle \langle 0| \tau^i |m^0 \rangle \right) \]
\[+ \sum_{n,m} \mathcal{R}_4(\varepsilon_n, \varepsilon_m) \langle m^0 | z \rangle \sigma^3 \tau^j \langle z | n \rangle \langle n | m^0 \rangle \] (E.144\varepsilon)

\[+ \frac{2}{\sqrt{3}} K m \delta D^{(8)}_{ij} D^{(8)}_{\chi^0} \left( \sum_{n \neq 0} \frac{1}{\varepsilon_V - \varepsilon_n} \langle v | z \rangle \sigma^3 \langle z | n \rangle \langle n | \tau^j | v \rangle \right) - \frac{1}{2} \sum_{n,m} \mathcal{R}_5(\varepsilon_n, \varepsilon_m) \langle m | z \rangle \sigma^3 \langle z | n \rangle \langle n | \tau^j | m \rangle \] (E.144f)

\[+ 2K m \delta D^{(8)}_{ia} D^{(8)}_{\chi^0} d_{ab} \left( \sum_{n \neq 0} \frac{1}{\varepsilon_V - \varepsilon_n} \langle v | z \rangle \sigma^3 \tau^j \langle z | n \rangle \langle n | \gamma^0 | v \rangle \right) - \sum_{n,m} \mathcal{R}_5(\varepsilon_n, \varepsilon_m) \langle m | z \rangle \sigma^3 \langle z | n \rangle \langle n | \gamma^0 | m \rangle \] (E.144g)

\[+ 2 \left( M_1 D^{(8)}_{ij} + \frac{1}{\sqrt{3}} M_8 D^{(8)}_{i\alpha} D^{(8)}_{\chi^0} \right) \left( \sum_{n \neq 0} \frac{1}{\varepsilon_V - \varepsilon_n} \langle v | z \rangle \sigma^3 \tau^j \langle z | n \rangle \langle n | \gamma^0 | v \rangle \right) + \frac{1}{2} \sum_{n,m} \mathcal{R}_5(\varepsilon_n, \varepsilon_m) \langle m | z \rangle \sigma^3 \langle z | n \rangle \langle n | \gamma^0 | m \rangle \mathcal{R}_2(\varepsilon_n, \varepsilon_m) \] (E.144h)

\[+ \frac{2}{\sqrt{3}} M_8 D^{(8)}_{i\alpha} D^{(8)}_{\chi^0} \left( \sum_{n \neq 0} \varepsilon_{ij} - \varepsilon_n \langle v | z \rangle \sigma^3 \langle z | n \rangle \langle n | \gamma^0 \tau^j | v \rangle \right) + \frac{1}{2} \sum_{n,m} \mathcal{R}_5(\varepsilon_n, \varepsilon_m) \langle m | z \rangle \sigma^3 \langle z | n \rangle \langle n | \gamma^0 \tau^j | m \rangle \] (E.144i)

\[+ 2M_8 D^{(8)}_{ia} D^{(8)}_{\chi^0} d_{ab} \left( \sum_{m \neq 0} \frac{1}{\varepsilon_V - \varepsilon_n} \langle v | z \rangle \sigma^3 \tau^j \langle z | m^0 \rangle \langle m^0 | \gamma^0 | v \rangle \right) + \sum_{n,m} \mathcal{R}_5(\varepsilon_n, \varepsilon_m) \langle m | z \rangle \sigma^3 \tau^j P \langle z | m^0 \rangle \langle m^0 | \gamma^0 | n \rangle \] (E.144j)
F Collective matrix elements

F.1 Collective wave functions

The collective baryonic wave functions are given by
\[ \langle A | \Psi(n; Y, T, T_3; Y', J, J_3) \rangle = \Psi^{(n)}_{(Y, T, T_3)(Y', J, J_3)}(A) \]
\[ = \sqrt{\dim(n)} \langle -Y'/2 + J_3 \rangle \left( \langle Y, T, T_3 | D^{(n)}(A) | -Y', J, -J_3 \rangle \right)^* \]
\[ = \sqrt{\dim(n)} \langle -Y'/2 + J_3 \rangle D^{(n)*}_{(Y, T, T_3)(-Y', J, -J_3)}(A) \]  \hspace{1cm} (F.1).

In a basis in which the states are labeled by the quantum numbers. The right hypercharge means that the wave function in linear order in \( \delta m \) is a superposition of different components coming from different SU(3) representations given to linear order in \( \delta m \) by (2.57)
\[ |B(S_3 T_3)\rangle = |8; (S_3 T_3)\rangle + c_{TT} |10; (S_3 T_3)\rangle + c_{27} |27; (S_3 T_3)\rangle \]  \hspace{1cm} (F.7)

For the proton with spin up (F.3), using (F.4a) and (F.4b),
\[ \chi_{\nu_1}^{(n)*} (A) D^{(n)}_{\nu_1 \nu_2} (A) = \frac{1}{\dim(n)} \delta_{\nu_1 \nu_1} \delta_{\nu_2 \nu_2} \]  \hspace{1cm} (F.2)

where \( dA \) is the Haar measure over the elements of the group.

As an example, the wave function for the proton with spin up is
\[ |p \uparrow\rangle = \sqrt{8} D^{(8)*}_{(1/2, 1/2)} (A). \]  \hspace{1cm} (F.3)

One can use an alternative labeling of the states based on the following correspondence [269] to the SU(3) ‘spherical’ basis.

\[ |8; 1/2, 1/2 \rangle = - (|4\rangle + i |5\rangle) / \sqrt{2} \]  \hspace{1cm} (F.4a)
\[ |8; 1/2, -1/2 \rangle = (|6\rangle + i |7\rangle) / \sqrt{2} \]  \hspace{1cm} (F.4b)
\[ |8; -1/2, 1/2 \rangle = - (|6\rangle - i |7\rangle) / \sqrt{2} \]  \hspace{1cm} (F.4c)
\[ |8; -1/2, -1/2 \rangle = (|4\rangle - i |5\rangle) / \sqrt{2} \]  \hspace{1cm} (F.4d)
\[ |8; 0, 1, 1 \rangle = - (|1\rangle + i |2\rangle) / \sqrt{2} \]  \hspace{1cm} (F.4e)
\[ |8; 0, 1, 0 \rangle = |3\rangle \]  \hspace{1cm} (F.4f)
\[ |8; 0, 1, -1 \rangle = (|1\rangle - i |2\rangle) / \sqrt{2} \]  \hspace{1cm} (F.4g)
\[ |8; 0, 0, 0 \rangle = |8\rangle \]  \hspace{1cm} (F.4h)

