Convexity properties of moment maps of real forms acting on Kählerian manifolds

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1 Introduction

A moment map for an action of a compact Lie group $U$ on a symplectic manifold $X$ is by definition a $U$-equivariant map $\mu: X \to u^*$ such that

$$d\mu_\xi = \iota_\xi \omega$$

holds for all $\xi \in u$. Here $\mu_\xi: X \to \mathbb{R}$ is given by $\mu_\xi(x) = \mu(x)(\xi)$ and $\iota_\xi \omega$ denotes the contraction of the symplectic form $\omega$ along the vector field $\xi_X$ induced by $\xi \in u$.

Moment maps have remarkable convexity properties which have been studied intensively.

In 1954, Horn ([Hor54]) proved that the image of the set of Hermitian $(n \times n)$-matrices with given eigenvalues $\lambda_1, ..., \lambda_n$ under the projection onto their diagonal entries is given by the convex hull of the points $S_n \cdot (\lambda_1, ..., \lambda_n)$, where $S_n$ denotes the symmetric group acting on $\mathbb{C}^n$ by permutations. A far reaching generalization of Horn’s result appeared in [Kos73]. Kostant considered a semisimple group $G$ with Iwasawa decomposition $g = \mathfrak{t} \oplus a \oplus \mathfrak{n}$ of $g = \operatorname{Lie}(G)$ and proved that the image of the projection of the orbit $K \cdot \xi \subset \mathfrak{p} \subset g$ through $\xi \in a$ onto $a$ is given by the convex hull of the Weyl group orbit $N_K(a) \cdot \xi \subset a$.

The case where $G$ is the complexification of a compact group turned out to be a special case of the following convexity theorem for moment maps with respect to Hamiltonian torus actions which was proven independently by Atiyah ([Ati82]) and Guillemin-Sternberg ([GS82]).

**Theorem.** Let $X$ be a compact connected symplectic manifold with a Hamiltonian action of a compact torus $T$ and a moment map $\mu_T: X \to \mathfrak{t}^*$. Then the set $\mu_T(X)$ is a convex polytope in $\mathfrak{t}^*$. Moreover, the vertices of this polytope are given by images of fixed points of $T$ in $X$.

Guillemin and Sternberg proved actually more. For an arbitrary compact Lie group $K$ acting in a Hamiltonian fashion on a compact symplectic manifold $X$ they showed that the intersection of the image of the moment map $\mu: X \to \mathfrak{t}^*$ with the positive Weyl chamber $\mathfrak{t}^+_+ \subset \mathfrak{t}^*$ of some maximal torus $t$ in $\mathfrak{t} = \operatorname{Lie}(K)$ is a finite union of convex polytopes. Moreover, they proved that in the integral Kähler case this intersection is one single convex polytope (see also [GS84]). This was also proven by Mumford in the appendix of [Nes84] using the notion of semistability with respect to line bundles.

The first proof of the convexity theorem for arbitrary compact connected symplectic manifolds was given by Kirwan. In [Kir84b] she proved the following.

**Theorem.** Let $X$ be a compact connected symplectic manifold with a Hamiltonian action of a compact connected Lie group $K$. Then the image of the corresponding moment map intersects the positive Weyl chamber $\mathfrak{t}^+_+ \subset \mathfrak{t}^*$ in a convex polytope.

The proof uses the result of Guillemin-Sternberg about the decomposition of the set $\mu(X) \cap \mathfrak{t}^+_+$ into finitely many convex polytopes and results of Morse type which Kirwan
developed in [Kir84a]. In particular, she used the fact that the fiber over the closest point to the origin in \( \mu(X) \cap t^*_+ \) is connected.

For further results including convexity properties of moment maps on non compact manifolds and orbifolds we refer the reader to [AP83], [BS00], [Bri87], [CDM88], [Del88], [Dui83], [FR96], [GS05], [GS06], [Hil94], [HH96], [HN98] [HNP94], [Kno02], [KO06], [LMTW98], [LR91], [Nei99], [OS00], [Pra94], [Sja98], [Wei01] and references therein.

Our goal here is to generalize Kostant’s convexity theorem to the following situation. Let \( G \) be a real closed subgroup of a complex reductive group \( U^C \) which is compatible with the Cartan decomposition \( U^C = U \cdot \exp(iu) \) of \( U^C \), i.e.

\[
G = K \cdot \exp(p),
\]

where \( K = G \cap U \) and \( p \) is a \( K \)-stable linear subspace of \( iu \). Further, we assume that there is a holomorphic \( U^C \)-action on a compact Kähler manifold \( X \) such that the induced \( U \)-action is Hamiltonian. We then get a moment map

\[
\mu: X \to u^*
\]

with respect to the \( U \)-action on \( X \). Moreover, the two linear subspaces \( \mathfrak{k} = \text{Lie}(K) \) and \( i\mathfrak{p} \) of \( u \) define the \( K \)-equivariant maps

\[
\mu_{\mathfrak{k}}: X \to \mathfrak{k}^* \quad \text{and} \quad \mu_{i\mathfrak{p}}: X \to i\mathfrak{p}^*
\]

given by restriction. By the results of Heinzner-Schwarz ([HS05a]) and Heinzner-Stötzel ([HS05b]) the \( i\mathfrak{p} \)-component \( \mu_{i\mathfrak{p}} \) of the moment map \( \mu \) is closely related to the structure of the \( G \)-action on \( X \).

In this monograph, we study convexity properties of this map for closed \( G \)-stable subsets \( Y \) of \( X \). Taking the positive Weyl chamber \( i\mathfrak{a}^*_+ \) of a maximal subalgebra \( i\mathfrak{a}^* \) of \( i\mathfrak{p} \) as a convex slice for the \( K \)-action on \( \mu_{i\mathfrak{p}}(Y) \), we consider the question whether the set \( \mu_{i\mathfrak{p}}(Y) \cap i\mathfrak{a}^*_+ \) is a convex polytope or not.

There is a positive answer to this question by O’Shea-Sjamaar ([OS00]) in the case where \( Y \) is a Lagrangian submanifold of \( X \) given as the fixed point set of an antisymplectic involution \( \tau \) on \( X \) fulfilling the condition

\[
\mu(\tau(x)) = -\sigma(\mu(x))
\]

for all \( x \in X \). They prove that in this situation the set \( \mu_{i\mathfrak{p}}(Y) \cap i\mathfrak{a}^*_+ \) coincides with the set \( \mu(X) \cap i\mathfrak{a}^*_+ \), which is a convex polytope by Kirwan’s convexity theorem.

Here we will give another sufficient condition on \( Y \) such that \( \mu_{i\mathfrak{p}}(Y) \cap i\mathfrak{a}^*_+ \) is a convex polytope.

After introducing the main notations and properties of reductive groups and moment maps in chapter 2 we show in chapter 3 that the set \( \mu_{i\mathfrak{p}}(Y) \cap i\mathfrak{a}^*_+ \) is a finite union of
convex polytopes for every closed $G$-stable subset $Y$ of $X$. This is done by applying the above convexity theorem of Atiyah and Guillemin-Sternberg to the moment map with respect to the action of the compact torus $T = \exp(i\alpha)$ which gives conditions for the inclusion of $\mu_{ip}(Y) \cap i\alpha^*_+ \subset \mu_{ia}(X)$.

In chapter 4, we follow the ideas of [Kir84a] and consider the gradient flow of the function $\eta_{ip} = \|\mu_{ip}\|^2$ on $X$. This gives a decomposition of $X$ into disjoint subsets $S_\beta$ labeled by the finite set

$$B_{ip} = \mu_{ip}(\{x \in X \mid d\eta_{ip}(x) = 0\}) \cap i\alpha^*_+.$$

In this step one main difference to Kirwan’s case appears. For a moment map with respect to a compact group the subsets which form the decomposition with respect to the gradient flow of $\eta = \|\mu\|^2$ are symplectic. This immediately implies the existence of an open and dense connected set in this decomposition. But this is false in our situation. Even in the easiest examples the sets $S_\beta$ are not symplectic and there exist open sets $S_\beta$ which are not connected. Even so there may exists a set $S_\beta$ which is open and dense in $X$. We will see in chapter 6 that the question of existence of such a $S_\beta$ is highly related to the convexity question we are interested in. In order to get a more explicit understanding of the sets $S_\beta$, we give an alternative description independent of the gradient flow of $\eta_{ip}$. Moreover, we show that some of these sets are given in terms of semistable points with respect to $\mu_{ip}$. Here a point $x \in X$ is called semistable with respect to $\mu_{ip}$ if the closure of the orbit $G \cdot x$ intersects the zero fiber of $\mu_{ip}$.

To transfer properties of the zero fiber to arbitrary fibers of the moment map, there exists a method called shifting of the moment map. In chapter 5 we introduce the right analogue for the map $\mu_{ip}$ by considering the special case of moment maps on complex flag manifolds.

Using the previous results we obtain the following convexity theorem.

**Theorem.** Let $Y$ be a closed $G$-stable subset of $X$ such that the set

$$S_G(M_{ip,\beta}(\alpha)) := \{(y, \xi) \in Y \times K \cdot \beta \mid G \cdot (y, \xi) \cap (\mu_{ip,\beta})^{-1}(\alpha) \neq \emptyset\}$$

is either open and dense in $Y \times K \cdot \beta$ or empty for every $\alpha, \beta \in i\alpha^*_+$. Then the set $\mu_{ip}(Y) \cap i\alpha^*_+$ is a convex polytope.

In chapter 7, we show that this theorem can be applied to the following class of examples. Let $V$ be a complex finite dimensional $U^C$-representation space and $\mathbb{P}(V)$ the corresponding projective space with the induced action of $U^C$ and moment map $\mu_{\mathbb{P}(V)} : \mathbb{P}(V) \rightarrow u^*$. In this case the set of semistable points with respect to the $ip$-component of $\mu_{\mathbb{P}(V)}$ is given by the complement of the projection of the null cone

$$\mathcal{N}_G := \{v \in V \mid 0 \in G \cdot v\}$$

onto $\mathbb{P}(V)$. Using real algebraicity of this cone and the above theorem, we get the following.
Theorem. Let $Y$ be a closed connected $G$-stable semialgebraic subset of $\mathbb{P}(V)$ such that the real Zariski closure of $Y$ in $\mathbb{P}(V)$ is an irreducible real algebraic subset of $\mathbb{P}(V)$. Then the set $\mu_{ip}(Y)$ intersects the positive Weyl chamber $i\mathfrak{a}^*_+$ in a convex polytope.

As a special case we both obtain the convexity theorem of Kostant and the convexity theorem of Guillemin-Sternberg and Mumford in the integral Kähler case. Note that in the second case, where $G = U^\mathbb{C}$, the theorem shows even more. Another consequence of the above theorem is the following.

Corollary. If $G$ is an algebraic group which acts algebraically on $\mathbb{P}(V)$, then the set

$$\mu_{ip}(G \cdot x) \cap i\mathfrak{a}^*_+$$

is a convex polytope for every $x \in \mathbb{P}(V)$.

Moreover, for a real $G$-representation $G \to GL(V_\mathbb{R})$ one obtains the following corollary by considering the complexified representation.

Corollary. Let $\mathbb{P}(V_\mathbb{R})$ denote the real projective space corresponding to $V_\mathbb{R}$. Then

$$\mu_{ip}(\mathbb{P}(V_\mathbb{R})) \cap i\mathfrak{a}^*_+$$

is a convex polytope.

Since a complex flag manifold can in general not be embedded symplectically in a projective spaces $\mathbb{P}(V)$ equipped with the Fubini-Study metric coming from a $U$-invariant hermitian form on $V$, we have to use a limit argument to prove the following.

Theorem. Let $U^\mathbb{C}/Q$ be a complex flag manifold with moment map $\mu$. Then the sets

$$\mu_{ip}(U^\mathbb{C}/Q) \cap i\mathfrak{a}^*_+ \quad \text{and} \quad \mu_{ip}(G \cdot x) \cap i\mathfrak{a}^*_+$$

are convex polytopes.

We compute these convex polytopes for some complex flag manifolds in chapter 8.

If $G$ is given by an antiholomorphic involution which commutes with the Cartan involution on $U^\mathbb{C}$ and $\beta$ is the closest point to the origin in $\mu_{ip}(U^\mathbb{C}/Q) \cap i\mathfrak{a}^*_+$, then the set

$$S_G(M_{ip}(\beta)) = \{ x \in U^\mathbb{C}/Q \mid G \cdot x \cap (\mu_{ip})^{-1}(\beta) \neq \emptyset \}$$

coinsides with the union of all open $G$-orbits in $U^\mathbb{C}/Q$. 

4
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2 Preliminaries

In this chapter we introduce the basic notation and state some properties of reductive groups and moment maps which we will need later on.

Let $U$ be a connected compact Lie group with complexification $U^C$ and let $u$ and $u^C$ denote the corresponding Lie algebras. Then we have a Cartan decomposition

$$U^C = U \cdot \exp(iu)$$

of $U^C$ with corresponding Cartan involution $\theta$. We call a real Lie subgroup $G$ of $U^C$ compatible with the Cartan decomposition of $U^C$ if $G = K \cdot \exp(p)$, where $K$ is a Lie subgroup of $U$ and $p$ is a $K$-stable linear subspace of $iu$. In particular, we have a $(K \times \tilde{K})$-equivariant diffeomorphism

$$K \times p \to G, \quad (k, \xi) \mapsto k \cdot \exp(\xi).$$

Note that $G$ is a closed subgroup of $U^C$ if and only if $K$ is a compact subgroup of $U$. If $G$ is connected, the following lemma holds.

Lemma 2.1. ([HS05b]) Let $G$ be a connected Lie subgroup of $U^C$ which is compatible with the Cartan involution of $U^C$ and let $\tilde{U}$ be the smallest closed subgroup of $U$ which contains $\exp(\mathfrak{t} + ip)$. Then the Zariski closure of $G$ in $U^C$ equals $\tilde{U}^C = \tilde{U} \cdot \exp(i\tilde{u})$, where $\tilde{u}$ denotes the Lie algebra of $\tilde{U}$. In particular, $G$ is compatible with the Cartan decomposition of $\tilde{U}^C$.

From now on we fix a closed Lie subgroup $G$ of $U^C$ which is compatible with the Cartan involution of $U^C$. Since all results of this monograph can be proven by reduction to the identity component of $G$, we assume that $G$ is connected. Using the above lemma, we can further make the assumption that $G$ is Zariski dense in $U^C$.

2.1 Iwasawa decomposition and parabolic subgroups

We want to recall some basic facts about the structure theory of reductive groups which we will need in the following. All proofs can be found in Knapp’s book ([Kna96]), so we omit them here. We use the following definition of a reductive group.

Definition 2.1. A reductive group is a Lie group $G$ together with a triple $(K, \theta, \kappa)$ consisting of a compact subgroup $K$ of $G$, a Lie algebra involution $\theta$ of the Lie algebra $\mathfrak{g}$ of $G$, and a nondegenerate, $\text{Ad}(G)$-invariant, $\theta$-invariant bilinear form $\kappa$ on $\mathfrak{g}$ such that

1. the Lie algebra $\mathfrak{g}$ is a reductive Lie algebra, i.e. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z_\mathfrak{g}$, where $Z_\mathfrak{g}$ denotes the center of $\mathfrak{g}$. 

2. the decomposition of $\mathfrak{g}$ into $+1$ and $-1$ eigenspaces of $\theta$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of $K$.

3. the eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to $\kappa$, and $\kappa$ is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$.

4. the multiplication map $K \times \exp(\mathfrak{p}) \to G$ is a diffeomorphism onto, and

5. for all $g \in G$ the automorphism $\text{Ad}(g)$ of $\mathfrak{g}^c$ is inner, i.e., is given by some $x$ in $\text{Int}(\mathfrak{g}^c)$.

Actually all Lie groups mentioned above are reductive groups in the above sense. For the compact groups $U$ and $K$ one can take the 3-tuples $(U, \text{Id}_U, - <, >_u)$ and $(K, \text{Id}_K, - <, >_t)$, respectively. Here $<, >_u$ and $<, >_t$ denotes invariant inner products on $u$ and $t$, respectively. For the non compact group $U^C$ the 3-tuple $(U, \theta, \kappa_{\mathfrak{u}^C})$ satisfies the above properties, where $\kappa_{\mathfrak{u}^C}$ denotes the natural extension of $- <, >_u$ to $\mathfrak{u}^C$ such that $\kappa_{\mathfrak{u}^C}$ is positive definite on $\mathfrak{i}u$ and negative on $u$. Restricting this invariant bilinear form from to $\mathfrak{g}$, the Lie algebra of $G$, we get that $G$, together with the compact subgroup $K$, the involution $\theta|_g$ and this bilinear form, is also a reductive group in the above sense.

Now let $G$ be any reductive Lie group. We fix a maximal subalgebra of $\mathfrak{p}$ which we call $\mathfrak{a}$. Since the commutator $[\mathfrak{p}, \mathfrak{p}]$ is contained in $\mathfrak{k}$, this subalgebra is automatically commutative. By the next proposition, every $K$-orbit in $\mathfrak{p}$ intersects this subalgebra $\mathfrak{a}$.

**Proposition 2.1.** If $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ are two maximal subalgebras of $\mathfrak{p}$, then there exists an element $k \in K$ such that $\text{Ad}(k)\mathfrak{a} = \tilde{\mathfrak{a}}$. Moreover, this element $k$ of $K$ can be chosen in the semisimple part $K_{ss}$ of $K$. Consequently, the space $\mathfrak{p}$ satisfies the equation

$$\mathfrak{p} = \bigcup_{k \in K_{ss}} \text{Ad}(k)\mathfrak{a}.$$ 

Furthermore we can define a restricted root system with respect to $\mathfrak{a}$ as follows. For each nonzero linear functional $\lambda$ on $\mathfrak{a}$, let

$$\mathfrak{g}_{\lambda} = \{ \xi \in \mathfrak{g} \mid [H, \xi] = \lambda(H)\xi \text{ for all } H \in \mathfrak{a} \}.$$ 

If $\mathfrak{g}_{\lambda}$ is nonzero, we call $\lambda$ a restricted root of $\mathfrak{g}$ and $\mathfrak{g}_{\lambda}$ the corresponding restricted root space. The set of all restricted roots is denoted by $\Sigma$. Reflections in the restricted roots generate a group $W(\Sigma)$ which we call the Weyl group of the restricted root system $\Sigma$.

**Proposition 2.2.** The Weyl group $W(\Sigma)$ of the restricted root system $\Sigma$ coincides with the group $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. 

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After introducing a notion of positivity of the restricted root system $\Sigma$, we can define a positive Weyl chamber $a_+$ which is a fundamental domain for the action of the Weyl group. Moreover, every $K$-orbit in $p$ intersects this positive Weyl chamber in precisely one point.

If $G$ is a compatible subgroup of $U^C$, the above statements also hold for the maximal subalgebra $i_a$ of $ip \subset u$. In particular, $i_a$ is a slice for the $K$-action on $ip$.

Defining $n$ to be the sum of all root spaces corresponding to positive roots, we get a decomposition

$$g = k \oplus a \oplus n$$

of $g$ which is called the Iwasawa decomposition. The next proposition shows that one also has an Iwasawa decomposition on the group level.

**Proposition 2.3.** Let $G$ be a reductive group and let $g = k \oplus a \oplus n$ be the Iwasawa decomposition of the Lie algebra $g$ of $G$. Let $A$ and $N$ denote the analytic subgroups of $G$ with Lie algebras $a$ and $n$, respectively. Then the multiplication map

$$K \times A \times N \to G, \quad (k, a, n) \mapsto k \cdot a \cdot n$$

is a diffeomorphism onto. Moreover, the groups $A$ and $N$ are simply connected.

Note also that for such an abelian subspace $a$ there exists a maximal abelian subspace $h$ of the centralizer $m := z_k(a)$ of $a$ in $k$ such that $h \oplus a$ is a $\theta$-stable Cartan subalgebra of $g$. The roots defined above are then just the nonzero restrictions to $a$ of the roots relative to this Cartan subalgebra.

In the case of a compatible subgroup $G$ of $U^C$ this also shows that $h \oplus i_a$ is a maximal torus of the Lie algebra $u$. Note that $\exp(ia)$ is in general not closed in $U$. So we define $T$ to be the closure of $\exp(ia)$. Its Lie algebra will be denoted by $i\tilde{a}$

**Definition 2.2.** A subalgebra $q$ of $g$ is called a minimal parabolic subalgebra if $q$ is conjugate to the subalgebra

$$q_0 := m \oplus a \oplus n.$$  

Using the Iwasawa decomposition one can assume that the conjugacy is via $Ad(K)$. Since $q_0$ contains the $\theta$-stable Cartan subalgebra $h \oplus a$ and $Ad(K)$ sends any such Cartan subalgebra onto another $\theta$-stable maximally noncompact Cartan subalgebra, every minimal parabolic subalgebra contains such a Cartan subalgebra.