In this case, e.g., using (F.4h)
\[ D^{(8)}_{88} (A) \equiv \langle 8 | D^{(8)} (A) | 8 \rangle \rightarrow \langle 8; 0, 0, 0 | D^{(8)} (A) | 8; 0, 0, 0 \rangle = D^{(8)}_{(0,0,0),(0,0,0)} (A). \]  \hspace{1cm} (F.5)

For the proton with spin up (F.3), using (F.4a) and (F.4b),
\[ |p \uparrow\rangle = \sqrt{8} D^{(8)*}_{(1/2, 1/2)} (A) = \sqrt{8} \langle 8; 1/2, 1/2 | D^{(8)} (A) | 8; 1/2, -1/2 \rangle^* \]
\[ = \sqrt{2} \left( (|4\rangle + i |5\rangle) D^{(8)} (A) (|6\rangle + i |7\rangle) \right)^* = \sqrt{2} D^{(8)*}_{4+i5,6+i7} (A) \]  \hspace{1cm} (F.6)

which defines objects like \( D^{(8)*}_{4+i5,6+i7} (A) \).

F.2 Matrix elements with symmetry breaking

The symmetry breaking perturbs, however, the collective wave function of the nucleon. This means that the wave function in linear order in \( \delta m \) is a superposition of different components coming from different SU(3) representations given to linear order in \( \delta m \) by (2.57)
\[ |B(S_3 T_3)\rangle = |8; (S_3 T_3)\rangle + c_{TT} |10; (S_3 T_3)\rangle + c_{27} |27; (S_3 T_3)\rangle \]  \hspace{1cm} (F.7)
with the coefficients
\[ c_{10}(Y,T) = -M_8 \frac{I_2}{\sqrt{15}} \left( \frac{1}{m} \sum_{SU(2)} - \frac{K_1}{I_1} \right) \left( \frac{Y}{2} + \frac{Y^2}{8} + \frac{1}{2} T (T + 1) \right) \]  
(F.8a)
\[ c_{27}(Y,T) = -M_8 \frac{I_2}{25\sqrt{3}} \left( \frac{1}{m} \sum_{SU(2)} + \frac{K_1}{I_1} - \frac{4K_2}{I_2} \right) \times \left[ \sqrt{6}Y^2 + 3 \left( 1 - \frac{7}{8}Y^2 \right) - \frac{1}{2} T (T + 1) \right], \]  
(F.8b)
both of order \( \delta m = M_1 + M_8 \lambda_3 \). For the baryon octet,
\[ \frac{Y}{2} + \frac{Y^2}{8} + \frac{1}{2} T (T + 1) = \delta_{T,2}^\gamma \delta_{Y,1} + \delta_{T,1}^\gamma \delta_{Y,0}, \]
\[ \sqrt{6}Y^2 + 3 \left( 1 - \frac{7}{8}Y^2 \right) - \frac{1}{2} T (T + 1) = \sqrt{6}\delta_{T,2}^\gamma (\delta_{Y,1} + \delta_{Y,-1}) + \delta_{Y,0} (3\delta_{T,0} + 2\delta_{T,1}). \]  
(F.9a, F.9b)

The collective matrix elements are calculated also up to linear order in \( \delta m \) according to (\( O_{coll} \) is a generic collective operator) and letting for simplicity \( B \) represent the baryon quantum numbers, \( i.e. B \equiv (TT_3)(SS_3) \)
\[ \langle B' | O_{coll} | B \rangle = \langle 8; B' | O_{coll} [8; B] + c_{10}(8) \langle 8; B' | O_{coll} [10; B] \rangle 
+ c_{27}(8) \langle 27; B' | O_{coll} [8; B] + \langle 8; B' | O_{coll} [27; B] \rangle \]  
(F.10)
In the case the matrix elements multiply terms for the mass corrections, then (\( \gamma = 1, 8 \))
\[ M_7 \langle B'(S_3'T_3') | O_{coll} [B(S_3T_3)] \rangle = M_7 \langle 8; B'(S_3'T_3') | O_{coll} [8; B(S_3T_3)] \rangle \]  
(F.11)
because \( M_6c_{10} \) and \( M_7c_{27} \) are of higher order in the strange quark mass.

### F.3 Matrix elements of collective operators

The basic collective operators are the angular moment operators \( J_\alpha, \alpha = 1, \ldots 8 \) and the Wigner SU(3) functions \( D_{\alpha J}^{(n)}(A) \). There may be compound operators made from these basic ones. It is important to notice that the basic operators do not always commute. One can list (sums over repeated \( a, b \) run from 4 to 7, and over repeated \( i \) from 1 to 3) the possible cases of the collective operators entering the calculation of form factors:

#### The operator \( J_\alpha \)

In this case it is found that \( \alpha = 3 \) for matrix element between a given baryon state which means that
\[ \hat{J}_3 \Psi_{(Y,T,T_3)(\gamma',J_3)}(A) = J_3 \Psi_{(Y,T,T_3)(\gamma,J_3)}(A). \]  
(F.12)
This is an example of how these operators act on the Wigner functions. One has, e.g. taking the case of the proton with spin up
\[ \hat{J}_3 D_{(1,\frac{1}{2},\frac{1}{2})}^{(8)*}(A) = \left[ \hat{J}_3, D_{(1,\frac{1}{2},\frac{1}{2})}^{(8)*}(1,\frac{1}{2},\frac{1}{2}) \right](A) \]
\[ = \frac{1}{2} \left[ \hat{J}_3, D_{4+5,6}^{(8)*}(A) - i \left[ \hat{J}_3, D_{4+5,7}^{(8)*}(A) \right] \right] \]  
(F.13)
Now, one can invoke the commutation relations between the collective operators, (D.74):
\[ [\hat{J}_{\alpha}, D_{\beta J}^{(8)*}] = if_{\alpha \beta \gamma} D_{\beta J}^{(8)*}. \]  
(F.14)
Then
\[ \left[ \hat{J}_3, D_{4+5,6}^{(8)*}(A) \right] = if_{36a} D_{4+5,6,a}^{(8)*}(A) = -i\frac{1}{2} D_{4+5,7}^{(8)*}(A) \]  
(F.15a)
\[ \left[ \hat{J}_3, D_{4+5,7}^{(8)*}(A) \right] = if_{37a} D_{4+5,7,a}^{(8)*}(A) = i\frac{1}{2} D_{4+5,6}^{(8)*}(A) \]  
(F.15b)
and finally
\[ \hat{J}_3 D^{(8)*}_{3,3}(A) = \frac{1}{4} D^{(8)*}_{4+5,6+17}(A) = \frac{1}{2} D^{(8)*}_{(1,\frac{1}{2}; \frac{1}{2},-\frac{1}{2})}(A). \] (F.16)

One \( D^{(8)}_{Q'Q} \) matrix (\( Q', Q = 3, 8 \))

In this case one has
\[ \langle B(S_3 T_3) \mid D^{(8)}_{Q'Q} \mid B(S_3 T_3) \rangle = \langle B; \mu' (Y'T'T'_3)(Y'S'S'_3) \mid D^{(8)}_{Q'Q} \mid B; \mu(YTT_3)(Y_RT_S) \rangle \]
\[ = \int dA \sqrt{\text{dim}(\mu')}(-Y'/2+S_3) D^{(\mu')}_{Y',T'}(A) \times \sqrt{\text{dim}(\mu)}(-Y_R/2+S_3) D^{(\mu)*}_{Y,T}(A) \]
\[ \times \sum_\gamma \left( \begin{array}{ccc} 8 & Q' & \mu' \\ 8 & Y', T', T'_3 & (Y, T, T_3) \end{array} \right) \left( \begin{array}{ccc} \mu & \mu' & \mu' \\ Q & (-Y', S', -S'_3) & (-Y_R, S, -S_3) \end{array} \right) \] (F.17)

using the relation [268]
\[ \int dA D^{(n)}_{\nu_\mu} D^{(n_1)}_{\nu_1 \mu_1} D^{(n_2)}_{\nu_2 \mu_2} = \frac{1}{\text{dim}(n)} \sum_\gamma \left( \begin{array}{ccc} n_1 & n_2 & n_3 \\ \nu_1 & \nu_2 & \nu_3 \end{array} \right) \left( \begin{array}{ccc} \mu_1 & \mu_2 & \mu_3 \\ n_4 & n_5 & n_6 \end{array} \right). \] (F.18)

That is to say that the matrix element reduces to the calculation of the Clebsch-Gordan coefficients in SU(3) [268, 270, 271]. The relevant matrix elements are contained in Tab. F.1-F.3.

<table>
<thead>
<tr>
<th>( Y )</th>
<th>( D^{(8)}_{33} )</th>
<th>( D^{(8)}_{38} )</th>
<th>( D^{(8)}_{35} )</th>
<th>( D^{(8)}_{88} )</th>
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<td>0</td>
<td>(\frac{\sqrt{5}}{3} J_3 )</td>
<td>(\frac{1}{10} )</td>
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<tr>
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<td>(-\frac{\sqrt{2}}{3} J_3 )</td>
<td>(-\frac{1}{3} )</td>
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<td>(\frac{1}{3 \sqrt{15}} J_3 )</td>
<td>(\frac{1}{2 \sqrt{3}} )</td>
</tr>
</tbody>
</table>

Table F.1: \( \langle 8(Y, T_3, J_3) \mid D^{(8)} \mid 8(Y, T_3, J_3) \rangle \)

<table>
<thead>
<tr>
<th>( Y )</th>
<th>( D^{(8)}_{35} )</th>
<th>( D^{(8)}_{38} )</th>
<th>( D^{(8)}_{35} )</th>
<th>( D^{(8)}_{88} )</th>
</tr>
</thead>
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<td>(-\frac{\sqrt{2}}{12} J_3 )</td>
<td>(\frac{\sqrt{6}}{10} )</td>
</tr>
<tr>
<td>0_\Lambda</td>
<td>0</td>
<td>0</td>
<td>(-\frac{\sqrt{2}}{12} J_3 )</td>
<td>(\frac{3}{10} )</td>
</tr>
<tr>
<td>0_\Sigma</td>
<td>(-\frac{2 \sqrt{6}}{3 \sqrt{3}} T_3 J_3 )</td>
<td>(-\frac{\sqrt{3}}{2} T_3 )</td>
<td>(-\frac{\sqrt{2}}{2 \sqrt{3}} J_3 )</td>
<td>(\frac{1}{10} )</td>
</tr>
<tr>
<td>1</td>
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<td>(-\frac{\sqrt{2}}{2 \sqrt{5}} T_3 )</td>
<td>(-\frac{\sqrt{2}}{2 \sqrt{5}} J_3 )</td>
<td>(\frac{\sqrt{5}}{2 \sqrt{10}} )</td>
</tr>
</tbody>
</table>

Table F.2: \( \langle 8(Y, T_3, J_3) \mid 10(Y, T_3, J_3) \rangle \)

<table>
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<tr>
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<th>( D^{(8)}_{38} )</th>
<th>( D^{(8)}_{35} )</th>
<th>( D^{(8)}_{88} )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>(-\frac{2 \sqrt{6}}{3 \sqrt{3}} T_3 J_3 )</td>
<td>(\frac{\sqrt{3}}{2} T_3 )</td>
<td>(-\frac{\sqrt{2}}{12} J_3 )</td>
<td>(\frac{\sqrt{6}}{10} )</td>
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<tr>
<td>0_\Lambda</td>
<td>0</td>
<td>0</td>
<td>(-\frac{\sqrt{2}}{12} J_3 )</td>
<td>(\frac{3}{10} )</td>
</tr>
<tr>
<td>0_\Sigma</td>
<td>(-\frac{2 \sqrt{6}}{3 \sqrt{3}} T_3 J_3 )</td>
<td>(-\frac{\sqrt{3}}{2} T_3 )</td>
<td>(-\frac{\sqrt{2}}{2 \sqrt{3}} J_3 )</td>
<td>(\frac{1}{10} )</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{2 \sqrt{3}}{2 \sqrt{10}} T_3 J_3 )</td>
<td>(-\frac{\sqrt{2}}{2 \sqrt{5}} T_3 )</td>
<td>(-\frac{\sqrt{2}}{2 \sqrt{5}} J_3 )</td>
<td>(\frac{\sqrt{5}}{2 \sqrt{10}} )</td>
</tr>
</tbody>
</table>

Table F.3: \( \langle 8(Y, T_3, J_3) \mid 27(Y, T_3, J_3) \rangle \)
The product \( \sum_{i=1}^{3} D_{Q_i}^{(8)} J_i \) \((Q = 3, 8)\)

Changing to the spherical basis,

\[ D_{Q_0}^{(8)} J_i = \sum_{\mu=1}^{1} (-)^\mu D_{Q_0}^{(8)} J_\mu = -D_{Q_1}^{(8)} J_1 + D_{Q_0}^{(8)} J_0 - D_{Q_1}^{(8)} J_1 \]

\[ = -D_{Q_0(0,1,1)} J_1 + D_{Q_0(0,0,0)} J_0 - D_{Q_0(0,1,0)} J_1, \]

this case reduces to the case of a single Wigner matrix, since the action of the angular momentum ladder operators is known:

\[ J_{3} \Psi_{\mu}^{(µ)}(Y_{TTT})_{(Y'SS)}(A) = S_{3} \Psi_{\mu}^{(µ)}(Y_{TTT})_{(Y'SS)}(A), \]

\( J_{3} \Psi_{\mu}^{(µ)}(Y_{TTT})_{(Y'SS)}(A) = \frac{1}{2} \sqrt{S(S+1) - S_{3}(S_{3} \pm 1)} \Psi_{\mu}^{(µ)}(Y_{TTT})_{(Y'SS_3 \pm 1)} \). \]

The relevant matrix elements for form factors are listed in Tab. F.4-F.6.