**Definition 2.3.** A parabolic subalgebra of $g$ is a subalgebra of $g$ which contains a minimal parabolic subalgebra.

In particular, any parabolic subalgebra contains a maximally noncompact $\theta$-stable Cartan subalgebra. So we can assume that the parabolic subalgebra $q$ contains a minimal parabolic subalgebra of the form

$$q_0 = m \oplus a \oplus n.$$  

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where \( a \) is any maximal subalgebra of \( p \) and \( m \) and \( n \) are constructed as above. Let \( \Pi \) denote the set of simple restricted roots in the set of positive restricted roots \( \Sigma^+ \) in \( \Sigma \). For a subset \( \Pi' \) of \( \Pi \) we set

\[
\Gamma = \Sigma^+ \cup \{ \beta \in \Sigma \mid \beta \in \text{span}(\Pi') \},
\]

which defines a parabolic subalgebra of \( g \) as follows

\[
q_\Gamma = m \oplus a \oplus \bigoplus_{\beta \in \Gamma} g_\beta.
\]

Every parabolic subalgebra \( q \) of \( g \) containing the minimal parabolic subalgebra \( q_0 = m \oplus a \oplus n \) is of this form for some subset \( \Pi' \) of \( \Pi \). This leads to another decomposition of the parabolic subalgebra \( q \). Defining

\[
a_q := \bigcap_{\beta \in \Gamma \cap -\Gamma} \ker \beta \subset a, \quad m_q := a_q^+ \oplus m \oplus \bigoplus_{\beta \in \Gamma \cap -\Gamma} g_\beta, \quad \text{and} \quad n_q = \bigoplus_{\beta \in \Gamma \setminus -\Gamma} g_\beta,
\]

we get the decomposition

\[
q = m_q \oplus a_q \oplus n_q
\]

which is called the Langlands decomposition of \( q \). Here \( a_q \) is abelian and \( n_q \) is a nilpotent subalgebra of \( G \). We define \( M_q \) to be the group \( Z_K(a_q) \cdot (Z_G(a_q))_{ss} \), where \( (Z_G(a_q))_{ss} \) denotes the semisimple part of the centralizer of \( a_q \) in \( G \). The next proposition leads to the definition of parabolic subgroups of \( G \).

**Proposition 2.4.** The subgroups \( M_q, A_q \) and \( N_q \) have the following properties.

1. \( M_q A_q \) normalizes \( N_q \), so that \( Q := M_q A_q N_q \) is a group.

2. \( Q = N_G(m_q \oplus a_q \oplus n_q) \), and hence \( Q \) is a closed subgroup.

3. \( Q \) has Lie algebra \( q = m_q \oplus a_q \oplus n_q \).

4. The multiplication map \( M_q \times A_q \times N_q \to Q \) is a diffeomorphism.

5. \( G = KQ \).

We call \( Q \subset G \) the parabolic subgroup associated to the parabolic subalgebra \( q \). If \( q \) is a minimal parabolic subalgebra, we call the associated group \( Q \) a minimal parabolic subgroup of \( G \). The decomposition \( Q = M_q A_q N_q \) is called the Langlands decomposition of \( Q \).
2.2 Cartan decomposition of the moment map

Let $X$ be a compact connected Kähler manifold with Kähler form $\omega$. We assume that there is a holomorphic $U^\mathbb{C}$-action

$$U^\mathbb{C} \times X \to X, \quad (g, x) \mapsto g \cdot x$$
on X$ such that $U$ acts symplectically on $X$, i.e. for every $u \in U$ the map

$$u : X \to X, \quad x \mapsto u \cdot x$$

fixes the symplectic form: $u^* \omega = \omega$.

A moment map for this $U$-action is a smooth $U$-equivariant map $\mu : X \to u^*$ fulfilling the following momentum condition. Let $\mu^\xi(x) = \mu(x)(\xi)$ for $\xi \in u$ and let $\xi_X$ denote the induced vector field given by

$$\xi_X(x) := \frac{d}{dt} \bigg|_{t=0} (\exp(t\xi) \cdot x).$$

Then the momentum condition can be formulated as follows:

$$d\mu^\xi = i_{\xi_X} \omega,$$

where $i$ denotes contraction along the vector field $\xi_X$, i.e.

$$d\mu^\xi(x)(v) = \omega_x(\xi_X(x), v).$$

If such a moment map exists for a given symplectic $U$-action, we call the action Hamiltonian. The momentum condition implies that for a given action and a fixed symplectic form the moment map $\mu$ is unique up to a constant vector in $u^*$. By the equivariance condition this vector has to be an element in the center of $u^*$. In particular, for semisimple groups the moment map is unique. Moment maps have the following useful property.

**Lemma 2.2.** Let $\mu : X \to u^*$ be a moment map. Then the restriction of $\mu$ to the fixed point set $X^H$ of a closed subgroup $H \subset U$ takes values in the dual of the Lie algebra $\mathfrak{z}_u(H)$ and is a moment map for the action of $Z_U(H)$.

**Proof.** Since $X^H$ is a symplectic submanifold of $X$ whose symplectic form is the restriction of the symplectic form $\omega$ on $X$ to the tangent space

$$T_x X^H = \{ v \in T_x X \mid h \cdot v = v \ \forall \ h \in H \},$$

the momentum condition is satisfied. By definition, the centralizer $Z_U(H)$ is given by those elements $u \in U$ which commute with all elements in $H$. Consequently, $Z_U(H)$ acts on $X^H$ and $\mu|_{X^H}$ is equivariant with respect to $Z_U(H)$. For $x \in X^H$ we have

$$h \cdot \mu(x) = \mu(h \cdot x) = \mu(x)$$
for all $h \in H$. Therefore, $\mu(x)$ is contained in $(\mathfrak{z}_u(H))^*$.

In the following, we identify $u^*$ and $u$ using the invariant inner product $<,>_u$ on $u$. To simplify notation we will use the same letter for an element in $u$ and its image under the above identification.

Note that every linear subspace $m$ of $u$ defines a map $\mu_m: X \to m^*$ given by restriction of $\mu$ to $m^*$. In particular, if $G = K \cdot \exp(\mathfrak{p})$ is a compatible subgroup of $U^C$, the two subspaces $\mathfrak{t} = \text{Lie}(K)$ and $i\mathfrak{p}$ of $u$ define two $K$-equivariant maps

$$\mu_{\mathfrak{t}}: X \to \mathfrak{t}^* \quad \text{and} \quad \mu_{i\mathfrak{p}}: X \to i\mathfrak{p}^*.$$ 

Since $K$ is compact, the map $\mu_{\mathfrak{t}}$ is just the moment map with respect to the $K$-action on $X$. As mentioned in the introduction, we are mostly interested in the properties of the map $\mu_{i\mathfrak{p}}$ which is no moment map in the above sense but has some similar properties.

We have

$$\mu_{i\mathfrak{p}}^\xi(x) := \mu_{i\mathfrak{p}}(x)(\xi) = \mu(x)(\xi) =: \mu^\xi(x)$$ 

for all $\xi \in i\mathfrak{p}$ which implies that

$$d\mu_{i\mathfrak{p}}^\xi(x)(v) = d\mu^\xi(x)(v) = \omega_x(\xi X(x), v)$$

for all $v$ in the tangent space $T_xX$. Let $(i\mathfrak{p})_x$ denote the stabilizer of $i\mathfrak{p}$ in $x$, i.e.

$$(i\mathfrak{p})_x = \{\xi \in i\mathfrak{p} \mid \xi X(x) = 0\}.$$

Then the image of $d\mu_{i\mathfrak{p}}(x)$ can be computed as follows.

**Lemma 2.3.** Let $x$ be a point in $X$. Then the image of $d\mu_{i\mathfrak{p}}(x): T_xX \to i\mathfrak{p}^*$ is the annihilator space

$$(i\mathfrak{p})^-_x := \{\eta \in i\mathfrak{p}^* \mid \eta(\xi) = 0 \text{ for all } \xi \in (i\mathfrak{p})_x\}$$

of $(i\mathfrak{p})_x$ in $i\mathfrak{p}^*$.

**Proof.** Consider the map

$$f: i\mathfrak{p} \to (T_xX)^*, \quad \xi \mapsto \omega(\xi X(x), \cdot).$$

Then the dual map $f^*$ of $f$ is given by

$$f^*(v)(\xi) = f(\xi)(v) = \omega(\xi X(x), v)$$

which, by the above computations, is equal to

$$d\mu_{i\mathfrak{p}}^\xi(x)(v) = d\mu_{i\mathfrak{p}}(x)(\xi)(v).$$

Therefore, $d\mu_{i\mathfrak{p}}(x)$ coincides with the dual map of $f$. So the image of $d\mu_{i\mathfrak{p}}(x)$ is the annihilator space of the kernel of $f$. The kernel of $f$ is given by the space of vectors $\xi \in i\mathfrak{p}$ for which $\xi X(x) = 0$, i.e. it is $(i\mathfrak{p})_x$. 

\]
2.3 Critical points of the norm squared of $\mu_{ip}$

Using the invariant inner product $<,>_u$ on $u$, we can define the function

$$\eta_{ip} : X \to \mathbb{R}, \quad x \mapsto \eta_{ip}(x) := \|\mu_{ip}(x)\|^2 := <\mu_{ip}(x), \mu_{ip}(x)>_u.$$ 

For any $x \in X$ the derivative of $\eta_{ip}$ is given by

$$d\eta_{ip}(x) = 2 \cdot <d\mu_{ip}(x), \mu_{ip}(x)>_u = 2 \cdot d\mu^\xi(x) = 2 \cdot \omega_x(\xi_X(x), \cdot),$$

with $\xi = \mu_{ip}(x)$. Consequently, a point $x \in X$ is a critical point of the function $\eta_{ip}$ if and only if

$$\omega_x(\xi_X(x), v) = 0$$

for all $v \in T_x X$. Since the symplectic form $\omega_x$ is non degenerate, we get the following lemma.

**Lemma 2.4.** A point $x \in X$ is a critical point of $\eta_{ip}$ if and only if the vector field $\xi_X$ with $\xi = \mu_{ip}(x)$ vanishes at $x$.

Since $\eta_{ip}$ is $K$-invariant, the set of critical points of $\eta_{ip}$ is a union of $K$-orbits. But each $K$-orbit in $ip^*$ intersects the positive Weyl chamber $i\mathfrak{a}^*_+$ in exactly one point. So on each of these critical $K$-orbits there exists a unique point $x_0$ such that $\beta := \mu_{ip}(x_0)$ is contained in $i\mathfrak{a}^*_+$. Let $B_{ip}$ denote the set of all points $\beta \in i\mathfrak{a}^*_+$ which appear in this way.

We will need the following statement about the second derivative of the function $\eta_{ip}$ which gives a relation to the second derivative of the component function $\mu^\xi : X \to \mathbb{R}$.

**Lemma 2.5.** Let $\gamma : (-1, 1) \to X$ be a smooth curve in $X$ with $\gamma(0) = x \in X$ and $\frac{d}{dt} \bigg|_{t=0} \gamma(t) = v \in T_x X$. Then

$$\frac{d^2}{dt^2} \bigg|_{t=0} \eta_{ip}(\gamma(t)) = 2 \cdot \frac{d^2}{dt^2} \bigg|_{t=0} \mu^\xi(\gamma(t)) + 2 \cdot \|d\mu_{ip}(x)(v)\|^2,$$

where $\xi := \mu_{ip}(x)$.

**Proof.** Let $\gamma : (-1, 1) \to X$ be as above. Then we have

$$\frac{d^2}{dt^2} \bigg|_{t=0} \eta_{ip}(\gamma(t)) = \frac{d^2}{dt^2} \bigg|_{t=0} <\mu_{ip}(\gamma(t)), \mu_{ip}(\gamma(t)>_u$$

$$= 2 \cdot \frac{d^2}{dt^2} \bigg|_{t=0} <\mu_{ip}(\gamma(t)), \mu_{ip}(x)>_u + 2 \cdot \left\| \frac{d}{dt} \bigg|_{t=0} \mu_{ip}(\gamma(t)) \right\|^2$$

$$= 2 \cdot \frac{d^2}{dt^2} \bigg|_{t=0} \mu^\xi(\gamma(t)) + 2 \cdot \|d\mu_{ip}(x)(v)\|^2.$$ 

□

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2.4 Semistable points with respect to $\mu_{ip}$

Another object of interest is the set of semistable points with respect to $\mu_{ip}$. It can be defined as follows. Let $\beta$ be an element in $\text{ip}$ and let $\mathcal{M}_{ip}(\beta)$ be the $\beta$-fiber of $\mu_{ip}$, i.e.

$$\mathcal{M}_{ip}(\beta) = (\mu_{ip})^{-1}(\beta).$$

Then we can define the set

$$S_G(\mathcal{M}_{ip}(\beta)) := \{x \in X \mid \overline{G \cdot x} \cap \mathcal{M}_{ip}(\beta) \neq \emptyset\},$$

where $\overline{G \cdot x}$ denotes the closure of $G \cdot x$ in $X$. To shorten notation we write $\mathcal{M}_{ip}$ instead of $\mathcal{M}_{ip}(0)$. The set $S_G(\mathcal{M}_{ip})$ is called the set of semistable points of $X$ with respect to $\mu_{ip}$. The sets $S_G(\mathcal{M}_{ip}(\beta))$ and $S_G(\mathcal{M}_{ip})$ have the following useful properties.

**Lemma 2.6.** ([HS05a]) Let $x$ be a point in $\mathcal{M}_{ip}$. Then the $G$-orbit through $x$ is closed in $S_G(\mathcal{M}_{ip})$.

**Theorem 2.1.** ([HS05b]) The set $S_G(\mathcal{M}_{ip}(\beta))$ is an open subset of $X$.

We will see in the following that the question whether $S_G(\mathcal{M}_{ip}(\beta))$ is a dense subset of $X$ or not is highly related to the convexity question we are interested in.

Note that for a closed $G$-stable subset $Y$ of $X$ the semistable points in $Y$ with respect to $\mu_{ip}$ can be defined by intersection with the semistable points in $X$. We use the following notation

$$S_G(\mathcal{M}_{ip}(Y)) = \{y \in Y \mid \overline{G \cdot y} \cap \mathcal{M}_{ip}(Y) \neq \emptyset\} = S_G(\mathcal{M}_{ip}) \cap Y,$$

where we set $\mathcal{M}_{ip}(Y) = \mathcal{M}_{ip} \cap Y$. 

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3 Decomposition of the image of $\mu_{i \alpha}$

Using the above notation, the goal of this chapter is to show that the intersection of $\mu_{i \alpha}(Y)$ with the dual of the maximal subalgebra $i \alpha$ of $i \mathfrak{p}$ is a finite union of convex polytopes for any closed $G$-stable subset $Y$ of $X$. To do this, we first discuss the moment map with respect to Hamiltonian torus actions on compact connected symplectic manifolds. By the results of Guillemin, Sternberg ([GS82]) and Atiyah ([Ati82]) the image of this moment map is a convex polytope and it can be described very explicitly. We use this general results to stratify the image of the moment map $\mu_{i \alpha}: X \to i \alpha^*$ induced by the action of the torus $T = \exp(i \alpha)$. In particular, we will show that the images of the strata under the map $\mu_{i \alpha}$ will induce the decomposition of

$$\mu_{i \alpha}(Y) \cap i \alpha^* \subset \mu_{i \alpha}(X)$$

into convex polytopes.

3.1 Moment maps with respect to Hamiltonian torus actions

Let $X$ be a compact connected symplectic manifold with a Hamiltonian action of a compact torus $T$. Then the corresponding moment map

$$\mu_T: X \to \mathfrak{t}^*$$

fulfills the following convexity property.

**Theorem 3.1.** ([GS82]) The set $\mu_T(X)$ is a convex polytope in $\mathfrak{t}^*$. Moreover, the vertices of this convex polytope are images of fixed points of $T$ in $X$.

This can be proven using the stratification of the manifold $X$ with respect to the $T$-isotropy. So let $\{T_\alpha | \alpha \in \Gamma\}$ be the set of subtori of $T$ which occur as stabilizer groups of points in $X$. Then the corresponding subsets

$$X_\alpha := \{ x \in X | T_x = T_\alpha \}$$

of $X$ form a disjoint decomposition of $X$ into $T$-stable symplectic submanifolds. In general, these submanifolds are not connected. So let $X'_\alpha$ be one of its finitely many connected components. Since $T$ is abelian, there are only finitely many subtori $T_\alpha$ appearing as stabilizer subgroups (see e. g. [Mos57]). So there are only finitely many of these sets $X'_\alpha$. Let $t_\alpha$ denote the Lie algebra of $T_\alpha$ and $t_\alpha^\perp$ the orthogonal complement of $t_\alpha$ in $\mathfrak{t}$ with respect to the Killing form. Then the image of $X'_\alpha$ under the moment map $\mu_T$ is given by the following theorem.

**Theorem 3.2.** ([GS82]) There exists an element $\beta \in \mathfrak{t}^*$ such that $\mu_T$ maps $X'_\alpha$ onto an open subset of the affine subspace $\beta + (t_\alpha^\perp)^*$ of $\mathfrak{t}^*$. 

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Proof. Using the analogue of Lemma 2.3 for the moment map \( \tilde{\mu}_T = \mu_T|_{X'_\alpha} \) one concludes that \( d\tilde{\mu}_T(x) \) maps the tangent space \( T_xX'_\alpha \) onto \( t^*_\alpha \) for all \( x \in X'_\alpha \). Therefore, the image \( \mu_T(X'_\alpha) \) is an open subset of an affine subspace \( \beta + (t^*_\alpha)^* \) of \( t^* \).

In particular, the dimension of \( \mu_T(X_\alpha) \subset t^* \) coincides with the dimension of the \( T \)-orbits in \( X_\alpha \). We define an ordering on the index set \( \Gamma \) in the following way. Let \( \alpha \) and \( \beta \) be elements in \( \Gamma \), then \( \alpha < \beta \) if and only if the dimension of the \( T \)-orbits in \( X_\alpha \) is smaller than the dimension of the \( T \)-orbits in \( X_\beta \). We obtain

\[
X'_\alpha \subset X'_\alpha \cup \bigcup_{\beta < \alpha} X_\beta
\]

which shows that the sets \( X_\alpha \) form a smooth stratification of \( X \) in the following sense.

**Definition 3.1.** A finite collection of subsets \( \{S_j \mid j \in J\} \) of \( X \) forms a stratification of \( X \) if \( X \) is the disjoint union of the sets \( S_j \) and if there exists a strict partial order on the index set \( J \) such that

\[
\overline{S}_j \subset \bigcup_{k \geq j} S_k
\]

for every \( j \in J \). The subsets \( S_j \) are called strata of \( X \). Such a stratification is called smooth if all strata are locally closed submanifolds in \( X \).

By induction over the dimension of the subtori \( T_\alpha \), we get the following corollary.

**Corollary 3.1.** The closure of each set \( \mu_T(X'_\alpha) \) is a convex polytope whose vertices are images of fixed points of \( T \) in \( X \).

Since the sets \( X_\alpha \) are symplectic manifolds, there is only one stratum \( X_\tilde{\alpha} \) such that \( \tilde{\alpha} \) is maximal with respect to the above ordering and this strata is connected. In particular, the closure of \( \mu_T(X_\tilde{\alpha}) \) coincides with \( \mu_T(X) \). Therefore, Theorem 3.1 follows from the above corollary applied to this open and dense stratum \( X_\tilde{\alpha} \).

### 3.2 Decomposition of \( \mu_{i\alpha}(Y) \cap i\alpha^* \) into convex polytopes

We now return to the situation of section 2.2. Let \( X \) be a compact connected Kähler manifold with a holomorphic action of the group \( U^\mathbb{C} \) such that the induced \( U \)-action is Hamiltonian. Let

\[
\mu: X \to u^*
\]

denote the corresponding moment map. Since \( T = \exp(i\alpha) \) is a closed abelian subgroup of \( U \), we can apply the general results stated above to the corresponding moment map

\[
\mu_{i\alpha}: X \to i\tilde{\alpha}^*.
\]
Using Theorem 3.1, we know that the set \( \mu_{ia}(X) \) is a convex polytope whose vertices are images of fixed points of \( T \) in \( X \) under the map \( \mu_{ia} \). Since \( ia \) is a linear subspace of \( i\tilde{a} \), the set \( \mu_{ia}(X) \) is the projection of \( \mu_{i\tilde{a}}(X) \) onto the subspace \( ia \) and therefore, it is also a convex polytope whose vertices are images of fixed points of \( T \) in \( X \) under the map \( \mu_{ia} \). But \( ia^* \) is a linear subspace of \( ip^* \). So we have

\[
\mu_{ip}(Y) \cap ia^* \subset \mu_{ia}(Y) \subset \mu_{ia}(X)
\]

for every subset \( Y \) of \( X \). In the following we want to describe this inclusions more explicitly if \( Y \) is a closed \( G \)-stable subset of \( X \). In particular, we want to prove the following theorem.

**Theorem 3.3.** The intersection \( \mu_{ip}(Y) \cap ia^* \) is a finite union of convex polytopes for every closed \( G \)-stable subset \( Y \) of \( X \).

So from now on let \( Y \) be a closed \( G \)-stable subset of \( X \). As in the general situation of the previous section we decompose \( X \) into the disjoint union of the symplectic submanifolds

\[
X_\alpha = \{ x \in X \mid T_x = T_\alpha \} \subset X,
\]

where the \( T_\alpha, \alpha \in \Gamma \), denote those subtori of \( T = \exp(i\alpha) \) which appear as stabilizer subgroups of points in \( X \). This defines a decomposition of \( \mu_{ia}(X) \) into subsets

\[
P_\alpha := \mu_{ia}(X_\alpha) \setminus \bigcup_{\beta < \alpha} \mu_{ia}(X_\beta),
\]

where the ordering on the index set is given as above. Let \( P^i_\alpha, i \in I \), denote the connected components of the set \( P_\alpha \). Since every point \( x \in X \) is contained in some \( X_\alpha \), the sets \( P^i_\alpha \) cover the whole image \( \mu_{ia}(X) \). In particular, \( \mu_{ip}(Y) \cap ia^* \) can be covered by the sets \( \mu_{ip}(Y) \cap P^i_\alpha \).

By the results of the previous section the closure of the set \( P_\alpha \) is a convex polytope since it is given as the image of the convex polytope \( \mu_{i\tilde{a}}(X_\alpha) \) under the projection onto \( ia \). This implies that the sets \( \overline{P^i_\alpha} \) are finite unions of convex polytopes since they are given by removing open convex polyhedral subsets from the sets \( P^i_\alpha \). Since \( Y \) is compact, we get the statement of Theorem 3.3 by proving the following proposition.

**Proposition 3.1.** The set \( \mu_{ip}(Y) \cap P^i_\alpha \) is either empty or it coincides with \( P^i_\alpha \).

For the proof of this proposition we need the following statement about the preimage of special boundary points of \( \mu_{ip}(Y) \cap ia^* \).

**Proposition 3.2.** Let \( p \in \mu_{ip}(Y) \cap ia^* \) be an isolated minimum of the function

\[
\psi_q : \mu_{ip}(Y) \to \mathbb{R}, \quad \xi \mapsto \|\xi - q\|^2,
\]

where \( q \) is some fixed point in \( ia^* \). Then \( (\mu_{ip})^{-1}(p) \cap Y \) is contained in the set

\[
Y^{p-q} = \{ y \in Y \mid \exp(t \cdot (p - q)) \cdot y = y \quad \forall t \in \mathbb{R} \}.
\]
Proof. We have chosen \( p \) and \( q \) in such a way that the inequality
\[
\|p - q\|^2 \leq \|\xi - q\|^2
\]
holds for all \( \xi \) in \( \mu_{ip}(Y) \). Since \( Y \) is by assumption closed and \( G \)-stable, every point \( y_0 \in (\mu_{ip})^{-1}(p) \cap Y \) is a critical point of the function
\[
\eta^q_{ip} \colon G \cdot y_0 \to \mathbb{R}, \quad y \mapsto \|\mu_{ip}(y) - q\|^2.
\]
Analogously to the computation of the derivative of \( \eta_{ip} \) in section 2.3, the derivative of the function \( \eta^q_{ip} \) can be calculated as follows
\[
d\eta^q_{ip}(y) = 2 \cdot <d\mu_{ip}(y), \mu_{ip}(y) - q> = 2 \cdot d\mu_{ip}(y) = 2 \cdot \omega_y(\xi_X(y), \cdot),
\]
where we set \( \xi := \mu_{ip}(y) - q \in \mathfrak{i}p^* \). Since \( y_0 \) is by assumption a critical point of the function \( \eta^q_{ip} \), we get
\[
0 = d\eta^q_{ip}(y_0)(v) = d\mu_{ip}(p-q)(y_0)(v) = \omega_{y_0}((p-q)_X(y_0), v)
\]
for every \( v \) in the tangent space to the \( G \)-orbit through \( y_0 \). Since \( p - q \) is contained in \( i\mathfrak{a} \), the vector \( (i \cdot (p-q))_X(y_0) \) is contained in the tangent space \( T_{y_0}(G \cdot y_0) \). But this implies
\[
0 = \omega_{y_0}((p-q)_X(y_0), (i \cdot (p-q))_X(y_0)) = \|(p-q)_X(y_0)\|^2,
\]
where \( \|\cdot\|^2 \) denotes the norm with respect to the Riemannian metric given by
\[
<v, w> = \omega_x(v, J_x w)
\]
and \( J_x \) denotes the complex structure on \( T_x X \). Therefore, the vector field \( (p-q)_X \) vanishes in \( y_0 \) which implies that \( (\mu_{ip})^{-1}(p) \cap Y \) is contained in the set \( Y^p-q \).

To prove Proposition 3.1, we need the above proposition under the weaker assumption that \( p \) is an isolated minimum of the function \( \psi|_{(\mu_{ip}(Y) \cap \mathfrak{i}\alpha^*)} \). Note that this is satisfied if we replace \( \mu_{ip} \) by \( \mu_{i\alpha} \). Therefore, the rest of this monograph is true for the group \( A = \exp(\mathfrak{a}) \) without using the next lemma. In particular, we can apply Theorem 7.2 to the following situation. Let \( \xi \) be an integral element in \( \mathfrak{i}\alpha^* \) in sense of Definition 5.3 and let \( K \cdot \xi \) denote the \( K \)-orbit through \( \xi \) with respect to the coadjoint action of \( K \) on \( ip^* \). Then the results of section 5.2 imply that this \( K \)-orbit is stable with respect to \( A \subseteq G \). Moreover, the \( K \)-orbit is a closed connected \( A \)-stable irreducible algebraic set and therefore, it satisfies the conditions of Theorem 7.2. This implies that \( \mu_{i\alpha}(K \cdot \xi) \) is a convex polytope whose vertices are images of fixed points of \( T = \exp(i\alpha) \) under \( \mu_{i\alpha} \). Using the methods of section 7.6, this can be generalized to coadjoint orbits through arbitrary points of \( \mathfrak{i}\alpha^* \). Since the images of the fixed points of \( T \) on \( K \cdot \xi \) are the points on the Weyl group orbit through \( \xi \), the convexity theorem of Kostant, which we mentioned in the introduction, turns out to be a special case of our convexity result. We mention this here because we need Kostant’s theorem in the proof of the following lemma.
Lemma 3.1. Let $\xi, \nu$ be elements in the positive Weyl chamber $i\alpha^*_+$. Then

$$\min \{ \| \xi - \alpha \|^2 \mid \alpha \in K \cdot \nu \} = \| \xi - \nu \|^2.$$ 

Proof. Using the $K$-invariance of the inner product we get

$$\| \xi - k \cdot \nu \|^2 = \| \xi \|^2 - 2 \cdot < \xi, k \cdot \nu >_u + \| \nu \|^2.$$ 

Therefore, we have to prove the inequality

$$< \xi, \alpha >_u \leq < \xi, \nu >_u$$

for all $\alpha \in K \cdot \nu$. Let $\pi_{ia}$ denote the orthogonal projection of $i\alpha^*$ onto $i\alpha^*$. Since $\xi$ is an element in $i\alpha^*_+$, we get

$$< \xi, \alpha >_u = < \xi, \pi_{ia}(\alpha) >_u$$

for every $\alpha \in i\alpha^*$. By the above considerations the projection of $K \cdot \eta$ onto $i\alpha^*$ is contained in the convex hull of the Weyl group orbit $W \cdot \eta$. So we have to prove the inequality

$$< \xi, \alpha >_u \leq < \xi, \nu >_u$$

only for elements $\alpha$ of this convex polytope. Since a linear functional on a convex polytope becomes extreme at its vertices, it suffices to prove

$$< \xi, \nu >_u \geq < \xi, w \cdot \nu >_u$$

for all $w$ in the Weyl group $W(\Sigma)$. But $w \cdot \nu - \nu$ is a positive linear combination of negative roots (see [Hum78] Lemma B in section 10.3), so $< \xi, w \cdot \nu - \nu >_u \leq 0$ which proves the above inequality.

So if $p$ is an isolated minimum of the function $\psi_q|_{(\mu_{ip}(Y) \cap i\alpha^*_+)}$, it is also an isolated minimum of $\psi_q$ since $\mu_{ip}(Y) = K \cdot (\mu_{ip}(Y) \cap i\alpha^*_+)$. Therefore, Proposition 3.2 also holds under this weaker assumption. Now we are able to prove Proposition 3.1.

Proof of Proposition 3.1. Let us assume that the set $\mu_{ip}(Y) \cap P^i_\alpha$ is a non empty proper subset of $P^i_\alpha$ for some $\alpha \in \Gamma$. Let $\hat{\alpha}$ be maximal among such $\alpha$ with respect to the given order on $\Gamma$. Since $\mu_{ip}(Y)$ is compact, the set $\mu_{ip}(Y) \cap P^i_\alpha$ is a closed subset of $P^i_\alpha$. So every point $q'$ in the complement has a fixed distance $d$ to $\mu_{ip}(Y) \cap P^i_\alpha$. Since we have chosen $\hat{\alpha}$ to be maximal, we can choose $q'$ such that the closed ball of radius $d$ centered in $q'$ intersects $\mu_{ip}(Y)$ in $\mu_{ip}(Y) \cap P^i_\alpha$. We take a point $p$ in this intersection. By possibly taking a point $p'$ on the line joining $p$ and $q'$ we can arrange that $p$ is an isolated minimum of the function

$$\psi_q|_{(\mu_{ip}(Y) \cap i\alpha^*_+)} : (\mu_{ip}(Y) \cap i\alpha^*_+) \to \mathbb{R}, \quad x \mapsto \| x - q \|^2.$$
By Lemma 3.1, it follows that \( p \) and \( q \) are contained in the same Weyl chamber. Without loss of generality we can assume that this is the positive Weyl chamber. So we can apply Proposition 3.2. Therefore, \((\mu_{ip})^{-1}(p) \cap Y\) is contained in \(Y^{p-q}\). But \(p - q\) is not contained in \(t_\alpha^+\) which implies that the set \((\mu_{ip})^{-1}(p) \cap Y\) is contained in a smaller stratum than \(X_\alpha\) which is a contradiction to our assumption.

\[\square\]

In general one cannot expect that the set \(\mu_{ip}(X) \cap i a^*\) is a convex polytope as one can see in the following example which we discuss in detail in chapter 8. But we find a sufficient condition on \(Y\) such that the intersection of \(\mu_{ip}(Y)\) with the positive Weyl chamber is a convex polytope.

**Example 3.1.** Let \(X\) be the complex flag manifold of full flags in \(\mathbb{C}^3\) given as the quotient of the group \(U^\mathbb{C} = SL_3(\mathbb{C})\) by the Borel subgroup of upper triangular matrices. Further, let \(G\) be the group \(SL_3(\mathbb{R})\) of real matrices in \(U^\mathbb{C}\). Then we can choose \(i a\) to be the set of diagonal matrices with imaginary entries and zero trace. In this case \(\mu_{ip}(X) \cap i a^*\) has the following form.

\[\text{The darker part marks the intersection with the positive Weyl chamber which is actually a convex polytope.}\]
4 Decomposition of $X$ with respect to $\eta_{ip}$

In this chapter we want to transfer some results of Kirwan’s book ([Kir84a]) about moment maps on compact Kähler manifolds to the map $\mu_{ip}$. Using the gradient flow of $\eta_{ip}$ we get a decomposition of $X$ into disjoint subsets $S_\beta$ labeled by the set $B_{\eta_{ip}} := \mu_{ip}(\{ x \in X \mid d\eta_{ip}(x) = 0 \}) \cap i\alpha_+^*$ which turns out to be a finite set. Afterwards, we use the notion of minimally degenerate Morse functions to get an alternative description of the sets $S_\beta$ independent of the gradient flow of $\eta_{ip}$. Moreover, we show that those sets $S_\beta$ which correspond to global minima $x_0$ of $\eta_{ip}$ coincide with the sets $S_G(M_{ip}(\beta)) := \{ x \in X \mid G \cdot x \cap (\mu_{ip})^{-1}(\beta) \neq \emptyset \}$, where $\beta$ is the intersection of the $K$-orbit through $\mu_{ip}(x_0)$ with the positive Weyl chamber $i\alpha_+^*$.

4.1 Critical points and the gradient flow of $\eta_{ip}$

As we saw in section 2.3 the image of the set of critical points of the function

$$\eta_{ip} : X \to \mathbb{R}, \quad x \mapsto ||\mu_{ip}(x)||^2$$

under $\mu_{ip}$ is a union of $K$-orbits which can be labeled by their intersection points with the positive Weyl chamber $i\alpha_+^*$. As before let $B_{ip}$ denote the set of all these intersection points. We first want to show that the set $B_{ip}$ is finite and that the set of critical points of $\eta_{ip}$ can be recovered from this set.

So for every point $\beta \in i\alpha$ we consider the function

$$\mu^\beta : X \to \mathbb{R}, \quad x \mapsto \mu^\beta(x) := \mu(x)(\beta) = \mu_{ip}(x)(\beta).$$

By the momentum condition, the set of critical points of $\mu^\beta$ on $X$ is given by the set of points $x \in X$ such that the vector field $\beta_x$ vanishes in $x$. But the later set is precisely the fixed point set $X^{T_\beta}$ of the subtorus

$$T_\beta := \{ \exp(t\beta) \mid t \in \mathbb{R} \} \subset T$$

in $X$. Let $Z_\beta$ be the union of those connected components of $X^{T_\beta}$ on which $\mu^\beta$ takes the value $||\beta||^2$, i.e.

$$Z_\beta := \{ x \in X \mid T_\beta \cdot x = x \text{ and } \mu^\beta(x) = ||\beta||^2 \}.$$ 

In particular, a point $x \in X$ is a critical point of the function $\eta_{i\alpha} = ||\mu_{i\alpha}||^2$ if and only if $x$ is contained in $Z_{i\beta}$ for $\beta = \mu_{i\alpha}(x)$. This leads to the following Lemma.
Lemma 4.1. The set $B_{ip}$ is a finite subset of $i\alpha^*_+$. 

Proof. Let $x_0 \in X$ be such that $\beta := \mu_{ip}(x_0)$ is contained in $B_{ip}$. By Lemma 2.4, the vector field $\beta_X$ vanishes at the point $x_0$. Since $\beta$ is contained in $i\alpha^*_+ \subset i\tilde{a}$, this implies that $x_0$ is also a critical point for the function

$$\eta_{i\tilde{a}} := \|\mu_{i\tilde{a}}\|^2 : X \to \mathbb{R}.$$ 

Therefore, $x_0$ is contained in $Z_\beta$. Using Corollary 3.1 we get that $\mu_{i\tilde{a}}(Z_\beta)$ is a convex polytope in the affine plane $\beta \oplus t_\beta \subset i\tilde{a}$ whose vertices are images of fixed points of $T$ in $X$. In particular, $\beta$ is the closest point to the origin in this polytope. Consequently, $B_{ip}$ is contained in the set which consists of the closest points to the origin of convex hulls of points in $\mu_{i\tilde{a}}(X^T)$. Since the fixed point set $X^T$ of $T$ in $X$ has only finitely many connected components, the set $\mu_{i\tilde{a}}(X^T)$ contains only finitely many points. Therefore, $B_{ip}$ is contained in a finite set which proves the lemma. 

Note that Kirwan calls the set of closest points to the origin of convex hulls of points in $\mu_{i\tilde{a}}(X^T)$ minimal combinations of weights. 

We are now able to reconstruct the set of critical points of $\eta_{ip}$ from the finite set $B_{ip}$.

Lemma 4.2. The set of critical points of $\eta_{ip}$ is the finite union of the disjoint closed subsets

$$C_\beta := K \cdot (Z_\beta \cap (\mu_{ip})^{-1}(\beta)) \subset X$$

with $\beta \in B_{ip}$. 

Proof. Let $x$ be a critical point of $\eta_{ip}$ and let $k \in K$ be such that $\beta = \mu_{ip}(k \cdot x)$ is contained in the positive Weyl chamber $i\alpha^*_+$. By definition, $\beta$ is an element in $B_{ip}$ and by Lemma 2.4, the vector field $\beta_X$ vanishes at $k \cdot x$. Therefore, $k \cdot x$ is contained in the set

$$X^T \cap (\mu_{ip})^{-1}(\beta) = Z_\beta \cap (\mu_{ip})^{-1}(\beta).$$

In particular, the set of critical points of the function $\eta_{ip}$ is contained in the finite union of the closed sets

$$C_\beta := K \cdot (Z_\beta \cap (\mu_{ip})^{-1}(\beta)) \subset X.$$

These sets are disjoint because the intersection of $\mu_{ip}(K \cdot x)$ with the positive Weyl chamber is unique. 

On the other hand, every point $x \in Z_\beta \cap (\mu_{ip})^{-1}(\beta)$ is a critical point of $\eta_{ip}$ since the vector field $\beta_X$ vanishes in $x$. Since the $K$-orbit through a critical point is also critical, the lemma follows. 

\[\square\]
We now want to compute the gradient flow of \( \eta_{ip} \) with respect to the Riemannian metric \( <, > \) on \( X \) induced by the Kähler structure, i.e.

\[
<v, w>_x = \omega_x(v, J_x w) \quad \forall v, w \in T_x X,
\]

where \( J \) denotes the complex structure on \( X \). Using the results of section 2.3, we get

\[
<v, (\text{grad } \eta_{ip})(x) > = d\eta_{ip}(x)(v) = 2 \cdot \omega_x(\beta_X(x), v) = < v, 2i \cdot \beta_X(x) >
\]

for any \( x \in X \) and \( v \in T_x X \), where we have set \( \beta = \mu_{ip}(x) \). For a given \( x \in X \), let \( \gamma(t, x) \) denote the integral curve through \( x \) for \( -\text{grad } \eta_{ip} \). Since \( 2i \cdot \beta = 2i \cdot \mu_{ip}(x) \) is contained in \( p \), these integral curves are contained in orbits of the group \( G \).

We call a point \( y \in X \) a limit point of the gradient flow of \( \eta_{ip} \) through a point \( x \in X \) if for every open neighborhood \( U \) of \( y \) there exists an increasing unbounded sequence \( (t_n)_{n \in \mathbb{N}} \subset \mathbb{R} \) such that \( \gamma(t_n, x) \) is contained in \( U \) for all \( n \in \mathbb{N} \). Let \( L_x \) denote the set of all these limit points.

**Lemma 4.3.** Let \( x \) be a point in \( X \) and let \( L_x \) be the limit set of the gradient flow of \( \eta_{ip} \) through \( x \). Then \( L_x \) is a nonempty closed connected subset of \( X \) which lies in the set of critical points of the function \( \eta_{ip} \).

**Proof.** Since \( X \) is compact, the set \( L_x \) is non empty and closed. To prove that \( L_x \) is connected assume that there are two disjoint open subsets \( U_1 \) and \( U_2 \) in \( X \) such that \( L_x \) is contained in \( U_1 \cup U_2 \). By definition of \( L_x \), for every point \( y \in X \setminus (U_1 \cup U_2) \) there exists a number \( t_y \in \mathbb{R} \) and an open neighborhood \( W(y) \) of \( y \) such that \( \gamma(t, y) \) does not meet \( W(y) \) for all \( t \geq t_y \). Moreover, since \( X \setminus (U_1 \cup U_2) \) is compact there exists a \( T > 0 \) such that \( \gamma(t, y) \) is contained in \( U_1 \cup U_2 \) for all \( t \geq T \). But the set

\[
\{ \gamma(t, y) \mid t \geq T \}
\]

is connected. So it is contained either in \( U_1 \) or in \( U_2 \) which shows that \( L_x \) is contained either in \( U_1 \) or in \( U_2 \). Therefore, \( L_x \) is connected.

Let us assume the limit set \( L_x \) is not contained in the set of critical points of \( \eta_{ip} \). Then there exists a point \( y \in L_x \) such that \( \text{grad } \eta_{ip} \) does not vanish in \( y \). But this implies that for a sufficiently small neighborhood \( U(y) \) of \( y \) there exists a \( t_0 > 0 \) such that \( \gamma(t, x) \) is not contained in \( U(y) \) for \( t \geq t_0 \) and every \( x \in U(y) \). This is a contradiction to the definition of a limit point.

As we saw in Lemma 4.2 the set of critical points of \( \eta_{ip} \) is the disjoint union of finitely many closed subsets \( C_\beta \). Therefore, the previous lemma implies that for every point \( x \in X \) there exists a unique point \( \beta \in \mathcal{B}_{ip} \) such that \( L_x \) is contained in \( C_\beta \). So \( X \) can be written as the disjoint union of the subsets

\[
S_\beta := \{ x \in X \mid L_x \subset C_\beta \}.
\]
4.2 Minimally degenerate Morse functions

To get a better description of the sets $S_{\beta}$, we first show that $\eta_{ip}$ is a minimally degenerate Morse function in the following sense.

**Definition 4.1.** A smooth function $f : X \to \mathbb{R}$ on a compact manifold $X$ is called a minimally degenerate Morse function if the following conditions hold.

1. The set of critical points for $f$ on $X$ is a finite union of disjoint closed subsets $C_j$ on each of which $f$ takes constant value.

   The subsets $C_j$ are called critical subsets of $f$.

2. For every $C_j$ there is a locally closed submanifold $\Sigma_j$ containing $C_j$ such that
   (a) every $C_j$ is the subset of $\Sigma_j$ on which $f$ takes its minimum value,
   (b) at every point $x \in C_j$ the tangent space $T_x\Sigma_j$ is maximal among all subspaces of $T_xX$ on which the Hessian $H_x(f)$ is positive semi-definite.

A submanifold satisfying these properties is called a minimizing manifold for $f$ along $C_j$.

Note that these function are not Morse functions in the sense of Bott since the critical sets have in general singularities.

Using Lemma 4.2 and the fact that $\eta_{ip}$ has value $\|\beta\|^2$ on the set $C_{\beta}$, we get that the sets $C_{\beta}$ are critical sets of $\eta_{ip}$ in the sense of the above definition. To prove that $\eta_{ip}$ also fulfills the second requirement of the above definition, we have to construct minimizing manifolds for $\eta_{ip}$ along the sets $C_{\beta}$.

Therefore, we again consider the function $\mu^\beta$. Note that this function is a nondegenerate Morse function on $X$ in the sense of Bott (see e.g. [Ati82] or [Aud91]), which means that the set of critical points is a submanifold of $X$ and its second derivative is a nondegenerate quadratic form in the transverse direction to this manifold. For Morse functions in the sense of Bott it is known that the gradient flow induces a smooth stratification in the sense of Definition 3.1.