The product \( \sum_{\alpha=3}^{7} D_{Q_\alpha}^{(8)} J_\alpha \) \((Q = 3, 8)\)

Although one could follow the preceding case, having to resort in this case to ladder operators for \(U\) and \(V\) spin as in the case of the next operator, there is a more economical way. In fact, for \(Q = 8\) one may take advantage of the relation between the right generators \(R\) and the left generator \(L_3\)

\[ \sum_{\alpha=3}^{7} D_{Q_\alpha}^{(8)} R_\alpha = L_3 \rightarrow \sum_{\alpha=3}^{7} D_{Q_\alpha}^{(8)} J_\alpha = -\frac{\sqrt{3}Y}{2}, \]

since \( R_\alpha = -J_\alpha, \) \( J_3 = -R_8 = -N_c/(2\sqrt{3}) \) and \( L_3 = \sqrt{3}Y/2. \) This means that

\[ \langle B'(S_3 T_3') | \sum_{a=3}^{7} D_{Q_\alpha}^{(8)} J_a | B(S_3 T_3) \rangle = -\frac{\sqrt{3}}{2} Y \delta_{YY'} \]

\[ - \langle B'(S_3 T_3') | D_{Q_8}^{(8)} J_1 | B(S_3 T_3) \rangle + \frac{N_c}{2\sqrt{3}} \langle B'(S_3 T_3') | D_{Q_8}^{(8)} | B(S_3 T_3) \rangle. \]

For \(Q = 3\) one takes advantage of a similar relation but now for the left generator \(L_3\)

\[ \sum_{\alpha=3}^{7} D_{Q_\alpha}^{(8)} R_\alpha = L_3 \rightarrow \sum_{\alpha=3}^{7} D_{Q_\alpha}^{(8)} J_\alpha = -T_3, \]

which corresponds to the third component of the isospin operator \(L_3 = T_3.\) It follows that

\[ \langle B'(S_3 T_3') | \sum_{a=3}^{7} D_{Q_\alpha}^{(8)} J_a | B(S_3 T_3) \rangle = -T_3 \delta_{T_3 T_3'} \]

\[ - \langle B'(S_3 T_3') | D_{Q_8}^{(8)} J_i | B(S_3 T_3) \rangle + \frac{N_c}{2\sqrt{3}} \langle B'(S_3 T_3') | D_{Q_8}^{(8)} | B(S_3 T_3) \rangle. \]

The relevant matrix elements for form factors are listed in Tab. F.4-F.6.

<table>
<thead>
<tr>
<th>( Y )</th>
<th>( D_{3}^{(8)} J_{i} )</th>
<th>( D_{3a}^{(8)} J_{i} )</th>
<th>( D_{8}^{(8)} J_{i} )</th>
<th>( D_{8a}^{(8)} J_{i} )</th>
<th>( d_{3ab}^{(8)} D_{3a}^{(8)} J_{b} )</th>
<th>( d_{3ab}^{(8)} D_{3b}^{(8)} J_{a} )</th>
<th>( d_{3ab}^{(8)} D_{3c}^{(8)} J_{c} )</th>
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<td>0</td>
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<td>-( \frac{2\sqrt{3}}{20} )</td>
<td>-( \frac{2\sqrt{3}}{20} )</td>
<td>0</td>
<td>-( \frac{2\sqrt{3}}{10} ) ( J_3 )</td>
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<td>-( \frac{\sqrt{3}}{5} )</td>
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<td>-( \frac{2\sqrt{3}}{10} ) ( T_3 )</td>
<td>-( \frac{2\sqrt{3}}{15} ) ( J_3 )</td>
</tr>
</tbody>
</table>

Table F.4: \( \langle 8(Y, T_3, J_3) | D^{(8)} J | 8(Y, T_3, J_3) \rangle \)
Inverting the ladder operators for $|Y,T,J\rangle$ indicates just the hypercharge-isospin part of the wave function upon which these operators act.

\[ U_{\alpha a} D^{(8)}_{\alpha a} J_b = d_{3ab} D^{(8)}_{\alpha a} J_b. \]

In the strict case of the octet baryons, the operator takes the form

\[ D^{(8)}_{\alpha a} J_b = \sqrt{\frac{2}{3}} \delta_{ab} \chi_{a} J_b. \]

Taking into account that the symmetric structure constants, Table A.1, fulfill

\[ d_{3ab} = \delta_{ab} (\delta_{a4} + \delta_{a5} - \delta_{a6} - \delta_{a7})/2, \]

the operator takes the form

\[ \sum_{a,b=4}^7 d_{3ab} D^{(8)}_{\alpha a} J_b = \frac{1}{2} \left( D^{(8)}_{\chi4} J_4 + D^{(8)}_{\chi5} J_5 - D^{(8)}_{\chi6} J_6 - D^{(8)}_{\chi7} J_7 \right) \]

\[ = \frac{1}{4} \left( D^{(8)}_{\chi\bar{4}} V_+ + D^{(8)}_{\chi\bar{4}} V_- - D^{(8)}_{\chi\bar{4}} U_+ - D^{(8)}_{\chi\bar{4}} U_- \right) \]

\[ = \frac{1}{2\sqrt{2}} \left( D^{(8)}_{\chi\bar{4}} V_+ - D^{(8)}_{\chi\bar{4}} V_- + D^{(8)}_{\chi\bar{4}} U_+ + D^{(8)}_{\chi\bar{4}} U_- \right). \]

Indicating just the hypercharge-isospin part of the wave function upon which these operators act.