So for each $\beta \in B_{ip}$ there exists a smooth stratum $Z_{\beta}(\cdot)$ associated to $Z_{\beta}$ which consists of all points in $X$ such that the gradient flow of the function $\mu^\beta$ has limit points in the critical set $Z_{\beta}$. Again, we can compute the gradient with respect to the Riemannian structure on $X$. We get

$$<v, (\text{grad } \mu^\beta)(x) > = d\mu^\beta(x)(v) = \omega_x(v, \beta_X(x)) = <v, i\beta_X(x) >$$

for any $x \in X$ and $v \in T_xX$. In particular, the integral curves through $x$ are given by

$$\gamma(t, x) = \exp(-it\beta) \cdot x.$$ 

Before constructing the minimizing manifolds as open subsets of $G \cdot Z_{\beta}(\cdot)$ we need the following properties of the strata $Z_{\beta}(\cdot)$.
Lemma 4.4. Let $x$ be a point in $Z_{\beta}(+)$. Then $\eta_{ip}(x) \geq \|\beta\|^2$ and equality holds if and only if $\mu_{ip}(x) = \beta$.

Proof. By definition, the image of $Z_{\beta}(+)$ under $\mu_{ia}$ is contained in the half space given by
\[
\{\xi \in i\mathfrak{a}^* \mid <\beta, \xi> \geq \|\beta\|^2\}.
\]
Consequently, $\beta$ is the unique point in this half space with norm equal to $\|\beta\|^2$. Since we have $\eta_{ia}(x) \leq \eta_{ip}(x)$ for all $x \in X$ and equality holds if and only if $\mu_{ip}(x)$ is contained in $i\mathfrak{a}^*$, the lemma follows.

To prove the next property, we consider the group
\[
Q_- (\beta) := \{g \in G \mid \lim_{t \to -\infty} \exp(it\beta) \cdot g \cdot \exp(-it\beta) \text{ exists in } G\}
\]
for $\beta \in B_p \subset i\mathfrak{a}_p^*$. Using the notation of section 2.1, this is the parabolic subgroup of $G$ corresponding to the parabolic subalgebra
\[
q_\Gamma = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\beta \in \Gamma} \mathfrak{g}_\beta,
\]
where $\Gamma$ is given by those simple roots which vanish on $\beta$. Note that the limit point
\[
s_g := \lim_{t \to -\infty} \exp(it\beta) \cdot g \cdot \exp(-it\beta)
\]
which exists for $g \in Q_- (\beta)$ is contained in the centralizer of $\beta$ in $G$. This leads to the following lemma.

Lemma 4.5. The subset $Z_{\beta}(+)$ of $X$ is invariant under the parabolic subgroup $Q_- (\beta)$.

Proof. Let $g$ be an element in $Q_- (\beta)$ and let $y$ be a point in $Z_{\beta}(+)$. By definition, $y$ lies in $Z_{\beta}(+)$ if and only if the limit point
\[
x = \lim_{t \to -\infty} \exp(it\beta)y
\]
exists and is contained in $Z_{\beta}$. But this limit point exists if and only if the limit point
\[
s_g \cdot x = \lim_{t \to -\infty} \exp(it\beta) \cdot g \cdot y
\]
exists, where $s_g$ is defined as above. Since $s_g$ lies in the centralizer of $\beta$ in $G$, it preserves $Z_{\beta}$. Therefore, $Z_{\beta}(+)$ is invariant under the parabolic subgroup $Q_- (\beta)$.

Since the group $G$ decomposes into the product $K \cdot Q_- (\beta)$ by Proposition 2.4, we have
\[
G \cdot Z_{\beta}(+) = K \cdot Z_{\beta}(+)
\]
In particular, the subgroup $Z_K(\beta) = K \cap Q_- (\beta)$ of $K$ stabilizes $Z_{\beta}(+)$. 

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Lemma 4.6. Let $x$ be a point in the set $Z_\beta \cap (\mu_{ip})^{-1}(\beta)$. Then we have

$$\{ k \in K \mid k \cdot x \in Z_\beta(+)\} = Z_K(\beta)$$
and

$$\{ \xi \in \mathfrak{k} \mid \xi_X(x) \in T_x(Z_\beta(+)\} = \mathfrak{z}_x(\beta).$$

Proof. By the previous lemma we know that the centralizer $Z_K(\beta)$ lies in the set

$$\{ k \in K \mid k \cdot x \in Z_\beta(+)\}.$$

Therefore, the same inclusion holds on the Lie algebra level. To prove the opposite inclusions let $k \cdot x \in Z_\beta(+)$. Then the gradient flow of $\mu^\beta$ through this point has a limit point in $Z_\beta$. Since $\mu^\beta$ has constant value $\|\beta\|^2$ on $Z_\beta$, we have

$$\mu^\beta(k \cdot x) = \mu_{ip}(k \cdot x)(\beta) \geq \|\beta\|^2.$$ 

Since $\eta_{ip}$ is $K$-invariant, we have $\eta_{ip}(k \cdot x) = \|\beta\|^2$. Together with the above inequality we get

$$\mu_{ip}(k \cdot x) = k \cdot \mu_{ip}(x) = k \cdot \beta = \beta$$
which shows that $k$ is contained in the centralizer $Z_K(\beta)$. We now prove this inclusion on the Lie algebra level. Let $\xi$ be an element in $\mathfrak{k}$ such that $\xi_X(x)$ is contained in $T_xZ_\beta(+)$. Since $\mu_{ip}$ is $K$-equivariant, we get

$$d\mu_{ip}(x)(\xi_X(x)) = \frac{d}{dt}\bigg|_{t=0} \mu_{ip}(\exp(t\xi) \cdot x)$$

$$= \frac{d}{dt}\bigg|_{t=0} \text{Ad}(\exp(t\xi)) (\mu_{ip}(x)) = [\xi, \mu_{ip}(x)] = [\xi, \beta].$$

Using again the fact that the norm is $K$-invariant, we get that the function

$$t \mapsto \eta_{ip}(\exp(t\xi) \cdot x)$$

is constant. Together with the equation of Lemma 2.5 we get

$$0 = \frac{d^2}{dt^2}\bigg|_{t=0} \eta_{ip}(\exp(t\xi) \cdot x) = \frac{d^2}{dt^2}\bigg|_{t=0} \mu^\beta(\exp(t\xi) \cdot x) + 2 \cdot \|d\mu_{ip}(x)(\xi_X(x))\|^2.$$ 

But by assumption, $\xi_X(x)$ is contained in $T_xZ_\beta(+)$ which is the sum of the non negative eigenspaces of the Hessian $H_x(\mu^\beta)$, since $\mu^\beta$ is a nondegenerate Morse function in the sense of Bott. Therefore, we get

$$d\mu_{ip}(x)(\xi_X(x)) = [\xi, \beta] = 0$$
which proves that $\xi$ is contained in $\mathfrak{z}_x(\beta)$. \qed
We are now able to construct the minimizing manifolds for $\eta_p$ along $C_\beta$.

**Lemma 4.7.** There exists an open $K$-invariant neighborhood $U$ of $C_\beta$ in $X$ such that the intersection $\Sigma_\beta = U \cap K \cdot Z_\beta(\beta)$ is a smooth submanifold of $X$.

**Proof.** Since $Z_\beta(\beta)$ is invariant under $Z_K(\beta)$ by the previous lemma, the map

$$f: K \times Z_\beta(\beta) \to X, \quad (k, x) \mapsto k \cdot x$$

induces a map $\tilde{f}: K \times Z_K(\beta) Z_\beta(\beta) \to X$ whose image is $K \cdot Z_\beta(\beta)$.

Let $[e, x]$ be a point in $K \times Z_K(\beta) Z_\beta(\beta)$ such that $e$ is the identity element in $K$ and $x$ is a point in $Z_\beta \cap (\mu_p)^{-1}(\beta)$. We first want to prove that $\tilde{f}$ maps an open neighborhood $U$ of such a point $[e, x]$ onto an open neighborhood of $x$ in $K \cdot Z_\beta(\beta)$.

Let us assume this is not true. Then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $K \cdot Z_\beta(\beta)$ which converges to $x$ but is not contained in the image of $U$ under $\tilde{f}$. Since $K$ is compact, we can assume that this sequence has the form $(k \cdot y_n)_{n \in \mathbb{N}}$ for fixed $k \in K$ and a sequence $(y_n)_{n \in \mathbb{N}}$ in $Z_\beta(\beta)$. In particular, the sequence $(y_n)_{n \in \mathbb{N}}$ converges to $k^{-1} \cdot x$ which therefore has to be contained in $Z_\beta(\beta)$. Consequently, $k$ is contained in $Z_K(\beta)$. But this shows that the sequence $(x_n)_{n \in \mathbb{N}}$ is contained in the image of $U$ for sufficiently large $n$. Therefore, the image of $U$ under $\tilde{f}$ is open in $K \cdot Z_\beta(\beta)$.

Again, let $x$ be a point in $Z_\beta \cap (\mu_p)^{-1}(\beta)$. Then the tangent space of the quotient $K \times Z_K(\beta) Z_\beta(\beta)$ at the point $[e, x]$ is the quotient of the vector space $T \times T_x Z_\beta(\beta)$ by the subspace

$$\{([\xi, -\xi_x(x)]) \in T \times T_x Z_\beta(\beta) : \xi \in \mathfrak{z}_v(\beta)\}$$

which is the tangent space to the orbit $Z_K(\beta) \cdot [e, x]$ in the point $[e, x]$. The derivative of $\tilde{f}$ in $[e, x] \in K \times Z_K(\beta) Z_\beta(\beta)$ which is given by

$$d\tilde{f}([e, x]): T_{[e, x]}(K \times Z_K(\beta) Z_\beta(\beta)) \to T_x X, \quad [\xi, v] \mapsto \xi_x(x) + v$$

is therefore injective. So this is also true in some open neighborhood $V$ of this point in $K \times Z_K(\beta) Z_\beta(\beta)$. By the previous considerations this open neighborhood $V$ is mapped by $\tilde{f}$ onto an open neighborhood of $x$ in $K \cdot Z_\beta(\beta)$. It follows from the inverse function theorem that the image $K \cdot Z_\beta(\beta)$ of $\tilde{f}$ is smooth in some open neighborhood of $Z_\beta \cap (\mu_p)^{-1}(\beta)$. Consequently, the set $K \cdot Z_\beta(\beta)$ is smooth in a $K$-stable open neighborhood of $C_\beta = K \cdot (Z_\beta \cap (\mu_p)^{-1}(\beta))$.

The next proposition shows that these locally closed submanifolds are minimizing manifolds for the function $\eta_p$ along $C_\beta$.

**Proposition 4.1.** The locally closed submanifolds $\Sigma_\beta$ of $X$ are minimizing manifolds for $\eta_p$ along $C_\beta$.
Proof. Since the critical points $C_\beta$ in $G \cdot Z_\beta(+)\rangle$ are minimal points by Lemma 4.4, it follows that the Hessian of $\eta_\beta$ at these critical points is positive semidefinite. This proves the first requirement in the definition of a minimizing manifold.

To prove the second requirement, we have to show that the restriction of the Hessian $H_x(\eta_\beta)$ to the orthogonal complement $(T_x \Sigma_\beta)^\perp$ is negative definite. Since $K \cdot Z_\beta(+)\rangle$ is $G$-stable by Lemma 4.5, the tangent space $T_x \Sigma_\beta$ contains the subspace

$$p \cdot x = \{ \xi X(x) \in T_x X \mid \xi \in p \}.$$

So the orthogonal complement $(T_x \Sigma_\beta)^\perp$ is contained in the subspace

$$(p \cdot x)^\perp = (ip \cdot x)^\perp = \ker(d\mu_\beta(x))$$

of $T_x X$, where $\perp$ denotes the orthogonal complement with respect to the symplectic form $\omega_x$. Now let $\gamma: (-1, 1) \to X$ be a curve in $X$ such that $\gamma(0) = x$ and

$$v := \frac{d}{dt} \bigg|_{t=0} \gamma(t) \in (T_x \Sigma_\beta)^\perp \subset \ker(d\mu_\beta(x)).$$

Then Lemma 2.5 implies that

$$\frac{d^2}{dt^2} \bigg|_{t=0} \eta_\beta(\gamma(t)) = 2 \cdot \frac{d^2}{dt^2} \bigg|_{t=0} \mu_\beta(\gamma(t)).$$

Therefore, the Hessians of $\eta_\beta$ and $\mu_\beta$ coincide up to the scalar 2 on the subspace $(T_x \Sigma_\beta)^\perp$. Since $\Sigma_\beta$ is an open subset of $K \cdot Z_\beta(+)\rangle$, we have

$$(T_x \Sigma_\beta)^\perp = (T_x (K \cdot Z_\beta(+)\rangle))^\perp \subset (T_x Z_\beta(+)\rangle)^\perp.$$

By definition, $Z_\beta(+)\rangle$ is the Morse stratum of the function $\mu_\beta$ associated to the critical subset $Z_\beta$. Consequently, the restriction of the Hessian of $\mu_\beta$ to $(T_x Z_\beta(+)\rangle)^\perp$ is negative definite which implies the second requirement in the definition and completes the proof of the proposition.

So $\eta_\beta$ satisfies the conditions of Definition 4.1 and we get the following.

**Lemma 4.8.** The function $\eta_\beta$ is a minimally degenerate Morse function.

Actually, the above lemma together with the fact that the gradient flow is tangent to the minimizing manifolds implies that the sets $S_\beta$ form a smooth stratification of $X$ and that the minimizing manifolds locally coincide with the strata around the critical sets. Since we do not need the smoothness of the sets $S_\beta$, we will not use this fact in the following.
4.3 An alternative description of the sets $S_\beta$

In this section, we want to give a description of the sets $S_\beta$ independent of the gradient flow of the function $\eta_{\pi p}$. In particular, this will lead to a connection between the sets $S_\beta$ corresponding to local minima of $\eta_{\pi p}$ and sets of semistable points with respect to $\mu_{\pi p}$.

Let $Z_{U^C}(\beta)$ denote the centralizer of $\beta$ in $U^C$. Then $Z_{U^C}(\beta)$ acts on the fixed point set $X^{T_\beta}$ and therefore on $Z_\beta$ which is a union of connected components of $X^{T_\beta}$. By Lemma 2.2, we get a moment map for the action of the compact subgroup $Z_U(\beta)$ of $Z_{U^C}(\beta)$ by restriction of the moment map $\mu$ to $Z_\beta$. Since $\beta$ lies in the center of $Z_U(\beta)$, the map

$$\mu - \beta : Z_\beta \rightarrow (z_u(\beta))^*,$$

where $z_u(\beta)$ denotes the Lie algebra of $Z_U(\beta)$, is also a moment map for this action.

Since $\beta$ is contained in $\iota p^*$, we get a $Z_K(\beta)$ equivariant map

$$\mu_{\pi p} - \beta : Z_\beta \rightarrow (z_{\pi p}(\beta))^*$$

which is the $\pi p$-component of the moment map $\mu - \beta$. Let $Z^s_\beta$ denote the set of semistable points in $Z_\beta$ with respect to the map $\mu - \beta$. We are now going to show that the sets $G \cdot Z^s_\beta(+)$ coincide with the sets $S_\beta$ defined by the gradient flow of $\eta_{\pi p}$.

Since the sets $Z^s_\beta$ are invariant under the centralizer $Z_G(\beta)$, we can apply the proof of Lemma 4.5 to show that $Z^s_\beta(+)$ is stable under the parabolic subgroup $Q_-(\beta)$. Using the decomposition $G = K \cdot Q_-(\beta)$, we get

$$G \cdot Z^s_\beta(+) = K \cdot Z^s_\beta(+) = K \cdot Z^s_\beta(+) = K,$$

**Lemma 4.9.** Let $x$ be a point in $G \cdot Z^s_\beta(+)$. Then $\eta_{\pi p}(x) \geq \|\beta\|^2$ and equality holds if and only if $\mu_{\pi p}(x)$ lies in the coadjoint orbit $K \cdot \beta$.

**Proof.** Let $x$ be a point in $G \cdot Z^s_\beta(+) = K \cdot Z^s_\beta(+)$. Since $\eta_{\pi p}$ is $K$-invariant, we have

$$\|\mu_{\pi p}(k \cdot x)\|^2 = \|\mu_{\pi p}(x)\|^2$$

for all $k \in K$. So we can assume that $x$ is contained in $Z^s_\beta(+) \subset Z_\beta(+)$. The result now follows from Lemma 4.4.

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This leads to the fact that the sets $K \cdot Z_{\beta}^{ss}(+) \subset Z_{\beta}(+)$ are disjoint which is the first step in proving that they coincide with the sets $S_{\beta}$.

**Lemma 4.10.** Let $x$ be a point in $Z_{\beta}^{ss}(+)$. Then $\beta$ is the unique closest point to the origin in the set

$$\mu_{ip}(G \cdot x) \cap i\Lambda_\dagger.$$

In particular, the sets $G \cdot Z_{\beta}^{ss}(+)$ are disjoint.

**Proof.** Let $x$ be a point in $Z_{\beta}^{ss}(+) \subset Z_{\beta}(+)$. The limit point $y = \lim_{t \to \infty} \exp(-it\beta) \cdot x$ exists and is contained in $Z_{\beta}^{ss}$. Since $\{\exp(it\beta) \mid t \in \mathbb{R}\}$ is a subgroup of $G$, the closure of the orbit $G \cdot y$ is contained in the closure of the orbit $G \cdot x$. But by the definition of $Z_{\beta}^{ss}$, the point $\beta$ is contained in the image of the closure of the orbit $Z_G(\beta) \cdot y$ under $\mu_{ip}$. So we have

$$\beta \in \mu_{ip}(Z_G(\beta) \cdot y) \subset \mu_{ip}(G \cdot y) \subset \mu_{ip}(G \cdot x).$$

By Lemma 4.9, the point $\beta$ is the unique closest point to 0 in the set $\mu_{ip}(G \cdot x) \cap i\Lambda_\dagger$. This uniqueness implies that the sets $G \cdot Z_{\beta}^{ss}(+)$ are disjoint.

We now want to show that the sets $S_{\beta}$ and $G \cdot Z_{\beta}^{ss}(+)$ coincide. Therefore, we need the following properties of the gradient flow near the critical sets $C_{\beta}$. As $C_{\beta}$ is contained in the minimizing manifolds $\Sigma_{\beta}$ we can choose local coordinates $(v, w) \in \mathbb{R}^n \times \mathbb{R}^m$ around a point $x_0 \in C_{\beta}$ such that the first coordinate corresponds to $\Sigma_{\beta}$, the second coordinate corresponds to the normal bundle of $\Sigma_{\beta}$ in $X$ and 0 corresponds to the point $x_0$. Fixing an inner product on $\mathbb{R}^{n+m}$ with norm $\| \cdot \|^2$ we get the following.

**Lemma 4.11.** There exists an endomorphism $L$ of $\mathbb{R}^n \simeq \mathbb{R}^n \times \{0\}$ such that

$$-\frac{1}{2} \cdot \text{grad}_i\mu_{ip}(v, w) = (-i\mu_{ip}(v, w))_X(v, w)$$

$$= d(\xi_X)(0)(v, w) + L(v) + O(\|v\|^2 + \|w\|^2),$$

where $\xi = -i \cdot \mu_{ip}(0) = -i\beta$.

**Proof.** The first equality was already proven in section 4.1. To construct $L$ and prove the second equation, let $\gamma(t) = t \cdot (v, w)$. Then we get

$$\left. \frac{d}{dt} \right|_{t=0} (-i\mu_{ip}(\gamma(t))_X(\gamma(t)) = d(\xi_X)(0)(v, w) + \left( \left. \frac{d}{dt} \right|_{t=0} -i\mu_{ip}(\gamma(t)) \right)_X(0)$$

$$= d(\xi_X)(0)(v, w) + (-i \cdot d\mu_{ip}(0)(v, w))_X(0).$$
Since the orthogonal complement \((T_{x_0} \Sigma_\beta)^\perp \cong \{0\} \times \mathbb{R}^m\) is contained in the kernel of \(d\mu_{ip}(x_0)\), which in our local coordinates corresponds to \(d\mu_{ip}(0)\), the second term of the above equation does not depend on the coordinate \(w\). Therefore, the linear map
\[
(v, w) \mapsto (-i \cdot d\mu_{ip}(0)(v, w))_X(0)
\]
can be seen as a linear map from \(\{0\} \times \mathbb{R}^n \cong \mathbb{R}^n\) to \(\mathbb{R}^{n+m}\). But \(i \cdot d\mu_{ip}(0)(v, w)\) is contained in \(\mathfrak{p}\). Since \(\Sigma_\beta\) is an open subset of the \(G\)-stable set \(G \cdot Z_{ss}^\Sigma(+)\), the image of this linear map is contained in the subspace \(\mathbb{R}^n \times 0 \cong \mathbb{R}^n\). This defines the linear map \(L: \mathbb{R}^n \to \mathbb{R}^n\).

Defining the function \(\rho: \mathbb{R}^{n+m} \to \mathbb{R}\) by \(\rho(v, w) = \|w\|^2\), we get the following property of the gradient of \(\eta_{ip}\) near the critical set \(C_\beta\).

**Lemma 4.12.** There exists a neighborhood \(U\) of \(0\) in \(\mathbb{R}^{n+m}\) such that
\[
d\rho(-\nabla \eta_{ip}(v, w))(v, w) > 0
\]
for all \((v, w) \in U \cap (\mathbb{R}^n \times (\mathbb{R}^m \setminus \{0\}))\).