The product $\sum_{a,b=4}^7 d_{3ab} D^{(8)}_{\alpha a} J_b$ ($Q = 3, 8$)

Inverting the ladder operators for $U$ and $V$ spin

\[ V_{\pm} = J_{\pm} \pm i J_{5}, U_{\pm} = J_{6} \pm i J_{7}, \]

and taking into account that the symmetric structure constants, Tab. A.1, fulfill

\[ d_{3ab} = \delta_{ab} (\delta_{a4} + \delta_{a5} - \delta_{a6} - \delta_{a7})/2, \]

the operator takes the form

\[ \sum_{a,b=4}^7 d_{3ab} D^{(8)}_{\alpha a} J_b = \frac{1}{2} \left( D^{(8)}_{\chi4} J_4 + D^{(8)}_{\chi5} J_5 - D^{(8)}_{\chi6} J_6 - D^{(8)}_{\chi7} J_7 \right) \]

\[ = \frac{1}{4} \left( D^{(8)}_{\chi\bar{4}} V_+ + D^{(8)}_{\chi\bar{4}} V_- - D^{(8)}_{\chi\bar{4}} U_+ - D^{(8)}_{\chi\bar{4}} U_- \right) \]

\[ = \frac{1}{2\sqrt{2}} \left( D^{(8)}_{\chi\bar{4}} V_+ - D^{(8)}_{\chi\bar{4}} V_- + D^{(8)}_{\chi\bar{4}} U_+ + D^{(8)}_{\chi\bar{4}} U_- \right). \]

Indicating just the hypercharge-isospin part of the wave function upon which these operators act.

\[ |N, (Y, T, J_3), (-1, J, J_3) \rangle \rightarrow |8, Y, T, J_3 \rangle \rightarrow |Y, T, J_3 \rangle \]

and in the strict case of the octet baryons, representation $N = 8$ one finds (e.g. [272]):

\[ V_+ |Y, T, J_3 \rangle = \sqrt{\frac{T + T_3 + 1}{2I + 1}} F^{(8)} (-Y, I + 1) \left| Y + 1, T + \frac{1}{2}, T_3 + \frac{1}{2} \right\rangle \]

\[ + \sqrt{\frac{T - T_3}{2I + 1}} F^{(8)} (Y, I) \left| Y + 1, T - \frac{1}{2}, T_3 + \frac{1}{2} \right\rangle \]

\[ V_- |Y, T, J_3 \rangle = \sqrt{\frac{T - T_3 + 1}{2I + 1}} F^{(8)} (Y, I + 1) \left| Y - 1, T + \frac{1}{2}, T_3 - \frac{1}{2} \right\rangle \]

\[ + \sqrt{\frac{T + T_3}{2I + 1}} F^{(8)} (-Y, I) \left| Y - 1, T - \frac{1}{2}, T_3 - \frac{1}{2} \right\rangle \]

\[ U_+ |Y, T, J_3 \rangle = \sqrt{\frac{T - T_3 + 1}{2I + 1}} F^{(8)} (-Y, I + 1) \left| Y + 1, T + \frac{1}{2}, T_3 - \frac{1}{2} \right\rangle \]

\[ - \sqrt{\frac{T + T_3}{2I + 1}} F^{(8)} (Y, I) \left| Y + 1, T - \frac{1}{2}, T_3 - \frac{1}{2} \right\rangle \]
\[
U_- |Y, T, T_3\rangle = -\sqrt{\frac{T + T_3 + 1}{2I + 1}} F^{(8)}(Y, I + 1) |Y - 1, T + \frac{1}{2}, T_3 + \frac{1}{2}\rangle \\
+ \sqrt{\frac{T - T_3}{2I + 1}} F^{(8)}(-Y, I) |Y - 1, T - \frac{1}{2}, T_3 + \frac{1}{2}\rangle 
\]

with the definition
\[
F^{(8)}(Y, I) = \frac{1}{2I} \sqrt{(I - \frac{Y}{2})(2 - I + \frac{Y}{2})(2 + I - \frac{Y}{2})}.
\]

**The products** \(D^{(8)}_{\nu_1\nu_1} D^{(8)}_{8\nu_2}\)

This case, and the similar ones in Tab. F.7, can be reduced to the case of one Wigner matrix using the relation [268]

\[
D^{(8)}_{\nu_1\nu_1} D^{(8)}_{8\nu_2} = \frac{1}{\text{dim}(N)} \sum_{N, \mu} \sum_{\nu'_{\mu'}} \left( \begin{array}{ccc} 8 & 8 & N_{\mu} \\ \nu_1 & \nu_2 & \nu \end{array} \right) \left( \begin{array}{ccc} 8 & 8 & N_{\mu} \\ 0 & 0 & 0 \end{array} \right) D^{(N)}_{\nu\nu'}
\]

where the sum over \(N\) means the representations appearing in the decomposition of \(8 \otimes 8\), and \(\mu\) distinguishes between (degenerate) representations with the same dimension.

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<td>(\frac{10}{16})</td>
<td>(\frac{10}{16})</td>
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<td>0</td>
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<tr>
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</tr>
<tr>
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<td>(\frac{1}{5} J_3)</td>
<td>(-\frac{1}{5} J_3)</td>
<td>(\frac{1}{5} J_3)</td>
<td>(-\frac{2}{5} J_3)</td>
</tr>
<tr>
<td>(d_{3ab} D^{(8)}<em>{80} D^{(8)}</em>{8b})</td>
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<td>(-\frac{\sqrt{3}}{15} J_3 T_3)</td>
<td>(\frac{4\sqrt{3}}{15} J_3 T_3)</td>
</tr>
</tbody>
</table>

Table F.7: \(\langle 8(Y, T_3, J_3) | D^{(8)} | 8(Y, T_3, J_3) \rangle\).
G Assorted details

G.1 Fixing of the model parameters

G.1.1 Pion Mass

For the purpose of calculating the pion mass \(m_\pi\) and the pion decay constant \(f_\pi\) it is convenient to write the action explicitly in terms of the pion and sigma fields with the Dirac operators (2.8b) now including the diagonal current quark mass matrix in the case of isospin symmetry \(m = m_11, m_1 = (m_u + m_d)/2\). The action is then, in Euclidean space,

\[
S[\sigma, \vec{\pi}] = -N_c \left[ \text{Tr} \ln D(\sigma, \vec{\pi}) - \text{Tr} \ln D_0 \right] + \frac{\lambda}{8} \text{Tr} \left[ \sigma^2 + \vec{\pi}^2 - M^2 \right]
\]

where \(\lambda\) is a Lagrange multiplier enforcing the chiral circle restriction (the factor 8 is introduced to simplify the same value coming from the part of the trace over Dirac and isospin matrices.)