**Proof.** Using the previous lemma and the fact that the function \(\rho\) does not depend on the first coordinate, we get
\[
d\rho(-\nabla \eta_{ip}(v, w))(v, w) = <d\xi_X(0)(w), w> + O(\|v\|^3 + \|w\|^3),
\]
where we have defined
\[
d\xi_X(0)(w) = d\xi_X(0)(0, w).
\]
Using the curve \(\gamma(t) = (0, t \cdot w)\), we also have
\[
d^2\eta_{ip}(\gamma(t)) = \frac{d}{dt} \bigg|_{t=0} d\eta_{ip}(\gamma(t))(\gamma'(t))
\]
\[
= \frac{d}{dt} \bigg|_{t=0} <\gamma'(t), \nabla \eta_{ip}(\gamma(t))> 
\]
\[
= \frac{d}{dt} \bigg|_{t=0} <\gamma'(t), (i \cdot \mu_{ip}(\gamma(t)))_X(\gamma(t))> 
\]
\[
= <\gamma''(0), (i\beta)_X(0)> + <w, \frac{d}{dt} \bigg|_{t=0} (i\mu_{ip}(\gamma(t)))_X(\gamma(t))> 
\]
\[
= <w, d(i\beta)_X(0)(w)> = -<w, d\xi_X(0)(w)>. 
\]
Since $\Sigma_\beta$ is a minimizing manifold for $\eta_p$ along $C_\beta$ by Proposition 4.1, this value is strictly negative for $w \neq 0$. Therefore, there exists a neighborhood $U$ of $0$ in $\mathbb{R}^{n+m}$ such that $d\rho(-\nabla \eta_p(v, w))(v, w)$ is strictly positive for $(v, w) \in U \cap \mathbb{R}^n \times (\mathbb{R}^m \setminus \{0\})$.

Since $C_\beta$ is compact, there exist finitely many of such neighborhoods which cover $C_\beta$. So there exists a neighborhood of $C_\beta$ where the above lemma is valid. This leads to the following.

**Proposition 4.2.** The sets $G \cdot Z^{ss}(\beta)$ coincide with the sets $S_\beta$ defined by the gradient flow of the function $\eta_p$ on $X$.

**Proof.** Since the sets $G \cdot Z^{ss}(\beta)$ are disjoint and the sets $S_\beta$ cover the whole manifold $X$, it is sufficient to show that $S_\beta$ is contained in $G \cdot Z^{ss}(\beta)$. Let us assume that this is false. Then there exists a point $x \in X$ which is not contained in $G \cdot Z^{ss}(\beta)$ but the limit point of the gradient flow $\gamma(t)$ of $\eta_p$ through $x$ is contained in $C_\beta$. Since the gradient flow is contained in a $G$-orbit, the curve $\gamma(t)$ does not intersect $G \cdot Z^{ss}(\beta)$. But for every neighborhood $U$ of $C_\beta$ there exists an $t_0 > 0$ such that $\gamma(t)$ is contained in this neighborhood for all $t \geq t_0$. Using the above lemma one can choose $U$ such that $d\rho(\gamma(t)) > 0$ for all $t \geq t_0$. In particular, the distance between $\gamma(t)$ and $C_\beta$ is strictly increasing which is a contradiction to the assumption that $\gamma(t)$ has a limit point in $C_\beta$.

Let $Y$ be a closed $G$-stable subset of $X$. Then we get a stratification of $Y$ into the disjoint subsets $Y \cap G \cdot Z^{ss}(\beta)$. Together with Lemma 4.10, this gives a good description of the sets $S_\beta$ corresponding to global minima of $\eta_p$.

**Corollary 4.1.** Let $Y$ be a closed $G$-stable subset of $X$ and let $x_0$ be a global minimum of the function $\eta_p|_Y$ such that $\beta = \mu_p(x_0)$ is contained in $i\alpha^*_+$. Then the set $Y \cap G \cdot Z^{ss}(\alpha)$ coincides with the set

$$S_G(M_{ip}(\beta)) \cap Y = \{ y \in Y \mid G^* y \cap (\mu_p)^{-1}(\beta) \neq \emptyset \}.$$  

**Proof.** By Lemma 4.10, the set $G \cdot Z^{ss}(\beta)$ is contained in $S_G(M_{ip}(\beta))$. So this is also true for the intersection with $Y$. To prove the opposite inclusion, note that Proposition 4.2 implies that the sets $G \cdot Z^{ss}(\beta)$ form a disjoint decomposition of the manifold $X$. So let $x_1$ be a point in $Y \cap S_G(M_{ip}(\beta))$ which is not contained in $Y \cap G \cdot Z^{ss}(\beta)$. Then $x_1$ is contained in some set $Y \cap G \cdot Z^{ss}(\alpha)$ with $\alpha \neq \beta$. Since $x_0$ is a global minimum of $\eta_p|_Y$, we obtain

$$\|\beta\|^2 \leq \|\alpha\|^2.$$  

Using Lemma 4.10 again, we get that $\alpha$ is the unique closest point to $0$ of $\mu_p(G \cdot x_1)$. Since $x_1$ is contained in $Y \cap S_G(M_{ip}(\beta))$ and $Y$ is closed and $G$-stable, this is a contradiction to the assumption that $x_0$ is a global minimum of the function $\eta_p|_Y$.

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This leads to the following consequence which is important for the proof of the convexity theorem.

**Corollary 4.2.** Let \( Y \) be as above and let \( x_1 \) and \( x_2 \) be two global minima of the function \( \eta_{ip}|_Y \) such that \( \alpha_1 \neq \alpha_2 \) for \( \alpha_i = (K \cdot \mu_{ip}(x_i)) \cap \mathfrak{a}_+^* \). Then we have

\[
(Y \cap S_G(M_{ip}(\alpha_1))) \cap (Y \cap S_G(M_{ip}(\alpha_2))) = \emptyset.
\]

**Proof.** By the previous corollary the sets \( Y \cap S_G(M_{ip}(\alpha_j)) \) coincide with the sets \( Y \cap G \cdot Z_{\alpha_j}^{ss}(+) \). But these are disjoint since \( \alpha_1 \neq \alpha_2 \).

\[\square\]
5 The moment map for complex flag manifolds

In this chapter we discuss the special case of moment maps for complex flag manifolds \( Z = U^C / Q \), where \( U^C \) is the complexification of the compact connected Lie group \( U \) and \( Q \) is a parabolic subgroup of \( U^C \). After introducing the notion of complex flag manifolds, we identify them with coadjoint \( U \)-orbits and show that this identification is in fact the moment map with respect to the Kähler form on \( Z \) induced by the Kirilov form on the coadjoint orbit. As a standard reference we take [FHW06].

After that we study coadjoint orbits \( U \cdot \beta \), where \( \beta \) is an element in \( i \mathfrak{p}^* \). It turns out that in this case the \( K \)-orbit \( K \cdot \beta \) is actually a \( G \)-orbit, where the \( G \)-action is given by the identification with a complex flag manifold. This leads to a shifting method for the map \( \mu : X \to i \mathfrak{p}^* \). In the last part of this chapter we discuss projective embeddings of integral coadjoint orbits which will be relevant in the proof of the convexity theorem in the projective case.

5.1 Flag manifolds and coadjoint orbits

Let \( U^C \) be the complexification of the compact connected group \( U \). Then \( U^C \) is a complex reductive group and we have the notion of minimal parabolic subgroups and parabolic subgroups, respectively.

Since we are interested in quotients by parabolic subgroups and the radical of \( U^C \) is contained in each parabolic subgroup, there is no loss of generality in assuming that \( U^C \) is semisimple. In particular, the notion of minimal parabolic subalgebras and subgroups coincides with the following notion of Borel subalgebras and Borel subgroups.

**Definition 5.1.** A subalgebra \( \mathfrak{b} \subset \mathfrak{u}^C \) is called a Borel subalgebra if \( \mathfrak{b} \) is a maximal solvable subalgebra of \( \mathfrak{u}^C \). A subgroup \( B \subset U^C \) is called a Borel subgroup if the Lie algebra of \( B \) is a Borel subalgebra of \( \mathfrak{u}^C \).

Fixing a maximal \( \theta \)-stable Cartan subgroup \( S \) of \( U^C \) with Lie algebra \( \mathfrak{s} \), every Borel subalgebra is conjugated to

\[
\mathfrak{b} = \mathfrak{s} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{u}^C_{\alpha},
\]

where \( \Delta^+ \) is a set of positive roots in the root system \( \Delta(\mathfrak{u}^C, \mathfrak{s}) \). Let \( \Pi \) denote the set of simple roots in \( \Delta^+ \). Then every parabolic subalgebra containing \( \mathfrak{b} \) is given by a subset \( \Pi' \) of \( \Pi \) in the following way. Defining

\[
\Delta_{\Pi'} := \text{span}_\mathbb{Z}(\Pi') \cap \Delta(\mathfrak{u}^C, \mathfrak{s}),
\]

we get a parabolic subalgebra

\[
\mathfrak{q} := \mathfrak{s} \oplus \bigoplus_{\beta \in \Delta_{\Pi'}} \mathfrak{u}^C_{\beta} \oplus \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_{\Pi'}} \mathfrak{u}^C_{\alpha}.
\]
This construction also defines a decomposition of the parabolic subalgebra $q$, given by

$$q = l \oplus r,$$

where $l = s \oplus \bigoplus_{\alpha \in \Delta^{\prime}} u^C_\alpha$ and $r = \bigoplus_{\alpha \in \Delta^{+} \setminus \Delta^{\prime}} u^C_\alpha$,

which is called the Levi decomposition of $q$.

Let $Q$ be the parabolic subgroup corresponding to $q$. Then $Q$ is the semidirect product of $L$ and $R$, where $R$ is the analytic group $\exp(r)$ and $L$ is the connected reductive group with Lie algebra $l$. This defines a Levi decomposition on the group level. We call $R$ the unipotent radical and $L$ the Levi factor of $Q$.

Since we have chosen $s$ to be stable under the involution $\theta$, we get an action of $\theta$ on the root system $\Delta(u^C,s)$ given by $\theta(\alpha) = -\alpha$ for $\alpha \in \Delta(u^C,s)$. By definition, the Lie algebra $l$ of the Levi factor $L$ is stable under this action and we get a decomposition of $u^C$ of the form

$$u^C = \tilde{r} \oplus l \oplus r,$$

where $\tilde{r} := \theta(r) = \bigoplus_{\alpha \in (-\Delta^{+}) \setminus \Delta^{\prime}} u^C_\alpha$.

**Definition 5.2.** A complex flag manifold is a homogeneous space $Z = U^C/Q$ where $Q$ is a parabolic subgroup of $U^C$.

Giving $U^C$ the canonical algebraic structure, the parabolic subgroup $Q$ is an algebraic subgroup of $U^C$ and the normalizer of $Q$ in $U^C$ coincides with $Q$. In particular, $U^C/Q$ is a projective variety.

Since $N_{U^C}(Q) = Q$, one can identify $Z = U^C/Q$ with the $U^C$-conjugacy class of the parabolic subgroup $Q$. So for every $z \in Z$ we get a corresponding parabolic subgroup $Q_z$ of $U^C$. Let $z_0$ denote the base point in $Z$ which corresponds to the parabolic subgroup $Q$ under this identification.

Let $G_0$ be a real form of $U^C$. Then $G_0$ acts on $Z = U^C/Q$ and we have the following statements about the $G_0$-orbits in $Z$.

**Lemma 5.1.** There are only finitely many $G_0$-orbits in $Z$.

If $G_0 = U$, this has the following important consequence.

**Corollary 5.1.** The maximal compact subgroup $U$ of $U^C$ acts transitively on $Z$.

So $Z$ coincides with the quotient $U/L_U$, where $L_U$ denotes the isotropy subgroup of $U$ at the base point $z_0$. We want to compute this subgroup $L_U$ more explicitly.
Lemma 5.2. The subgroup $L_U$ coincides with $U \cap L$.

Proof. By definition, $L_U$ is given by $U \cap Q$. Since $U$ is the fixed point set of the involution $\theta$, the elements in $L_U$ are the fixed points of $\theta$ in $Q$. But the intersection of $Q$ and $\theta Q$ is by construction the Levi factor $L$. Therefore, the subgroup $U \cap Q$ coincides with the group $U \cap L$.

As mentioned above, we want to identify $Z = U^C/Q$ with a coadjoint orbit of $U$ in $u^*$. For this we need an element $\beta$ in $u^*$ such that the centralizer $Z_{U^C}(\beta)$ of $\beta$ in $U$ coincides with the subgroup $L_U$ of $U$. In particular, $\beta$ has to lie in the center of $(u \cap l)^*$. We construct this element $\beta$ as follows.

Let $\Pi = \{\alpha_1, ..., \alpha_n\}$ denote the set of simple roots in $\Delta^+$ and let $\Pi' \subset \Pi$ denote the defining subset for the parabolic subgroup $Q$ given above. After relabeling the $\alpha_i$ we can assume that $\Pi \setminus \Pi'$ is given by the elements $\alpha_1, ..., \alpha_k$. For an element $\zeta \in s_u := s \cap u$ we have
\[-\alpha_i(\zeta) = \theta(\alpha_i)(\zeta)) = \alpha_i(\theta(\zeta)) = \alpha_i(\zeta) \in \Pi'.\]

Therefore, the $\alpha_i$ restrict to elements in $(is_u)^*$ which can be considered as a real subspace of $(u^c)^*$. Now let $\beta$ be an element in the cone
\[C_Q := \{\alpha \in s_u^* \mid \alpha = \sum_{j=1}^k i c_j \cdot \alpha_j, c_j \in \mathbb{R}^+\}.
\]

Then by definition, $\beta_{u^c} = s \oplus \bigoplus_{\alpha \in \Delta_{u^c}} u_{\alpha}^c = l$ and therefore, $\beta_u = u \cap l$. The group $L$ is by definition connected and $Z_{U^c}(\beta)$ is connected as the centralizer of the torus $\exp(C \cdot \beta)$ in a connected algebraic Lie group. So we get $Z_{U^c}(\beta) = L_U$.

We obtain the following identification of flag manifolds with coadjoint orbits
\[Z = U^C/Q = U/L_U \rightarrow U \cdot \beta =: \mathcal{O}, \quad z = gQg^{-1} = uQu^{-1} \rightarrow \text{Ad}(u)\beta =: \beta_z.
\]

We want to use this identification to define a Kähler form on $U^C/Q$ such that the above map is the moment map with respect to this structure. Note that there is a natural symplectic form on the coadjoint orbit $\mathcal{O}$, called the Kirilov form. It is given by
\[\omega_{\beta_z}(\xi_{\mathcal{O}}(\beta_z), \nu_{\mathcal{O}}(\beta_z)) = \beta_z([\xi, \nu]).\]

Here $\xi$ and $\nu$ are elements in $u$ and $\xi_{\mathcal{O}}$ and $\nu_{\mathcal{O}}$ denote the corresponding vector fields on the coadjoint orbit $\mathcal{O}$. To define the induced symplectic form on $Z$, we need an isomorphism between the tangent spaces $T_zZ = u^C/\mathfrak{q}_z$ and $T_{\beta_z}\mathcal{O} = u/(u \cap l_z)$. Here $\mathfrak{q}_z$ denotes the Lie algebra of $Q_z = uQu^{-1}$ and $l_z$ the Lie algebra of the Levi factor $L_z := uLu^{-1}$ in $Q_z$. Using the decomposition
\[u^C = \tilde{r}_z \oplus l_z \oplus r_z\]
of $u^C$, where $r_z := \text{Ad}(u)\mathfrak{r}$ is the Lie algebra of the unipotent radical $R_z := uRu^{-1}$ in $Q_z$ and $\tilde{r}_z$ is given by $\theta(r_z) = \text{Ad}(u)\tilde{r}$, we get the following.

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**Lemma 5.3.** The mapping $I_z : \mathfrak{u}^C / q_z \rightarrow \mathfrak{u} / (\mathfrak{u} \cap I_z)$ defined by

$$I_z(\zeta + q_z) := (\zeta_z + \theta \zeta_z) + (\mathfrak{u} \cap I_z) = (\zeta_z + \zeta_z) + (\mathfrak{u} \cap I_z)$$

is the inverse of the mapping $\mathfrak{u} / (\mathfrak{u} \cap I_z) \rightarrow \mathfrak{u}^C / q_z$ induced by the inclusion $\mathfrak{u} \subset \mathfrak{u}^C$. Here $\zeta_z$ denotes the projection of $\zeta$ onto the component $\tilde{r} z$ of the above decomposition.

Especially, for an element $\xi$ in $\mathfrak{u}^C$ the two vector fields $\xi Z$ and $(\xi_z + \theta \xi_z) Z$ coincide, where $\xi_z + \theta \xi_z$ is an element in $\mathfrak{u}$.

Let $\xi$ and $\nu$ be elements in $\mathfrak{u}^C$. Then one can define a symplectic form on $Z$ by

$$\omega_z(\xi Z(z), \nu Z(z)) := (\beta_z([\xi_z, \theta \nu_z]), [\theta(\xi_z), \nu_z])$$

The last equation follows from the fact that $\beta_z(\xi)$ vanishes for $\xi \in \mathfrak{r} z$ or $\xi \in \tilde{r} z$ by the construction of $\beta$. This calculation also shows that $\omega$ is invariant with respect to the complex structure $J$ on $T\mathcal{O}$ which is induced by the isomorphism $I_z$ and can be written as follows

$$\omega_z(\xi_z + \zeta_z) + (\mathfrak{u} \cap I_z) \xrightarrow{J_z} (i\xi_z - i\zeta_z) + (\mathfrak{u} \cap I_z).$$

Therefore, one can define a $U$-invariant pseudo-Kähler form $\omega$ on $Z$ given by

$$\omega_z(\xi Z(z), \nu Z(z)) := (J \xi Z(z), J \nu Z(z)) + i(\xi Z(z), \nu Z(z)).$$

for $\xi, \nu \in \mathfrak{u}^C$. By the choice of $\beta$, the symmetric part is positive and we get the following.

**Proposition 5.1.** The flag manifold $Z$ admits a $U$-invariant Kähler form given by

$$\omega_z(\xi Z(z), \nu Z(z)) = 2i \cdot \beta_z([\xi_z, \theta \nu_z]),$$

where $z \in Z$ and $\xi, \nu \in \mathfrak{u}^C$. The moment map corresponding to this Kähler form is then given by

$$\mu : Z = U^C/Q \rightarrow \mathcal{O} = U \cdot \beta, \quad z \mapsto \beta_z.$$

By the following proposition, these are the only possible moment maps for complex flag manifolds.

**Proposition 5.2.** The $U$-invariant Kähler metrics on $Z = U^C/Q$ compatible with the canonical complex structure are in 1-1 correspondence with the elements of the cone $C_Q$ via the above construction.
On the other hand, every coadjoint $U$-orbit in $u^*$ can be realized as the image of a complex flag manifold under such a moment map. Let $\alpha$ be an element in $u \simeq u^*$ and define

$$Q_-(\alpha) := \{ g \in U^C \mid \lim_{t \to -\infty} \exp(it\alpha) \cdot g \cdot \exp(-it\alpha) \text{ exists in } U^C \}.$$ 

Then $Q_-(\alpha)$ is a parabolic subgroup of $U^C$ and $\alpha$ is an element in the cone $C_{Q_-}(\alpha)$. So the identification $U^C/Q_-(\alpha) \simeq U \cdot \alpha$ is a moment map for $U^C/Q_-(\alpha)$. For details see [Stö99].

### 5.2 Shifting of the map $\mu_{ip}$

Let us first recall what we mean by shifting of a moment map. Let $\mu: X \to u^*$ be the moment map given by the Hamiltonian action of the compact Lie group $U$ on a compact Kähler manifold $X$. Then shifting with respect to an element $\beta$ in $u^*$ is defined as follows.

The coadjoint orbit $U \cdot \beta \subset u^*$ can be identified with the complex flag manifold $Z = U^C/Q_-(\beta)$. Taking the conjugate complex structure on $Z$, the negative of the Kirilov form is a Kähler form on $U \cdot \beta$. The corresponding moment map on $Z \simeq U \cdot \beta$ is then given by $-\text{id}(U \cdot \beta)$. Let $U$ act diagonally on $X \times U \cdot \beta$. This gives a moment map

$$\mu_\beta: X \times U \cdot \beta \to u^*, \quad (x, \xi) \mapsto \mu(x) - \xi$$

which we call the shifted moment map with respect to $\beta$. In particular, $\beta$ is contained in the image of $\mu$ if and only if $0$ is contained in the image of $\mu_\beta$. Shifting is a useful tool to generalize properties of the zero fiber to general fibers.

We want to generalize this method to the map $\mu_{ip}$. So let $\beta$ be a point in $ip^*$. By the above considerations we get an induced $U^C$-action on the coadjoint orbit $U \cdot \beta \subset u^*$ by identifying it with the complex flag manifold $U^C/Q_-(\beta)$. In particular, we can consider the $G$-orbit through $\beta$ with respect to this induced action. We want to show the following.

**Proposition 5.3.** For $\beta \in ip^*$ we have $G \cdot \beta = K \cdot \beta$.

To prove this proposition note that we can again assume that $U^C$ is semisimple. In particular, the compact form $\bar{U}$ is also semisimple and we can use the following proposition which is proven in [HS05b].

**Proposition 5.4.** Let $U$ be a semisimple compact Lie group and let $G$ be a connected Zariski dense subgroup of $U^C$ which is compatible with the Cartan decomposition of $U^C$. Then there exist two compact connected Lie subgroups $U_0$ and $U_1$ of $U$ which centralize each other, such that
1. \( U^C = U_0^C \cdot U_1^C \) and the intersection \( U_0^C \cap U_1^C \) is a finite subgroup of the center of \( U \),

2. \( G = G_0 \cdot U_1^C \), where \( G_0 \) is a real form of \( U_0^C \) which is compatible with the Cartan decomposition of \( U_0^C \).

Using the notation of this proposition, the coadjoint orbit \( U \cdot \beta \) can be biholomorphically and \( U \)-equivariantly identified with \( U_0 \cdot \beta_0 \times U_1 \cdot \beta_1 \). Here \( \beta = \beta_0 + \beta_1 \) denotes the decomposition of \( \beta \) with respect to the decomposition \( u^C = u_0^C \oplus u_1^C \), where \( u_0^C \) and \( u_1^C \) are the Lie algebras of \( U_0^C \) and \( U_1^C \), respectively. Moreover, we can again identify \( U_1 \cdot \beta_1 \) with the complex flag manifold \( U_1^C / Q^- (\beta_1) \) which implies that the statement of Proposition 5.3 holds for the second component of the product \( U_0 \cdot \beta_0 \times U_1 \cdot \beta_1 \).

So without losing generality we can assume \( U = U_0 \), so that \( G \) is a real form of \( U^C \).

Proof of Proposition 5.3. Let \( \sigma : u^C \to u^C \) be the antiholomorphic involution which defines the real form \( g \) in \( u^C \). Since \( \beta \) is contained in \( ip^* \), the Lie algebra \( q \) of \( Q^- (\beta) \) is stable under this involution. Therefore, we get an involution on the tangent space \( T_\beta (U \cdot \beta) \cong u^C / q \). Note that the tangent space \( T_\beta (U \cdot \beta) \) can also be written in the form

\[
T_\beta (U \cdot \beta) = u^C \cdot \beta = u \cdot \beta = \mathfrak{t} \cdot \beta + ip \cdot \beta = i\mathfrak{t} \cdot \beta + p \cdot \beta,
\]

where \( m \cdot \beta := \{ \xi \mid \mathcal{U} \cdot \beta (x) \mid \xi \in m \} \) for any subspace \( m \) of \( u^C \). Since we have

\[
\sigma |_{\mathfrak{g} \cdot \beta} = \text{id}_{\mathfrak{g} \cdot \beta} \quad \text{and} \quad \sigma |_{ip \cdot \beta} = -\text{id}_{ip \cdot \beta},
\]

the subspaces \( \mathfrak{t} \cdot \beta \) and \( p \cdot \beta \) both coincide with the +1 eigenspace of \( \sigma \) on \( T_\beta (U \cdot \beta) \) and must therefore coincide. This shows \( T_\beta (G \cdot \beta) = T_\beta (K \cdot \beta) \). Moreover, the orbit \( G \cdot \beta \) coincides with the orbit \( K \cdot \beta \).

This leads to the notion of shifting of the map \( \mu_{ip} : X \to ip^* \) with respect to a point \( \beta \in ip^* \).

By Proposition 5.3, the manifold \( X \times U \cdot \beta \) contains the closed and \( G \)-stable subset \( X \times K \cdot \beta \). Therefore, the \( ip \)-component of the moment map \( \mu_{\beta} \), defined above, can be restricted to this subset and we get a map

\[
\mu_{ip, \beta} : X \times K \cdot \beta \to ip^*, \quad (x, \xi) \mapsto \mu_{ip}(x) - \xi
\]

which we call the shifting of \( \mu_{ip} \) with respect to \( \beta \). As above, \( \beta \) is contained in the image of \( \mu_{ip} \) if and only if 0 is contained in the image of \( \mu_{ip, \beta} \).
5.3 Projective embeddings of complex flag manifolds

As we saw above, every element $\beta$ in $C_Q$ defines a Kähler structure on $U^C/Q$ such that the moment map is given by the identification $U^C/Q \simeq U \cdot \beta$. In this section we want to discuss the special situation where $\beta$ is an integral element of the torus $s_u^* = h^* \oplus i\alpha^* \subset s^*$. We use the following notion of integrability.

**Definition 5.3.** If $\chi: T \to S^1$ is a character of a torus $T$, then we have the following commuting diagram:

$$
\begin{array}{ccc}
T & \xrightarrow{\chi} & S^1 \\
\exp & \downarrow & \downarrow e \\
\exp(t) & \xrightarrow{\lambda} & \mathbb{R}
\end{array}
$$

where $e(t) := e^{2\pi it}$ and $\lambda = \chi_* \in t^*$ is the derivative of $\chi$. Elements $\lambda \in t^*$ of this form will be called integral. An element $\beta \in t^*$ is called rational if $\beta = \frac{1}{n} \cdot \alpha$ for some integral element $\alpha \in t^*$ and some $n \in \mathbb{N}$.

Given an integral functional $\beta$ in $s_u^*$, we get a character

$$
\chi_\beta: S_U \to S^1, \quad \chi_\beta(\exp(H)) = e^{2\pi i \beta(H)},
$$

where $S_U$ denotes the maximal torus in $U$ with Lie algebra $s_u$. Such an integral functional $\beta$ extends uniquely to a $C$-linear functional $\beta \in s^*$. Let $\chi_\beta: S \to \mathbb{C}^*$ denote the associated character of $S$. Extending this map trivially onto the parabolic subgroup $Q := Q_-(\beta)$ containing $S$, we get a $U^C$-homogeneous line bundle

$$
L^\beta = U^C \times_{\chi_\beta} \mathbb{C}
$$

over $U^C/Q$. Let $\Gamma(U^C/Q, L^\beta)$ denote the space of sections of $L^\beta$. We define the map

$$
\varphi_\beta: U^C/Q \to \mathbb{P}(\Gamma(U^C/Q, L^\beta)^*), \quad z \mapsto \pi(\{f \in \Gamma(U^C/Q, L^\beta)^* \mid f(H(z)) = 0\}),
$$

where

$$
H(z) := \{\sigma \in \Gamma(U^C/Q, L^\beta) \mid \sigma(z) = 0\}.
$$

This map is a well defined $U^C$-equivariant holomorphic map such that

$$
U^C \cdot \varphi_\beta(z_0) \simeq U^C/Q',
$$

where $Q'$ is a parabolic subgroup of $U^C$ containing $Q_{z_0} = Q$. By the theorem of Borel and Weil (see e.g. [Akh95] and [Huc01]), the space $\Gamma(U^C/Q, L^\beta)^*$ is an irreducible $U^C$-representation space with highest weight $\beta$ and the projective space $\mathbb{P}(\Gamma(U^C/Q, L^\beta)^*)$ contains exactly one complex $U$-orbit.
Therefore, the orbit
\[ U^C \cdot \varphi_\beta(z_0) = U \cdot \varphi_\beta(z_0) \]
which we constructed above is the only complex \( U \)-orbit in \( \mathbb{P}(\Gamma(U^C/Q, L^\beta)^*) \).

On the other hand let \( v_\beta \) be a highest weight vector in \( \Gamma(U^C/Q, L^\beta)^* \) and let \( \pi \) be the canonical projection
\[ \pi : \Gamma(U^C/Q, L^\beta)^* \to \mathbb{P}(\Gamma(U^C/Q, L^\beta)^*) . \]
Then the stabilizer of the point
\[ [v_\beta] = \pi(v_\beta) \]
contains a Borel subgroup of \( U^C \). Therefore, the orbit \( U^C \cdot \pi(v_\beta) \) is a complex \( U \)-orbit in \( \mathbb{P}(\Gamma(U^C/Q, L^\beta)^*) \) and we have
\[ U^C \cdot \pi(v_\beta) = U^C \cdot \varphi_\beta(z_0) . \]

Since every irreducible \( U \)-representation \( \rho : U \to GL_C(V) \) defines a unitary structure \( <,> \) on \( V \) which is unique up to a scalar multiple, we get a unitary structure on the space \( \Gamma(U^C/Q, L^\beta)^* \) and a moment map
\[ \mu : \mathbb{P}(\Gamma(U^C/Q, L^\beta)^*) \to \mathfrak{s}^* , \quad [v] \mapsto \left( \xi \mapsto \frac{\langle \xi \cdot v, v \rangle}{\|v\|^2} \right) \]
with respect to the induced Fubini-Study metric on \( \mathbb{P}(\Gamma(U^C/Q, L^\beta)^*) \). We want to compute \( \mu([v_\beta]) \). Note that the representation space \( V \) decomposes as the orthogonal sum of weight spaces \( V_\lambda \) with respect to \( S \). In particular, \( \xi \cdot V_\lambda \subset V_{\lambda + \alpha} \) for any element \( \xi \) in \( \mathfrak{u}^C_\alpha \). Therefore, orthogonality of the weight space decomposition implies
\[ \mu^\xi([v_\beta]) = 0 \] for all elements \( \xi \in \mathfrak{u}^C_\alpha \). Consequently, \( \mu([v_\beta]) \) is an element in \( \mathfrak{s}^* \). For \( \xi \in \mathfrak{s} \) we have
\[ \mu^\xi([v_\beta]) = \frac{1}{i} \frac{\langle \xi \cdot v_\beta, v_\beta \rangle}{\|v_\beta\|^2} = \frac{1}{i} \frac{\beta(\xi)v_\beta, v_\beta}{\|v_\beta\|^2} = \beta(\xi) . \]

Therefore, the image of \( U \cdot [v_\beta] \) under the \( U \)-equivariant moment map is the coadjoint orbit \( U \cdot \beta \). In particular, we have \( Q' = Q \) and the map \( \varphi_\beta \) gives an embedding of \( U^C/Q \) into the projective space \( \mathbb{P}(\Gamma(U^C/Q, L^\beta)^*) \) such that \( \mu(\varphi_\beta(U^C/Q)) = U \cdot \beta \).
6 The convexity theorem

Let $X$ be a Kähler $U^\mathbb{C}$-manifold with a $U$-equivariant moment map $\mu : X \to u^*$. Using the results of the previous sections we can formulate a sufficient condition on a subset $Y \subset X$ for the set $\mu_{ip}(Y) \cap i\alpha^*_+ \beta$ being a convex polytope. Here $i\alpha_+$ again denotes the positive Weyl chamber in the maximal subalgebra $i\alpha$ of $ip$.

**Theorem 6.1.** Let $Y$ be a closed $G$-stable subset of $X$ such that the set

$$
S_G(M_{ip,\beta}(\alpha)) := \{(y, \xi) \in Y \times K \cdot \beta \mid G \cdot (y, \xi) \cap (\mu_{ip,\beta})^{-1}(\alpha) \neq \emptyset\}
$$

is either open and dense in $Y \times K \cdot \beta$ or empty for every $\alpha, \beta \in i\alpha^*_+$. Then the set $\mu_{ip}(Y) \cap i\alpha^*_+$ is a convex polytope.

Since $Y$ is a closed $G$-stable subset of $X$, Theorem 3.3 implies that $\mu_{ip}(Y) \cap i\alpha^*_+$ is a finite union of convex polytopes. To prove that it is one convex polytope we first need the following geometric statement.

**Lemma 6.1.** ([Kir84b]) Let $D$ be a finite union of convex polytopes in $i\alpha^*_+$ which is non convex. Then for any sufficiently small $\epsilon > 0$ there exists a point $\beta \in i\alpha^*_+$ such that the closed ball of radius $\epsilon$ and center $\beta$ meets $D$ in precisely two points $\alpha_1$ and $\alpha_2$ neither of which lies in the interior of the ball.

Using this geometric property of non convex unions of convex polytopes, we can prove the above convexity theorem.

**Proof of Theorem 6.1.** Let us assume that $\mu_{ip}(Y) \cap i\alpha^*_+$ is non convex. Since $Y$ is closed and $G$-stable, this set is a finite union of convex polytopes an we can apply the previous lemma. So there exists an $\epsilon > 0$ and a point $\beta \in i\alpha^*_+$ such that the closed ball of radius $\epsilon$ and center $\beta$ meets $\mu_{ip}(Y) \cap i\alpha^*_+$ in precisely two points $\alpha_1$ and $\alpha_2$ on the boundary of this ball.

Using the shifting method introduced in section 5.2 we can consider the shifting of $\mu_{ip}$ with respect to the point $\beta$ and obtain a map

$$
\mu_{ip,\beta} : Y \times K \cdot \beta \to ip^*, \quad (x, k \cdot \beta) \mapsto \mu_{ip}(x) - k \cdot \beta.
$$

We want to compute the global minima of $||\mu_{ip,\beta}||^2$. Since $||\mu_{ip,\beta}||^2$ is $K$-invariant, it suffices to consider points of the form $(y, \beta)$. By Lemma 2.4 a point $(y, \beta)$ is a critical point of $||\mu_{ip,\beta}||^2$ if and only if the vector field

$$
(\mu_{ip}(y) - \beta)_{(y \times K \cdot \beta)}
$$

vanishes in $(y, \beta)$. But this implies that $\mu_{ip}(y) - \beta$ is contained in the centralizer of $\beta$ in $ip$. In particular, there exists an element $k \in Z_K(\beta)$ such that $k \cdot \mu_{ip}(y) = \mu_{ip}(k \cdot y)$ is contained in $i\alpha$. Using Lemma 3.1 and the definition of the points $\alpha_1$ and $\alpha_2$, the point
\( \mu_{ip}(k \cdot y) \) coincides with one of the two points \( \alpha_1 \) and \( \alpha_2 \). Consequently, the points in the sets \( K \cdot ((\mu_{ip})^{-1}(\alpha_j) \times \beta), j = 1, 2 \), are global minima of the function \( \|\mu_{ip,\beta}\|^2 \) fulfilling the conditions of Corollary 4.2. So the intersection

\[
S_G(M_{ip,\beta}(\alpha_1)) \cap S_G(M_{ip,\beta}(\alpha_2))
\]

is empty. But both sets are open and dense in \( Y \times K \cdot \beta \) and therefore have non empty intersection which leads to a contradiction. Therefore, the set \( \mu_{ip}(Y) \cap i\alpha_+^* \) is a convex polytope.

\( \square \)
7 Consequences of the convexity theorem

In this chapter, we want to construct a class of examples which satisfies the conditions of the above convexity theorem. So let $U^C$ be a complex reductive group and $G$ a real subgroup which is compatible with the Cartan decomposition of $U^C$. Furthermore, let $V$ be a finite dimensional complex $U^C$-representation space. Then there exists a $U$-invariant Hermitian inner product on $V$ which induces the Fubini-Study metric $\omega_{FS}$ on the projective space $\mathbb{P}(V)$. Taking the induced action of $U^C$ on $\mathbb{P}(V)$, we get a $U$-equivariant moment map

$$\mu_{\mathbb{P}(V)} : \mathbb{P}(V) \to u^*$$

with respect to $\omega_{FS}$. Let $\mu_{ip}$ denote the $ip$-component of this moment map. We want to show that every closed connected $G$-stable irreducible semialgebraic subset $Y$ of $\mathbb{P}(V)$ satisfies the conditions of Theorem 6.1.

7.1 Semialgebraic geometry

Let us first introduce the notion of irreducible semialgebraic sets and some of their important properties which will be relevant in the following.

**Definition 7.1.** A subset $A \subset \mathbb{R}^n$ is called a real algebraic subset of $\mathbb{R}^n$ if it is given as the zero set of finitely many real polynomials. We call a subset $S$ of $\mathbb{R}^n$ semialgebraic if it is a finite union of sets of the form

$$\{x \in M \mid P_j(x) = 0 \text{ for } 1 \leq j \leq k \text{ and } P_j(x) > 0 \text{ for } k < j \leq m\},$$

where the $P_j$ are real polynomials on $\mathbb{R}^n$.

The following proposition gives a notion of dimension for these semialgebraic sets which, as we will see, coincides with the Krull dimension if the semialgebraic set is a real algebraic set.

**Proposition 7.1.** ([BR90]) Let $S \subset A$ be a semialgebraic subset of a real algebraic set $A \subset \mathbb{R}^n$. Then there exists a disjoint decomposition of $S$ into finitely many subsets $S_i$ such that

1. each $S_i$ is a semialgebraic real analytic locally closed submanifold of $\mathbb{R}^n$ and
2. if $S_i \cap S_j \neq \emptyset$, then $S_j$ is contained in $S_i$ and $\dim(S_j) < \dim(S_i)$.

Moreover, if there exists two decompositions of $S$ in the above sense, then the dimension of the top-dimensional subset $S_i$ is the same in both decompositions.

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We define the dimension of a semialgebraic set to be the dimension of the maximal dimensional subset $S_i$ of such a decomposition. For a semialgebraic set $S \subset \mathbb{R}^n$ we define the Zariski closure to be the smallest real algebraic subset of $\mathbb{R}^n$ containing $S$. It will be denoted by $\text{cl}(S)$. We call $S$ irreducible if the Zariski closure $\text{cl}(S)$ is an irreducible real algebraic set, i.e. $\text{cl}(S)$ cannot be written as the union of two proper real algebraic subsets.

The Krull dimension of a real algebraic set $A$ is by definition the maximum length of chains of prime ideals in $\mathcal{P}(A)$. Here $\mathcal{P}(A)$ denotes the quotient ring

$$\mathbb{R}[x_1, \ldots, x_n]/\mathcal{I}(A)$$

with

$$\mathcal{I}(A) := \{P \in \mathbb{R}[x_1, \ldots, x_n] \mid P(x) = 0 \text{ for all } x \in A\}.$$

The next theorem gives a connection between the dimension of a semialgebraic set and the Krull dimension of its Zariski closure.

**Theorem 7.1.** ([Cos00]) Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Its dimension as a semialgebraic set is equal to the Krull dimension of its Zariski closure $\text{cl}(S)$.

In particular, for real algebraic sets the two definitions of dimension coincide. Since the intersection of an irreducible real algebraic set $A$ with a real algebraic set $B$ is either $A$ itself or a proper real algebraic subset of $A$ of lower dimension (see e.g. [BR90]), we get the following statement for irreducible semialgebraic sets.

**Remark 7.1.** Let $S \subset \mathbb{R}^n$ be an irreducible semialgebraic set and $A \subset \mathbb{R}^n$ a real algebraic set. Then the intersection $A \cap S$ is either $S$ itself or an semialgebraic subset of lower dimension in $S$.

In the following, we are interested in subsets of the projective space $\mathbb{P}(V)$ for a unitary vector space $V$. Note that one can realize $\mathbb{P}(V)$ as the algebraic set

$$\{L \in \text{End}(V) \mid L = L^*, L^2 = L \text{ and } Tr(L) = 1\}$$

in the real vector space $\text{End}(V)$, where $L^*$ denotes the adjoint of $L$ with respect to the unitary structure. Here an element $[v]$ of $\mathbb{P}(V)$ is identified with the linear map which projects $V$ orthogonally onto the corresponding complex line $\mathbb{C} \cdot v$. This gives the notion of algebraic and semialgebraic subsets of $\mathbb{P}(V)$. In particular, the image under the canonical projection $\pi$ of a real algebraic subset of $V$ which is invariant under scalar multiplication is an algebraic set of $\mathbb{P}(V)$.
7.2 Semistable points

We now return to the situation mentioned at the beginning of this chapter. Let $V$ be a finite dimensional complex $U^\mathbb{C}$-representation space and let $<,>$ denote the induced $U$-invariant Hermitian inner product on $V$. We first want to compute the moment map for the induced action of $U$ on the projective space $\mathbb{P}(V)$. Therefore, we consider the function

$$\rho: V \setminus \{0\} \to \mathbb{R}, \quad z \mapsto \log(\|z\|^2),$$

where $\| \cdot \|^2$ denotes the norm function on $V$ induced by the Hermitian inner product. This function defines the form $\omega = \frac{i}{2} \partial \bar{\partial} \rho$ which satisfies $\omega = \pi^*(\omega_{FS})$ for the Fubini-Study form $\omega_{FS}$ on $\mathbb{P}(V)$. The corresponding $U$-equivariant moment map $\mu_\rho$ on $V \setminus \{0\}$ is then given by

$$\mu_\rho: V \setminus \{0\} \to u^*, \quad v \mapsto \left( \xi \mapsto \frac{1}{i} \cdot \frac{<\xi \cdot v, v>}{\|v\|^2} \right).$$

Since this moment map is invariant under scalar multiplication, it induces a moment map on the projective space $\mathbb{P}(V)$ given by

$$\mu_{\mathbb{P}(V)}: \mathbb{P}(V) \to u^*, \quad [v] \mapsto \mu_\rho(v).$$

Let $\mu_{ip}: \mathbb{P}(V) \to i^p u^*$ denote its $i^p$-component. Then we want to compute the set

$$S_G(M_{ip}) = \{ [v] \in \mathbb{P}(V) \mid G \cdot [v] \cap M_{ip} \neq \emptyset \}$$

of semistable points in $\mathbb{P}(V)$ with respect to $\mu_{ip}$. Here $M_{ip}$ again denotes the zero fiber of the map $\mu_{ip}$. We can compute the set

$$S_G(M_{ip}(V \setminus \{0\})) = \{ v \in V \setminus \{0\} \mid \overline{G \cdot v} \cap M_{ip}(V \setminus \{0\}) \neq \emptyset \}$$

of semistable points in $V \setminus \{0\}$ with respect to the $i^p$-component $(\mu_\rho)_{ip}$ of $\mu_\rho$, where $M_{ip}(V \setminus \{0\})$ denotes the zero fiber of $(\mu_\rho)_{ip}$. We want to show that $S_G(M_{ip})$ is the image of the set $S_G(M_{ip}(V \setminus \{0\}))$ under the canonical projection $\pi$.