For the purpose of considering fluctuations around the field configuration which make the action stationary, it’s better to redefine the \(\sigma\) field in such a way that it absorbs the current quark mass:

\[
\sigma' = \sigma + m_1, \quad (G.2a)
\]

\[
M' = M + m_1. \quad (G.2b)
\]

Since in SU(2) the effective action has just a divergent real part, the proper-time regularized version of the action functional (G.1) becomes

\[
S[\sigma', \vec{\pi}] = \frac{1}{2} N_c \text{Tr} \int_0^\infty \frac{du}{u} \phi(u, \Lambda) \left( e^{-uD^1}(\sigma', \vec{\pi})D(\sigma', \vec{\pi}) - e^{-uD_0^1}D_0 \right)
\]

\[
+ \frac{\lambda}{8} \text{Tr} \left[ \sigma'^2 - 2m_1 (\sigma' - M') + \vec{\pi}^2 - M^2 \right]. \quad (G.3)
\]

In order to determine the masses of the fields it is necessary to expand the action functional up to terms quadratic in the fluctuations \(\tilde{\sigma}\) and \(\tilde{\pi}^a\) of these fields around their vacuum expectation values, which are chosen as \(\langle 0|\sigma'|0 \rangle = \sigma'_0 = M', \langle 0|\pi^a|0 \rangle = \pi'^a_0 = 0\) due to parity:

\[
\sigma' \rightarrow \sigma'_0 + \tilde{\sigma}, \quad (G.4a)
\]

\[
\pi^a \rightarrow \pi'^a_0 + \tilde{\pi}^a. \quad (G.4b)
\]

Using the Dyson expansion

\[
e^{A+B} = e^A + \int_0^1 d\alpha e^{\alpha A} B e^{(1-\alpha)A} + \int_0^1 d\beta \int_0^{1-\beta} d\epsilon e^{\alpha A} B e^{\epsilon A} B e^{(1-\alpha-\beta)A} + \ldots
\]

for operators \(A\) and \(B\) the action up to terms quadratic in the fluctuations \(\tilde{\sigma}\) and \(\tilde{\pi}^a\) can be written as

\[
S[\tilde{\sigma}, \tilde{\pi}^a] = S^{(1)}[\tilde{\sigma}] + S^{(2)}[\tilde{\sigma}, \tilde{\pi}^a] + \ldots
\]

The linear part in \(\tilde{\sigma}\) is

\[
S^{(1)}[\tilde{\sigma}] = -N_c M' \text{Tr} \int_0^\infty du \phi(u, \Lambda) e^{-uD_0^1}D_0 \tilde{\sigma} + 2\lambda \left( M' - m_1 \right) \int d^4x \tilde{\sigma}
\]

\[
= \left[ -8N_c M'I_1 (\Lambda, M') + 2\lambda (M' - m_1) \right] \int d^4x \tilde{\sigma}
\]

with the proper-time regularization integral

\[
I_1 (\Lambda, M') = \frac{1}{(4\pi)^2} \int_0^\infty du \phi(u, \Lambda) e^{-uM'^2}.
\]
The second order variation of the action can be computed along the same lines. For the quadratic part one finds

\[
S^{(2)}[\tilde{\sigma}, \tilde{\pi}^a] = \frac{1}{2} N_c \text{Tr} \int_0^1 d\beta \int_0^1 d\alpha e^{-u \alpha D_0^\dagger D_0} (\tilde{\sigma}^2 + \tilde{\pi}^a \tilde{\pi}^a) e^{-(1-\alpha)u D_0^\dagger D_0} + u^2 \int_0^1 d\beta \int_0^1 d\alpha e^{-u \alpha D_0^\dagger D_0} (i \tilde{\alpha} \tilde{\sigma}) e^{-(1-\alpha-\beta)u D_0^\dagger D_0} + u^2 \int_0^1 d\beta \int_0^1 d\alpha e^{-u \alpha D_0^\dagger D_0} (i \tilde{\alpha} \tilde{\pi}^a) e^{-(1-\alpha-\beta)u D_0^\dagger D_0} + u^2 M'^2 \int_0^1 d\beta \int_0^1 d\alpha e^{-u \alpha D_0^\dagger D_0} \tilde{\sigma} e^{-(1-\alpha-\beta)u D_0^\dagger D_0} \]

\[
\frac{\lambda}{8} \text{Tr} (\tilde{\sigma}^2 + \tilde{\pi}^a \tilde{\pi}^a)
\]

Taking as an example the calculation of the term in the derivatives of \( \sigma \)

\[
\frac{1}{8} \text{Tr} \int_0^1 d\beta \int_0^1 d\alpha e^{-u \alpha D_0^\dagger D_0} (i \tilde{\alpha} \tilde{\sigma}) e^{-(1-\alpha-\beta)u D_0^\dagger D_0} \]

\[
= \frac{1}{u^2} \int \frac{d^4 q}{(2\pi)^4} q^2 \tilde{\sigma} (-q) \tilde{\sigma} (q) \frac{1}{2} \frac{1}{(4\pi)^2} \int_0^1 d\beta e^{-u [M'^2 + \beta (1-\beta) q^2]}
\]

where \( q = k' - k \). The remaining terms are simplified along the same lines with the result (G.11).

For the second order on the variations

\[
S^{(2)}[\tilde{\sigma}, \tilde{\pi}^a] = 4 N_c \int \frac{d^4 q}{(2\pi)^4} \tilde{\sigma} (-q) \tilde{\sigma} (q) \left[ (q^2 + 4M'^2) I_2 (q^2, \Lambda, M') - I_1 (\Lambda, M') + \frac{\lambda}{4N_c} \right] + 4 N_c \int \frac{d^4 q}{(2\pi)^4} \tilde{\pi}^a (-q) \tilde{\pi}^a (q) \left[ (q^2 I_2 (q^2, \Lambda, M') - I_1 (\Lambda, M') + \frac{\lambda}{4N_c} \right]
\]

where the integral proper-time regularization integral \( I_2 \) is defined by

\[
I_2 (q^2, \Lambda, M') = \frac{1}{2} \frac{1}{(4\pi)^2} \int_0^\infty \frac{d\beta e^{-u [M'^2 + \beta (1-\beta) q^2]}}{u \phi (u, \Lambda)} \int_0^1 d\beta e^{-u [M'^2 + \beta (1-\beta) q^2]}
\]

The Lagrange multiplier \( \lambda \) is obtained from the condition of vanishing first order variation of the action:

\[
\frac{\lambda}{4N_c} = \frac{M'}{M' - m_1} I_1 (\Lambda, M')
\]

The inverse pion and sigma propagators, \( K_{\pi^a} \) and \( K_{\sigma} \), are obtained from the second order variations and are:

\[
K_{\sigma}^{-1} (q^2) = \frac{1}{M' - m_1} I_2 (q^2, \Lambda, M')
\]

\[
K_{\pi}^{-1} (q^2) = \frac{1}{M' - m_1} I_2 (q^2, \Lambda, M')
\]

The on-shell meson masses are given by the zeros of the corresponding inverse meson propagators, \( K_{\pi}^{-1} (q^2 = -m_\pi^2) = 0 \), leading for the pion to

\[
m_\pi^2 = \frac{m_1}{M' - m_1} I_2 (-m_\pi^2, \Lambda, M')
\]

Comparing with the expression for the sigma mass it follows that the pion mass vanishes in the chiral limit \( (m_1 \rightarrow 0) \), while the sigma mass remains finite in this limit \( (m_\sigma \rightarrow 2M) \).
G.1.2 Pion decay constant

It is important that the model should not disagree with experiment regarding the value of the pion decay constant. This makes the pion decay constant a good quantity to be used in the fixing of the model parameters. To calculate it one has simply to realize that under isovector axial rotations through $\alpha$ and up to first order in $\alpha$

$$MU^\gamma \rightarrow e^{-i\vec{a} \cdot \vec{r}} MU^\gamma e^{i\vec{a} \cdot \vec{r}} \sim \sigma - i\gamma^5 \vec{a} \cdot \vec{r} \sigma + i\gamma^5 \vec{r} \cdot \vec{r}$$  \hspace{1cm} (G.16)

and that the axial current by the Noether theorem can be obtained from the variation of the action with respect to the derivatives of $\alpha$. It is easy to see that the term linear in the derivatives of $\alpha$ is similar to one of the terms in $S^{(2)}$ above, so that, with a similar calculation, the required part of the action is

$$S^{(1)} [\partial_\mu (u, \Lambda)] = N_c M \text{Tr} \int_0^\infty du \phi (u, \Lambda) u^2$$

$$\times \int_0^1 d\beta \int_0^{1-\beta} d\alpha e^{-\alpha u D_0^a D_0} (\partial \pi^a) e^{-\beta u D_0^a D_0} (\partial \alpha^a) e^{-(1-\alpha-\beta) u D_0^a D_0}$$

$$= -8N_c M \int \frac{d^4q}{(2\pi)^4} \int d^4x (\partial_\mu \alpha_a (x)) (\partial_\mu \alpha_a (y)) e^{-iqy} I_0 (q^2, \Lambda, M')$$  \hspace{1cm} (G.17)

from which the axial current may be extracted

$$A_{\mu a} (y) = -8N_c M e^{-iqy} \int \frac{d^4q}{(2\pi)^4} \int d^4x (\partial_\mu \alpha_a (x)) e^{iqy} I_0 (q^2, \Lambda, M') .$$  \hspace{1cm} (G.18)

This axial current is then used, assuming a canonical quantization for the pion field and using the inverse pion propagator above, to determine the matrix element of this current between the physical vacuum and a physical one-pion state, with the result that

$$\langle 0 | A_\mu^a | \pi^b (\vec{q}) \rangle = \delta_{ab} i M \sqrt{8N_c I_0 (q^2, \Lambda, M')} q_\mu e^{-iqy}$$  \hspace{1cm} (G.19)

which, when compared with the definition of the pion decay constant $f_\pi$,

$$\langle 0 | A_\mu^a | \pi^b (\vec{q}) \rangle = \delta_{ab} i f_\pi q_\mu e^{-iqy} ,$$  \hspace{1cm} (G.20)

yields the decay constant in terms of the model parameters:

$$f_\pi^2 = 8N_c M^2 I_0 \left( q^2 = -m_\pi^2, \Lambda, M' \right) = M^2 Z_\pi \left( q^2 = -m_\pi^2, \Lambda, M' \right) .$$  \hspace{1cm} (G.21)

G.2 Free Plane Wave Basis

The free one-particle Hamiltonian is given by

$$h_0 = c\gamma^0 \vec{p} + \gamma^0 mc^2$$  \hspace{1cm} (G.22)

where

$$m = \overline{m} + M$$  \hspace{1cm} (G.23)

with $\overline{m}$ the diagonal chiral symmetry breaking current quark mass matrix and $M$ the diagonal constituent quark mass of the quarks, supposed to be the same for all the quarks, which is to be identified later with the hedgehog mass in the case of the soliton.

The one-particle Dirac Hamiltonian for the interacting case is

$$h = c\gamma^0 \vec{p} + \left( \overline{m} + M \right) e^{2\gamma^0 1_e P(\gamma) \gamma^5 \vec{r} \cdot \vec{r}}$$

$$= c\gamma^0 \vec{p} + \left( \overline{m} + M \right) e^{2\gamma^0 \left( \cos P(r) + i\gamma^5 \vec{r} \cdot \vec{r} \sin P(r) \right)} .$$  \hspace{1cm} (G.24)

The first question regarding the diagonalization of this Hamiltonian is connected with a suitable basis, which is itself connected with the set of commuting observables which include $h$.

Solving the Dirac equation for the free particle in spherical coordinates yields the plane wave basis function. Combining these functions with the grand-spin allows to write the set of grand-spin eigenfunctions of (G.24), as in Tab. G.1 where the normalization constant $N_k$ and the factor $F_k$
are given by
\[ N_k = \frac{1}{\sqrt{\frac{2|E_k|}{m+|E_k|} \sqrt{\frac{|E_k|}{2} j_{G+1}(KD)}}}, \quad F_k = \frac{k}{m + |E_k|}. \]  

\[ \begin{array}{c|c|c|c}
 G \geq 0 & \langle N1 \rangle = N_k \left( \begin{array}{c}
 i j_G(kr) |0\rangle \\
 F_k j_{G+1}(kr) |2\rangle
\end{array} \right) & \langle N2 \rangle = N_k \left( \begin{array}{c}
 i F_k j_G(kr) |0\rangle \\
 -j_{G+1}(kr) |2\rangle
\end{array} \right) \\
 & \langle U1 \rangle = N_k \left( \begin{array}{c}
 i j_{G+1}(kr) |2\rangle \\
 -F_k j_G(kr) |0\rangle
\end{array} \right) & \langle U2 \rangle = N_k \left( \begin{array}{c}
 i F_k j_{G+1}(kr) |2\rangle \\
 j_G(kr) |0\rangle
\end{array} \right) \\
 G \geq 1 & \langle N3 \rangle = N_k \left( \begin{array}{c}
 i j_G(kr) |1\rangle \\
 -F_k j_{G-1}(kr) |3\rangle
\end{array} \right) & \langle N4 \rangle = N_k \left( \begin{array}{c}
 i F_k j_{G}(kr) |1\rangle \\
 j_{G-1}(kr) |3\rangle
\end{array} \right) \\
 & \langle U3 \rangle = N_k \left( \begin{array}{c}
 i j_{G-1}(kr) |3\rangle \\
 F_k j_G(kr) |1\rangle
\end{array} \right) & \langle U4 \rangle = N_k \left( \begin{array}{c}
 i F_k j_{G-1}(kr) |3\rangle \\
 -j_{G}(kr) |1\rangle
\end{array} \right) \\
\end{array} \]