By definition of $\mu_{\mathbb{P}(V)}$, we have $\mu_{ip}([v]) = (\mu_\rho)_{ip}(v)$ for every element $v \in V \setminus \{0\}$. Therefore, we get

$$\pi(M_{ip}(V \setminus \{0\})) = M_{ip}.$$ 

To prove that we also have equality for the sets of semistable points, we first need a moment map on the vector space $V$. Consider the norm function

$$\tilde{\rho}: V \to \mathbb{R}, \quad v \mapsto \|v\|^2$$

on $V$. This is a $U$-invariant strictly plurisubharmonic exhaustion function on $V$ which induces a Kähler form $\omega = i\partial \bar{\partial} \tilde{\rho}$ and a moment map $\mu_{\tilde{\rho}}: V \to u^*$ given by

$$\mu_{\tilde{\rho}}(v)(\xi) = -2i \cdot <\xi \cdot v, v>.$$
Let $\tilde{\mathcal{M}}_{ip}$ denote the zero fiber of the $ip$-component $\tilde{\mu}_{ip}$ of this moment map. Note that this is the set of minimal vectors for $G$ in the sense of Kempf and Ness ([KN79]).

Since $\rho$ restricted to the closure of $U^c \cdot z$ attains a minimal value for every $z \in \mathcal{V}$, we get the following.

**Lemma 7.1.** Let $S_G(\tilde{\mathcal{M}}_{ip})$ be the set of semistable points in $\mathcal{V}$ with respect to $\tilde{\mu}_{ip}$. Then we have

$$S_G(\tilde{\mathcal{M}}_{ip}) = \mathcal{V}.$$ 

Since the two moment maps $\mu_\rho$ and $\mu_{\tilde{\rho}}$ only differ by the factor $\|v\|^{-1}$ on $\mathcal{V} \setminus \{0\}$, we get

$$\tilde{\mathcal{M}}_{ip} = \mathcal{M}_{ip}(\mathcal{V} \setminus \{0\}) \cup \{0\}.$$ 

This leads to the following decomposition of the space $\mathcal{V}$.

**Lemma 7.2.** Let $N_G := \{v \in \mathcal{V} \mid 0 \in G \cdot v\}$ denote the null cone in $\mathcal{V}$ with respect to the $G$-action. Then $\mathcal{V}$ decomposes as a disjoint union of $S_G(\mathcal{M}_{ip}(\mathcal{V} \setminus \{0\}))$ and $N_G$.

**Proof.** By Lemma 7.1, we have $\mathcal{V} = S_G(\tilde{\mathcal{M}}_{ip})$. Using the above decomposition of the zero set $\tilde{\mathcal{M}}_{ip}$ we get

$$\mathcal{V} = S_G(\mathcal{M}_{ip}(\mathcal{V} \setminus \{0\})) \cup \{0\} = S_G(\mathcal{M}_{ip}(\mathcal{V} \setminus \{0\})) \cup N_G.$$ 

It remains to show that the last union is disjoint. So let us assume $y$ is contained in the intersection $S_G(\mathcal{M}_{ip}(\mathcal{V} \setminus \{0\})) \cap N_G$.

Then there exists a point $x$ in $\mathcal{M}_{ip}(\mathcal{V} \setminus \{0\}) \subset \tilde{\mathcal{M}}_{ip}$ such that the closure of the $G$-orbit through $y$ contains $x$. But $G \cdot x$ is closed in $S_G(\mathcal{M}_{ip}) = \mathcal{V}$ by Lemma 2.6. This is a contradiction since $\{0\}$ is the only closed $G$-orbit in the closed $G$-stable subset $N_G$ of $\mathcal{V}$.

$\square$

Therefore, $S_G(\mathcal{M}_{ip}(\mathcal{V} \setminus \{0\}))$ is the complement of $N_G \setminus \{0\}$ in $\mathcal{V} \setminus \{0\}$. The next lemma gives the connection to the set of semistable points in $\mathbb{P}(\mathcal{V})$.

**Lemma 7.3.** The image of the semistable points in $\mathcal{V} \setminus \{0\}$ under $\pi$ coincides with the set of semistable points in $\mathbb{P}(\mathcal{V})$, i.e.

$$\pi(S_G(\mathcal{M}_{ip}(\mathcal{V} \setminus \{0\}))) = S_G(\mathcal{M}_{ip}).$$

**Proof.** Taking an element $z$ in $S_G(\mathcal{M}_{ip}(\mathcal{V} \setminus \{0\}))$ there exists, by definition, a sequence $(g_n)_{n \in \mathbb{N}} \subset G$ such that

$$\lim_{n \to \infty} g_n \cdot z = z_0 \in \mathcal{M}_{ip}(\mathcal{V} \setminus \{0\}).$$

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Since $\pi$ is $G$-equivariant and continuous, we get
\[ \lim_{n \to \infty} g_n \cdot \pi(z) = \lim_{n \to \infty} \pi(g_n \cdot z) = \pi(z_0) \in \pi(M_{ip}(V \setminus \{0\})) = M_{ip}. \]

This proves that $\pi(S_G(M_{ip}(V \setminus \{0\}))$ is contained in $S_G(M_{ip})$. To prove the opposite inclusion we take an element $z$ in the set
\[ \pi^{-1}(S_G(M_{ip})) \setminus S_G(M_{ip}(V \setminus \{0\})). \]

By Lemma 7.2, the point $z$ is thus contained in the null cone. But the set $N_G \setminus \{0\}$ is a closed $G$-stable subset of $V \setminus \{0\}$ which is also stable under scalar multiplication. So we get
\[ \pi^{-1}(M_{ip}) \cap N_G \setminus \{0\} \neq \emptyset \]
which is a contradiction to the fact that $V$ decomposes into the disjoint union of $S_G(M_{ip}(V \setminus \{0\}))$ and $N_G$.

\[ \square \]

For a closed $G$-stable subset $Y$ of $\mathbb{P}(V)$, this gives a criterion for $M_{ip}(Y)$ being empty or not. The set $M_{ip}(Y)$ is empty if and only if $Y$ lies in the image of the null cone under $\pi$. In the next section, we want to analyze this null cone in more detail. In particular, we want to show that it is a real algebraic subset of $V$.

### 7.3 The null cone $N_G$

For the algebraicity of the null cone we use results of [Bir71], [Lun75] and [RS90]. All of them appear as special cases in [HS05a].

Using the decomposition of $G$ into its semisimple part $G_s$ and its center $Z(G)$, we first prove the following.

**Lemma 7.4.** The null cone $N_{G_s}$ is a real algebraic subset of $V$.

**Proof.** The null cone $N_{G_s}$ is by definition the equivalence class of the point 0 under the equivalence relation on $V$ given by
\[ v \sim w \quad \text{if and only if} \quad G_s \cdot v \cap G_s \cdot w \neq \emptyset. \]

The quotient $V//G_s$ with respect to this equivalence relation, which exists by the results of [Lun75], separates the closed $G_s$-orbits on $V$. Moreover, by the results of [RS90], this is a Hausdorff space. We denote the quotient map by
\[ p: V \to V//G_s, \quad v \mapsto [v]_{G_s}. \]

To prove that $N_{G_s}$ is a real algebraic subset of $V$, we first consider the representation of the complexification $G_s^C$ on the complexified space $V^C$. Doing the same construction as above using the Zariski topology on $V^C$, we get a quotient
\[ p^C: V^C \to (V^C)//G_s^C, \quad v \mapsto [v]_{G_s^C} \]

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which parameterizes the Zariski-closed $G^C_s$-orbits in $V^C$. Note that in the complex case the quotient $V^C/G^C_s$ can be realized as the image of the map

$$F = (F_1, \ldots, F_d) : V^C \to \mathbb{C}^d,$$

where the functions $F_j$ generates the algebra $\mathbb{C}[V^C]^{G^C_s}$ of $G^C_s$-invariant polynomials on $V^C$. Without loosing generality we can assume that $F_j(0) = 0$ for all generators $F_j$.

By a result of [Bir71], the $G^C_s$-orbit through a point $x$ on a closed $G_s$-orbit is Zariski-closed in $V^C$. Consequently, two closed $G_s$-orbits have the same image under $p_C$ if and only if they are contained in the same closed $G^C_s$-orbit. Since $\{0\} \subset V^C$ is a closed $G^C_s$-orbit in $V^C$, the image of every closed $G_s$-orbit through a point $v \in V \setminus \{0\}$ under $p_C$ is different from $p_C(0)$. So there exists a $G^C_s$-invariant polynomial $F_j$ from the list of generators of the algebra $\mathbb{C}[V^C]^{G^C_s}$ such that

$$0 \neq F_j|_{G_s \cdot v}.$$

Since $G_s$ is Zariski dense in $G^C_s$, we obtain

$$\mathbb{C}[V^C]^{G^C_s} = \mathbb{C}[V^C]^{G_s}.$$

But we also have

$$\mathbb{C}[V^C]^{G_s} = (\mathbb{R}[V] \otimes \mathbb{C})^{G_s} = \mathbb{R}[V]^{G_s} \otimes \mathbb{R} \otimes \mathbb{C},$$

which shows that there exists real polynomials $f_1, \ldots, f_d$ which generate the algebra $\mathbb{R}[V]^{G_s}$ and whose canonical extensions to complex valued polynomials on $V^C$ generate the algebra $\mathbb{C}[V^C]^{G^C_s}$. Therefore, without losing generality, we can assume that the functions $F_j$ are the extensions of the functions $f_j$. In particular, we have

$$0 \neq f_j|_{G_s \cdot v}$$

which shows that the null cone is given as the real algebraic subset

$$\{v \in V \mid f_j(v) = 0, \; j = 1, \ldots, d\}$$

of $V$.

Since $Z(G)$ is abelian and $V$ is a complex $Z(G)$-representation space, we have a decomposition of $V$ into the direct sum of subspaces $V_j$ such that $Z(G)$ acts on $V_j$ by a real character $\lambda_j$. Since $G_s$ and $Z(G)$ commute, these subspaces are stable under $G_s$. In particular, we have

$$\mathbb{R}[V]^{G_s} = \bigotimes \mathbb{R}[V_j]^{G_s}.$$

Each factor $\mathbb{R}[V_j]^{G_s}$ has finitely many generators which can be chosen to be homogeneous polynomials. Let $f : V \to \mathbb{R}^k$ be the polynomial map which is given by all
these generators. Then $f$ is invariant with respect to $G_s$ and equivariant with respect to $Z(G)$. Here the action of $Z(G)$ on $\mathbb{R}^k$ is given by the action on each generator.

In particular, we can consider the null cone with respect to the $Z(G)$-action on $\mathbb{R}^k$. This null cone is a finite union of linear subspaces $H_j \subset \mathbb{R}^k$ (see [HS05a] Corollary 15.5). Therefore, the preimage of this null cone under $f$ is an algebraic subset of $V$ which we call $N'$. By construction we have $N_G \subset N'$. So the algebraicity of $N_G$ follows from the next lemma.

**Lemma 7.5.** The null cone $N_G$ coincides with the algebraic set $N'$.

To prove this lemma we need some more preparations. Let $G_U$ denote the maximal compact subgroup of $G_s^\mathbb{C}$. Then we consider a $G_U$-invariant positive definite Hermitian $H$ form on $V^\mathbb{C}$ such that the alternating part vanishes on $V$. Such a form exists by [RS90]. Let $\rho_C$ denote the norm function with respect to this form. Then $\rho_C$ induces a Kähler form on $V^\mathbb{C}$ such that the corresponding moment map is given by

$$\mu_{\rho_C} : V^\mathbb{C} \to g^*_C, \quad v \mapsto (\xi \mapsto -2i \cdot H(\xi \cdot v, v)).$$

Let $\mathcal{M}$ denotes its zero fiber. By construction we have $\tilde{\mathcal{M}}_{ip} = \mathcal{M} \cap V$. This leads to a proof of the above lemma.

**Proof of Lemma 7.5.** Let $v$ be a point in $N'$. By the definition of $N'$ there exists a sequence $(g_n)_{n \in \mathbb{N}} \subset Z(G)$ such that

$$\lim_{n \to \infty} g_n \cdot f(v) = 0.$$  

For every point $g_n \cdot f(v)$ let $\alpha_n$ be a point in a closed $G_s$-orbit in the fiber of $g_n \cdot f(v)$ with respect to $f$. By [RS90] every closed $G_s$-orbit intersects the set $\mathcal{M}_{ip}$. Therefore, we may choose $\alpha_n$ in this set $\tilde{\mathcal{M}}_{ip}$. Using again a result of [RS90] we get that the map $F$ which is given by the complex extension of $f$ to $V^\mathbb{C}$ is a proper map when restricted to $\mathcal{M}$. Using the fact that $\tilde{\mathcal{M}}_{ip} = \mathcal{M} \cap V$ the restriction of $F$ to $\tilde{\mathcal{M}}_{ip}$ which coincides with $f$ restricted to $\tilde{\mathcal{M}}_{ip}$ is also proper. But this implies that the above sequence $(\alpha_n)_{n \in \mathbb{N}}$ has a convergent subsequence. Let $\alpha$ denote the limit of this subsequence. Then $f(\alpha) = 0$ which by the results for the semisimple part implies that $\alpha$ is contained in $N_G$. But this shows $v \in N_G$.

This has the following consequence for the set of semistable points in $Y$ if $\mathcal{M}_{ip}(Y)$ is non empty and $Y$ is a closed connected $G$-stable irreducible semialgebraic subset of $\mathbb{P}(V)$.
Proposition 7.2. Assume that $M_{ip}(Y)$ is non empty for a closed connected $G$-stable irreducible semialgebraic subset $Y$ of $\mathbb{P}(V)$. Then $S_{G}(M_{ip}(Y))$ is an open and dense subset of $Y$.

Proof. Since $Y$ is closed and stable under the group $G$, Lemma 7.3 implies that the semistable points in $Y$ are given by the intersection $Y \cap \pi(V \setminus N_G)$. Since the null cone is an algebraic subset in $V$, the intersection of $\pi(N_G)$ with the irreducible semialgebraic set $Y$ is by Remark 7.1 either $Y$ itself or a proper semialgebraic subset of lower dimension in $Y$. Since we have assumed that $M_{ip}(Y)$ is non empty, the intersection $Y \cap \pi(N_G)$ is a proper semialgebraic subset of lower dimension in $Y$. In particular, its complement which is the set of semistable points in $Y$, is open and dense in $Y$.

To apply Theorem 6.1 to this subset $Y$, we also need statements about semistable points with respect to the shifted map $\mu_{ip,\beta}: Y \times K \cdot \beta \to \mathfrak{i} \mathfrak{p}^*$. Since there is in general no symplectic embedding of a coadjoint orbit $U \cdot \beta$ into a projective space $\mathbb{P}(W)$, equipped with the Fubini-Study metric coming from a $U$-invariant Hermitian form on $W$, the set $Y \times K \cdot \beta$ is in general not contained in the class of examples we are considering in this chapter. Therefore, we have to introduce a slight modification of the shifting method introduced in section 5.2.

7.4 Shifting with respect to rational points

By the results of section 5.3, there exists a symplectic embedding for integral elements $\beta$ in $s_u^* = \mathfrak{h}^* \oplus \mathfrak{a}^*$. Therefore, the shifting method works perfectly for integral points $\beta \in \mathfrak{i} \mathfrak{a}^*$. To define a version of shifting which also works for rational points, note that the moment map on the projective space has the following useful property.

Let $V$ and $W$ be two finite dimensional $U^\mathbb{C}$-representation spaces and let $\mu_{\mathbb{P}(V)}$ and $\mu_{\mathbb{P}(W)}$ denote the moment maps with respect to the Fubini-Study metrics on the corresponding projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(W)$, respectively. We consider the embedding

$$\mathbb{P}(V) \times \mathbb{P}(W) \hookrightarrow \mathbb{P}(V \otimes W), \quad ([v], [w]) \mapsto [v \otimes w]$$

which is called the Segre embedding. Then the unitary structures for $V$ and $W$ induce a unitary structure on the $U^\mathbb{C}$-representation space $V \otimes W$. This defines a moment map $\mu_{\mathbb{P}(V\otimes W)}$ on $\mathbb{P}(V \otimes W)$ and the following lemma holds.

Lemma 7.6. The above defined moment maps fulfill the equation

$$\mu_{\mathbb{P}(V\otimes W)}(x \otimes y) = \mu_{\mathbb{P}(V)}(x) + \mu_{\mathbb{P}(W)}(y).$$
This allows us to define a shifting method for rational points in the following way. Let \( \alpha \) be a rational element in \( i\mathbb{A}^* \). Taking the unique minimal integral element \( \beta \) on the half line \( \mathbb{R}^+ \cdot \alpha \), we get \( \alpha = \frac{p}{q} \cdot \beta \) for coprime natural numbers \( p \) and \( q \). Let \( U \cdot [v_{-\beta}] \) denote the unique complex \( U \)-orbit in the projective space

\[
\mathbb{P}(\Gamma_{-\beta}) := \mathbb{P}(\Gamma(G/Q_{-\beta}, L_{-\beta}^*))
\]

given in section 5.3. Then the moment map \( \mu_{\mathbb{P}(\Gamma_{-\beta})} \) restricted to this orbit is given by

\[
\mu_{\mathbb{P}(\Gamma_{-\beta})}([v]) = \mu_{\mathbb{P}(\Gamma_{-\beta})}(u \cdot [v_{\beta}]) = -u \cdot \beta =: -\beta_{[v]}.
\]

Since \( \alpha \) and \( \beta \) are related by the fixed numbers \( p \) and \( q \), we can define a unique embedding

\[
\Phi_\alpha : Y \times U \cdot [v_{-\beta}] \subset \mathbb{P}(V) \times \mathbb{P}(\Gamma_{-\beta}) \hookrightarrow \mathbb{P}(V^\otimes q \otimes (\Gamma_{-\beta})^\otimes p)
\]

using the above Segre embedding. To shorten notation, we write \( V_\alpha \) instead of the tensor product \( V^\otimes q \otimes (\Gamma_{-\beta})^\otimes p \). By Lemma 7.6, we get a moment map

\[
\mu_{\mathbb{P}(V_\alpha)} : \mathbb{P}(V_\alpha) \to u^*
\]

such that the restriction to \( \Phi_\alpha(Y \times U \cdot [v_{-\beta}]) \) is given by

\[
\mu_\alpha : \Phi_\alpha(Y \times U \cdot [v_{-\beta}]) \to u^*, \quad \Phi_\alpha(([x], [v])) \mapsto q \cdot \mu_{\mathbb{P}(V)}([x]) - p \cdot \beta_{[v]}.
\]

As mentioned above, the \( G \)-orbit \( G \cdot [v_{-\beta}] \) is closed and coincides with the \( K \)-orbit through this point. We denote by \( Y_\alpha \) the image of \( Y \times K \cdot [v_{-\beta}] \) under \( \Phi_\alpha \). Then we have a \( K \)-equivariant map

\[
\mu_{ip,\alpha} : Y_\alpha \to i\mathbb{P}^*, \quad \Phi_\alpha(([x], [v])) \mapsto q \cdot \mu_{ip}([x]) - p \cdot \beta_{[v]}
\]

which we call the shifting of \( \mu_{ip} \) with respect to the rational point \( \alpha \in i\mathbb{A}^* \). In particular, this is contained in the class of examples which we consider in this chapter.

Again, we know that a rational point \( \alpha \in i\mathbb{A}^* \) is contained in \( \mu_{ip}(Y) \) if and only if there exists a point \( y \in Y \) with \( \mu_{ip,\alpha}(y) = 0 \). By definition, the existence of such a point \( y \in (\mu_{ip,\alpha})^{-1} \cap Y \) is equivalent to the non-emptiness of the set of semistable points \( S_G(M_{ip}(Y_\alpha)) \) in \( Y_\alpha \).
7.5 Convexity in the projective case

Using the results of the previous section we get the following.

**Proposition 7.3.** Let $Y$ be a closed connected $G$-stable irreducible semialgebraic sub-
set of $\mathbb{P}(V)$. Then the set

$$S_G(M_{ip}(\alpha)) := \{ y \in Y \mid G \cdot y \cap (\mu_{ip})^{-1}(\alpha) \neq \emptyset \}$$

is either open and dense in $Y$ or empty for all rational points $\alpha \in i\alpha^*$.

**Proof.** Let $\mu_{ip,\alpha}$ be the shifting of $\mu_{ip}$ with respect to a rational point $\alpha = \frac{p}{q} \cdot \beta$ defined above. By the results of section 5.3, the set

$$Y_\alpha = \Phi_\alpha(Y \times K \cdot [v\beta])$$

is a closed connected $G$-stable irreducible semialgebraic subset of $\mathbb{P}(V_\alpha)$. Let us assume that

$$M_{ip}(Y_\alpha) = \left\{ \Phi_\alpha(([x],[v])) \in Y_\alpha \mid \mu_{ip}([x]) - \frac{p}{q} \cdot \beta[v] = 0 \right\}$$

is non empty. Applying Proposition 7.2, we get that the set of semistable points $S_G(M_{ip}(Y_\alpha))$ is open and dense in $Y_\alpha$. Since $\Phi_\alpha$ is an embedding, the set

$$\Phi_\alpha^{-1}(S_G(M_{ip}(Y_\alpha)))$$

is open and dense in $\Phi_\alpha^{-1}(Y_\alpha) = Y \times K \cdot \alpha$. But the image of this open and dense subset under the projection onto the first component is just $S_G(M_{ip}(\alpha))$ which follows to be open and dense in $Y$.