Table G.1: Free basis. States with \( N \) stands for natural parity and \( U \) for unnatural parity. The odd numbered ones have positive energy and the even ones have negative energy. The bi-spinores are given in eq. (G.26)

The bi-spinores are:
\[ |0\rangle = |(\ell = G) J = G + 1/2, GM_G \rangle, \]  
\[ |1\rangle = |(\ell = G) J = G - 1/2, GM_G \rangle, \]  
\[ |2\rangle = |(\ell = G + 1) J = G + 1/2, GM_G \rangle, \]  
\[ |3\rangle = |(\ell = G - 1) J = G - 1/2, GM_G \rangle. \]  

G.2.1 Relations for the \( G_5 \) transformation

The main transformation rules for \( G_5 \) are:
\[ G_5 \gamma^a \gamma_5 G_5^{-1} = - (\gamma^a)^T, \]  
\[ G_5 \gamma^\mu \gamma_5 G_5^{-1} = \gamma^\mu_T, \quad \mu = 0, 1, 2, 3, 5, \]  
\[ G_5 h G_5^{-1} = h^*, \quad G_5 \phi_n(x) = \phi_n^*(x). \]

The \( G_5 \) operator is defined as
\[ G_5 = C \gamma_5 \tau^2 = \gamma_5 C_T \tau^2 \]
\[ G_5^{-1} = C^{-1} \gamma_5 \tau^2 = \gamma_5 C^{-1} \tau^2 \]

It can also be written as
\[ G_5 = i \gamma^5 \gamma^2 \gamma^0 \gamma^2 = i \gamma^5 \gamma^2 \gamma^0 \gamma^2 = i \Sigma^2 \gamma^2 \]

The operator \( C \) is the charge conjugation operator defined by
\[ C = i \gamma^{0} \gamma^{T}, \quad C^{-1} = - C = C^T = C^\dagger = i \gamma^{0} \gamma^{2}. \]

It fulfills
\[ C \gamma^\mu C^{-1} = - \gamma^\mu T, \quad \mu = 0, 1, 2, 3, 4 \]
\[ C \gamma^5 C^{-1} = \gamma^5 = \gamma^5 T \]

Using the above properties of the \( G_5 \) operator it is possible to study the change in matrix elements under this transformation, e.g.
\[ \langle n | \tau^a | m \rangle = \int d^3 x \langle n | x \rangle^* (-\tau^a)^T \langle x | m \rangle^* = - \langle m | \tau^a | n \rangle. \]

Particular consequences of these result are
\[ \langle n | \tau^a | n \rangle = 0, \quad \langle v | \tau^a | v \rangle = 0. \]

Similarly, now with \( \gamma \) matrices,
\[ \langle n | \gamma^0 \gamma^a | m \rangle = \delta_{\mu 0} \langle m | \gamma^\mu \gamma^a | n \rangle - (1 - \delta_{\mu 0}) \langle m | \gamma^0 \gamma^a | n \rangle, \]  
\[ \langle n | \gamma^0 \gamma^a | m \rangle = - \delta_{\mu 0} \langle m | \gamma^0 \gamma^a | n \rangle + (1 - \delta_{\mu 0}) \langle m | \gamma^0 \gamma^a | n \rangle. \]
H Figures

This appendix collects figures showing the behaviour of the results against model parameters, as commented in the text.

![Figure H.1](image1.png)

**Figure H.1:** The dependence of the strange electric and magnetic form factors $G_E^s$, $G_M^s$ on the strange quark mass $m_s$. Conventions and model parameter as in Fig. 3.8.

![Figure H.2](image2.png)

**Figure H.2:** The dependence of the strange electric and magnetic form factors $G_E^s$, $G_M^s$ on the constituent quark mass $M$. Conventions and model parameter as in Fig. 3.8.
Figure H.3: Dependence of the form factors $G^0, G^3, G^8$ (the magnetic ones in physical n.m.) with the strange quark mass $m_s$. The constituent quark mass is $M = 420$ MeV.
Figure H.4: Dependence of the form factors $G^0, G^2, G^S$ (the magnetic ones in physical n.m.) with the constituent quark mass $M$. The strange quark mass is $M = 180$ MeV.
Figure H.5: The electric and magnetic (in n.m.) form factors $G^{(0)}/3$, $G^{(3)}$ and $G^{(8)}/\sqrt{3}$ as reconstructed from the flavor form factors. Conventions and model parameters as in Fig. 3.1.
Figure H.6: Effects on the u quark form factors from changing a d into a s quark. Model parameters and conventions as in Fig. (3.3).
Figure H.7: Effects on the u quark form factors from changing a u into a d quark. Model parameters and conventions as in Fig. (3.3).

Figure H.8: Effects on the s quark form factors from changing a d into a s quark. Model parameters and conventions as in Fig. (3.3).
Figure H.9: Effects on the $s$ quark form factors from changing a $u$ into a $s$ quark. Model parameters and conventions as in Fig. (3.3).

Figure H.10: Flavor form factors for $\Lambda$ a $\Sigma^0$. Model parameters and conventions as in Fig. (3.3).
Figure H.11: Dependence of the electric form factors of the nucleon on the constituent quark mass $M$. For each mass the curve for SU(2) is above the corresponding one for SU(3). The strange quark mass is $m_s = 180$ MeV. The proton experimental data (o) comes from $p(e, e')$ [116, 173, 174], and (△) from $d(e, e'p)$ [175], and the neutron the experimental (o) data comes from $d(\bar{e}, e'n)p$ [107, 176, 177], △ from $\bar{d}(e', e'n)p$ [178], and o from $^3\bar{H}_e(\bar{e}, e'n)$ [106, 179].

Figure H.12: Dependence of the nucleon electromagnetic form factors on the constituent quark mass $M$. Proton experimental data (o) is from $p(e, e')$ [174, 180], and (△) from $d(e, e'p)$ [175], and for the neutron the experimental data (o) is from $d(e, e'p)$ [109, 119, 181]. Conventions and parameters as in Fig. H.11.
Figure H.13: Dependence of the electric form factors of the nucleon on the strange quark mass $m_s$. The constituent quark mass is here $M = 420$ MeV. Experimental data as in Fig. H.11.

Figure H.14: Dependence of the magnetic form factors of the nucleon on the strange quark mass $m_s$. The constituent quark mass is here $M = 420$ MeV. Experimental data as in Fig. H.12.
Figure H.15: Charge and magnetization densities of the nucleon. Model parameters as in Fig. 3.11.
Bibliography

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