□

To prove the same proposition for arbitrary points $\alpha \in i\alpha^*$, we want to reduce this case to the case of a rational point by a limit argument. Therefore, we need that the rational points are dense in $\mu_{ip}(Y) \cap i\alpha^*$. This is given by the following lemma.

**Lemma 7.7.** Let $Y \subset \mathbb{P}(V)$ be as above. Then $\mu_{ip}(Y) \cap i\alpha^*$ is a union of rational polytopes. In particular, the rational points are dense in this set.

**Proof.** By the results of section 3.2, the set $\mu_{ip}(Y) \cap i\alpha^*$ is a finite union of convex polytopes given by the stratification of $X$ with respect to the $T$-isotropy. Since $\mu_{i\alpha}$ maps the closures of these strata onto convex polytopes, whose vertices are fixed points of $T$, it is sufficient to show that the images of these fixed points are rational.

Consider the $T$-representation on $V$. Since $V$ is a complex vector space, it decomposes into a direct sum of weight spaces

$$V_\chi := \{ v \in V \mid t \cdot v = \chi(t) \cdot v \text{ for all } t \in T \},$$

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where \( \chi : T \to S^1 \) is an irreducible character. Let \([v]\) be a fixed point of \( T \) in \( \mathbb{P}(V) \). Then \( v \) is contained in such a weight space and we have

\[
\mu_{ia}([v])(\xi) = \frac{\langle \xi \cdot v, v \rangle}{\|v\|^2} = \frac{\langle \chi_\ast(\xi) \cdot v, v \rangle}{\|v\|^2} = \chi_\ast(\xi)
\]

for every \( \xi \) in \( ia \). Therefore, the images of the fixed points of \( T \) under \( \mu_{ia} \) are integral elements in \( ia^* \) and \( \mu_{ip}(Y) \cap ia^*_+ \) is the finite union of rational convex polytopes.

We now can generalize Proposition 7.3 to the set of semistable points with respect to any \( \alpha \in ia^* \).

**Proposition 7.4.** Let \( Y \) be a closed connected \( G \)-stable irreducible semialgebraic subset of \( \mathbb{P}(V) \). Then the set

\[
S_G(M_{ip}(\alpha)) = \{ y \in Y \mid G \cdot (y, \xi) \cap (\mu_{ip})^{-1}(\alpha) \neq \emptyset \}
\]

is either open and dense in \( Y \) or empty for all \( \alpha \in ia^* \).

**Proof.** Let us assume there exists an \( \alpha \in ia^* \) such that \( S_G(M_{ip}(\alpha)) \) is a nonempty open subset of \( Y \) which is not dense in \( Y \). Then there exists a \( G \)-stable open subset \( U \) of \( Y \) such that the closure \( \bar{U} \) in \( Y \) does not intersect \( S_G(M_{ip}(\alpha)) \). In particular, the set

\[
(\mu_{ip}(Y) \setminus \mu_{ip}(\bar{U})) \cap ia^*_+
\]

contains an open neighborhood of \( \alpha \) in \( \mu_{ip}(Y) \cap ia^*_+ \). Since the rational points are dense in \( \mu_{ip}(Y) \), there exists a rational point \( \tilde{\alpha} \) in this neighborhood. The set of semistable points \( S_G(M_{ip}(\tilde{\alpha})) \) is by construction non empty and does not contain \( U \) which is a contradiction to Proposition 7.3.

To show that Theorem 6.1 can be applied to \( Y \), we also need density of the semistable points in \( Y \times K \cdot \beta \).

**Proposition 7.5.** Let \( Y \) be as above. Then for fixed \( \alpha, \beta \in ia^* \) the set

\[
S_G(M_{ip,\beta}(\alpha))) = \{ (y, \xi) \in Y \times K \cdot \beta \mid G \cdot (y, \xi) \cap (\mu_{ip,\beta})^{-1}(\alpha) \neq \emptyset \}
\]

is either open and dense in \( Y \times K \cdot \beta \) or empty.

**Proof.** Let us assume that this set is non empty. We have to distinguish three cases. If \( \beta \) is an integral point in \( ia^* \), the result follows from the previous proposition since \( Y \times K \cdot \beta \) is again a closed connected \( G \)-stable irreducible semialgebraic set of some projective space.

If \( \beta = \frac{1}{n} \cdot \xi \) is a rational point, we have a \( G \)-equivariant diffeomorphism

\[
\varphi : Y \times K \cdot \beta \to Y \times K \cdot \xi
\]
given by multiplication by the integer $n$ in the second component. In particular, we have
\[ M_{ip,\beta}(\alpha) = \varphi^{-1}(M_{ip,\beta}(n \cdot \alpha)). \]
Since $\varphi$ is $G$-equivariant, we get the same result for the sets of semistable points.

Therefore, the second case follows from the integral case.

To prove the general case we again use the fact that the rational points are dense in $\mu_{ip}(Y) \cap i\mathfrak{a}^*$. So let us assume that $S_G(M_{ip,\beta}(\alpha))$ is an open proper subset of $Y \times K \cdot \beta$ which is not dense in $Y \times K \cdot \beta$. Then there exists a $G$-stable open set $U$ in the complement. Let $\bar{U}$ denote its closure in $Y \times K \cdot \beta$. Again there exists a rational point $\delta$ near $\beta$ such that $S_G(M_{ip,\delta}(\alpha))$ is non-empty and does not contain $U$ which is a contradiction to the rational case proved above.

We can apply Theorem 6.1 to the subset $Y$ of $\mathbb{P}(V)$ and we get the following.

**Theorem 7.2.** Let $Y$ be a closed connected $G$-stable irreducible semialgebraic subset of $\mathbb{P}(V)$. Then the set $\mu_{ip}(Y)$ intersects the positive Weyl chamber $i\mathfrak{a}^*_+$ in a convex polytope.

This theorem has interesting consequences. As we mentioned in section 3.2 the above theorem can be applied to the map $\mu_{ia}$ and the closed connected $A$-stable irreducible semialgebraic set $Y = K \cdot \xi \subset i\mathfrak{a}^*$, where $\xi$ is some element in $i\mathfrak{a}^*$. By identifying $i\mathfrak{g}^*$ with $\mathfrak{a}^*$ this gives Kostant’s convexity theorem. In the special case where $G = U^C = U(n) \cdot \exp(i\mathfrak{u}(n))$ and $i\mathfrak{a}$ is the set of diagonal matrices in $i\mathfrak{u}(n)$ we obtain the convexity theorem of Horn.

If the group $G$ and the action on $\mathbb{P}(V)$ are algebraic, the orbits $G \cdot x$ are semialgebraic as well as its topological closures. But the orbits are smooth and therefore the sets $\overline{G \cdot x}$ are irreducible semialgebraic subsets of $\mathbb{P}(V)$, which are by definition closed and $G$-stable. Moreover, since $G$ is connected, the orbits are connected, too. Consequently, we obtain the following.

**Corollary 7.1.** If $G$ is an algebraic group which acts algebraically on $\mathbb{P}(V)$, then the set
\[ \mu_{ip}(\overline{G \cdot x}) \cap i\mathfrak{a}^* \]
is a convex polytope for every $x \in \mathbb{P}(V)$.

In the special case of $G = U^C$ we obtain the convexity theorem of Guillemin-Sternberg and Mumford in the integral Kähler case. This even shows the following.

**Corollary 7.2.** Let $V$ be a complex $U^C$-representation space. Then
\[ \mu_{\mathbb{P}(V)}(Y) \cap (\mathfrak{g}_u)^*_+ \]
is a convex polytope for every closed connected $U^C$-stable irreducible semialgebraic subset of $\mathbb{P}(V)$.
If $G$ is a real form of $U^C$ such that $G$ preserves the real part $V_R$ of the complex $U^C$-representation space $V$, then the real projective space $\mathbb{P}(V_R)$ is a closed connected $G$-stable irreducible algebraic subset of $\mathbb{P}(V)$ and we get the following.

**Corollary 7.3.** Let $V = V_R \oplus iV_R$ be a complex $U^C$-representation space and let $G$ be a compatible subgroup of $U^C$ such that $G$ preserves the real structure of $V$. Then the set 
$$\mu_{ip}(\mathbb{P}(V_R)) \cap i\mathfrak{a}_+^*$$
is a convex polytope.

### 7.6 Consequences for coadjoint orbits

Theorem 7.2 can be applied to a complex flag manifold $Z = U^C/Q$ whose image under the moment map $\mu$ is a coadjoint orbit through an integral point $\beta \in s_u^* = \mathfrak{h}^* \oplus i\mathfrak{a}^*$. In this situation, the set 
$$\mu_{ip}(Y) \cap i\mathfrak{a}_+^*$$
is a convex polytope if $\varphi_\beta(Y)$ is a closed connected $G$-stable irreducible semialgebraic subset in $\mathbb{P}(\Gamma_\beta)$. In this section we generalize this result to arbitrary moment maps for complex flag manifolds.

**Lemma 7.8.** Let $Z = U^C/Q$ be a complex flag manifold and $Y$ a closed $G$-stable subset such that $\varphi_\beta(Y)$ is an irreducible semialgebraic subset of $\mathbb{P}(\Gamma_\beta)$ for every integral $\beta$ in the cone $C_Q$. Then $\mu_{ip}(Y) \cap i\mathfrak{a}_+^*$ is a convex polytope for every moment map $\mu: Z \to \mathfrak{u}^*$ on $Z$.

**Proof.** By the results of section 5.1, every moment map on $Z$ with respect to the action of $U$ is of the form 
$$\mu: Z \to \mathfrak{u}^*, \quad x \mapsto \alpha_x,$$
where $\alpha$ is an element in the cone $C_Q$. By assumption, the lemma holds if $\alpha$ is an integral element in $C_Q$. If $\alpha$ is a rational point, it can be written in the form $\alpha = \frac{1}{n} \cdot \beta$ for some integral element $\beta$ in $s_u^*$. Therefore, we can write the moment map $\mu$ in the following form 
$$\mu: Z \to \mathfrak{u}^*, \quad x = u \cdot Q \cdot u^{-1} \mapsto \alpha_x = u \cdot \alpha \cdot u^{-1} = \frac{1}{n} (u \cdot \beta \cdot u^{-1}) = \frac{1}{n} \cdot \beta_x.$$In particular, $\mu_{ip}(Y)$ coincides with the set $\frac{1}{n} \cdot \tilde{\mu}_{ip}(Y)$, where $\tilde{\mu}$ is given by $\tilde{\mu}(x) = \beta_x$. By assumption, the set $\tilde{\mu}_{ip}(Y) \cap i\mathfrak{a}_+^*$ is a convex polytope. So this is also true for the set $\mu_{ip}(Y) \cap i\mathfrak{a}_+^*$.

We now consider the non-rational case. Since the rational points are dense in each cone $C_Q$, we can construct a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of rational points $\alpha_n \in s_u^*$ such that $Q(\alpha_n)$ coincides with $Q$ for all $n \in \mathbb{N}$ and
$$\lim_{n \to \infty} \alpha_n = \alpha.$$
We get a sequence of moment maps \((\mu^{(n)})_{n\in\mathbb{N}}\) given by
\[
\mu^{(n)} : Z \to u^*, \quad x \mapsto (\alpha_n)_x.
\]
Since the \(\alpha_n\) are rational, the sets \(\mu^{(n)}_{\text{ip}}(Y) \cap i\mathfrak{a}^*_+\) are convex polytopes for all \(n \in \mathbb{N}\).

To prove the same for the map \(\mu_{\text{ip}}\) with respect to the limit point \(\alpha\), let \(\gamma_1\) and \(\gamma_2\) be two elements in \(\mu_{\text{ip}}(Y) \cap i\mathfrak{a}^*_+\). By definition of the sequence \((\alpha_n)_{n\in\mathbb{N}}\) we have
\[
\gamma_j = \pi_{\text{ip}}(u_j \cdot \alpha \cdot u_j^{-1}) = \pi_{\text{ip}}(u_j \cdot (\lim_{n\to\infty} \alpha_n) \cdot u_j^{-1}) = \pi_{\text{ip}}(\lim_{n\to\infty} u_j \cdot \alpha_n \cdot u_j^{-1})
\]
for \(j = 1, 2\). Here \(u_1\) and \(u_2\) are suitable elements in \(U\). Since \(\mu^{(n)}_{\text{ip}}(Y) \cap i\mathfrak{a}^*_+\) is convex for all \(n \in \mathbb{N}\), we get
\[
\alpha^{(n)}_{\lambda} := \lambda \cdot \pi_{\text{ip}}(u_1 \cdot \alpha_n \cdot u_1^{-1}) + (1 - \lambda) \cdot \pi_{\text{ip}}(u_2 \cdot \alpha_n \cdot u_2^{-1}) \subset \mu^{(n)}_{\text{ip}}(X) \cap i\mathfrak{a}^*_+
\]
for all \(\lambda \in [0, 1]\). So the points \(\alpha^{(n)}_{\lambda}\) can be written in the form
\[
\pi_{\text{ip}}(u^{(n)}_{\lambda} \cdot \alpha_n \cdot (u^{(n)}_{\lambda})^{-1})
\]
for sequences \((u^{(n)}_{\lambda})_{n\in\mathbb{N}} \subset U\) depending on \(\lambda\). Since \(U\) is compact, one can assume without loss of generality that these sequences converge. Let \(u_{\lambda}\) denote the limit of the sequence \((u^{(n)}_{\lambda})_{n\in\mathbb{N}}\). We then get
\[
\lambda \cdot \gamma_1 + (1 - \lambda) \cdot \gamma_2 = \lim_{n\to\infty} \alpha^{(n)}_{\lambda} = \lim_{n\to\infty} \pi_{\text{ip}}(u^{(n)}_{\lambda} \cdot \alpha_n \cdot (u^{(n)}_{\lambda})^{-1})
\]
\[
= \pi_{\text{ip}}(u_{\lambda} \cdot (\lim_{n\to\infty} \alpha_n) \cdot u_{\lambda}^{-1})
\]
\[
= \pi_{\text{ip}}(u_{\lambda} \cdot \alpha \cdot u_{\lambda}^{-1}).
\]
Consequently, \(\lambda \cdot \gamma_1 + (1 - \lambda) \cdot \gamma_2\) is contained in \(\mu_{\text{ip}}(Y) \cap i\mathfrak{a}^*_+\) for all \(\lambda \in [0, 1]\) which proves the convexity of \(\mu_{\text{ip}}(Y) \cap i\mathfrak{a}^*_+\). By Theorem 3.3 it is also a polytope.

In particular, the above lemma can be applied to the complex flag manifold \(Z\) itself and closures of \(G\)-orbits in \(Z\). Therefore, we have the following.

**Theorem 7.3.** Let \(Z\) be a complex flag manifold with moment map \(\mu\). Then the sets
\[
\mu_{\text{ip}}(Z) \cap i\mathfrak{a}^*_+ \quad \text{and} \quad \mu_{\text{ip}}(G \cdot x) \cap i\mathfrak{a}^*_+
\]
are convex polytopes.

There is another interesting consequence in the case where the real form \(G\) of \(U^C\) is given as the fixed point set of an antiholomorphic involution with commutes with the Cartan involution on \(U^C\).
Corollary 7.4. Let $Z$ be a complex flag manifold and let $\xi$ be the unique closest point to the origin in the convex polytope $\mu_{ip}(Z) \cap i\alpha^*_+$. Then $S_G(M_{ip}(\xi))$ coincides with the union of all open $G$-orbits in $Z$.

Proof. Since $\xi$ is the unique closest point to the origin in $\mu_{ip}(Z) \cap i\alpha^*_+$, the elements in $(\mu_{ip})^{-1}(\xi)$ are global minima of the function $\eta_{ip}$. But the $G$-orbits through these minima are open (see [BL02] and [MUV92]). So we need to show that all open $G$-orbits are contained in the set $S_G(M_{ip}(\xi))$.

Again, we start with the integral case. By Proposition 7.3, we know that $S_G(M_{ip}(\xi))$ is open and dense in $Z$. So $S_G(M_{ip}(\xi))$ contains all open $G$-orbits.

In the rational case the image $\mu(Z)$ is given by a coadjoint orbit $U \cdot (\frac{1}{n} \cdot \beta)$, where $\beta \in s_u^*$ is an integral element. In particular, the preimage of the point with minimal norm in $\mu_{ip}(Z) \cap i\alpha^*_+$ coincide with the preimage of the point with minimal norm in $\tilde{\mu}_{ip}(Z) \cap i\alpha^*_+$. Here $\tilde{\mu}$ is the moment map such that $\tilde{\mu}(Z)$ is the coadjoint orbit $U \cdot \beta$. Therefore, the statement is also true in the rational case.

To prove the non-rational case, we again take a sequence of rational points which converges to a non-rational point. Since the above statement is true for every point of this sequence, it is also true for the limit point.

$\square$
8 Examples

We want to give some examples of the convex polytopes \( \mu_{ip}(Z) \cap i\mathfrak{a}_\ast \), where \( Z \) is a complex flag manifold.

8.1 Action of \( SL_3(\mathbb{R}) \) on \( SL_3(\mathbb{C})/Q \)

We consider the compact group \( U = SU(3) \) with complexification \( U^C = SL_3(\mathbb{C}) \). The involution \( g \mapsto \overline{g} \), where \( \overline{g} \) denotes the complex conjugate of \( g \), commutes with the Cartan involution on \( SL_3(\mathbb{C}) \) and we get \( G = SL_3(\mathbb{R}) \) as the fixed point set of this involution. The Cartan decomposition of \( G \) is then given by

\[
SL_3(\mathbb{R}) = SO(3) \cdot \exp(p),
\]

where \( p \) denote the set of symmetric matrices in \( g = sl_3(\mathbb{R}) \). In particular, \( G \) is compatible with the Cartan decomposition of \( U^C \). We choose \( a \) to be the set of diagonal matrices in \( p \). Then \( i\mathfrak{a} \) is a maximal torus in \( u = su(3) \) and \( s := a \oplus i\mathfrak{a} \) is a maximal \( \theta \)-stable torus in \( u^C = sl_3(\mathbb{C}) \).

Let \( e_j : s \to \mathbb{C} \) be given by the projection onto the \( j \)th diagonal entry. Then the functionals \( \alpha_1 = e_1 - e_2 \) and \( \alpha_2 = e_2 - e_3 \) are the simple roots in a positive root system \( \Delta^+ \) in \( \Delta(u^C, s) \). So there are, up to conjugation, three possible parabolic subalgebras of \( u^C \). These are given by

\[
b := s \oplus \bigoplus_{\gamma \in \Delta^+} u_\gamma, \quad q_1 := b \oplus u^-_{-\alpha} \quad \text{and} \quad q_2 := b \oplus u^-_{-\beta}.
\]

The corresponding parabolic subgroups are of the form

\[
B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, \quad Q_1 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.
\]

We start with the case of the Borel subgroup \( B \). As we saw in section 5.1, we can identify \( Z = U^C / B \) with a coadjoint orbit \( U \cdot \xi \), where \( \xi \) is an element in the cone

\[
C_B = \{ \lambda \in i\mathfrak{a} \mid \lambda = ic_1 \cdot \alpha_1 + ic_2 \cdot \alpha_2, \ c_1, c_2 \in \mathbb{R}^+ \}.
\]

So let \( \xi \), for example, be given by

\[
\xi = 3i \cdot \alpha_1 + 6i \cdot \alpha_2.
\]

Actually, we want to draw the picture of \( \mu_{ip}(Z) \cap i\mathfrak{a}_\ast \) in \( i\mathfrak{a} \). Therefore, let \( \alpha_1^\ast \) and \( \alpha_2^\ast \) denote the dual elements in \( i\mathfrak{a} \) of the restrictions of the roots \( \alpha_1 \) and \( \alpha_2 \) given above. In this example they are given by

\[
\alpha_1^\ast = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \quad \text{and} \quad \alpha_2^\ast = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}.
\]
Consequently \( \xi^* \) is given by

\[
\xi^* = 3i \cdot \alpha_1^* + 6i \cdot \alpha_2^* = \begin{pmatrix} 4i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -5i \end{pmatrix}.
\]

Then \( \mu_{ip}(X) \cap i\mathfrak{a} \) has the following form.

The point \( \xi^* \) is the intersection of the image of the unique closed \( G \)-orbit in \( X \) with the positive Weyl chamber. In particular, this is an element in \( B_{ip} \). The elements of \( B_{ip} \) are marked as \( \bullet \) in the picture above. The two edges of \( \mu_{ip}(Z) \cap i\mathfrak{a}_+ \) which join \( \xi \) are images of \( G \)-orbits. By removing \( \xi \) and these two edges from the set \( \mu_{ip}(Z) \cap i\mathfrak{a}_+ \) one gets the image of the open \( G \)-orbits.

In the other cases \( Z = U^C/Q_j \) can be identified with the coadjoint orbit \( U \cdot \xi_j \), where \( \xi_j \) is an element in the cone

\[
C_{Q_j} = \{ \lambda \in i\mathfrak{a} \mid \lambda = ic_j \alpha_j, \ c_j \in \mathbb{R}^+ \}.
\]

So choosing, for example, the points

\[
\xi_1^* = 3i \cdot \alpha_1^* = \begin{pmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix} \quad \text{and} \quad \xi_2^* = 3i \cdot \alpha_2^* = \begin{pmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix}
\]

we get the following pictures.
Again the marked points represent the set $B_{ip}$. Note that these are examples where the intersection $\mu_{ip}(Z) \cap i\alpha_+^*$ has lower dimension than the set $\mu_{ia}(Z)$ which is the convex hull of the Weyl group orbit through the point $\xi_j$. In particular, this means that the image of the open stratum $Z_\alpha$ in the stratification with respect to the $T$-isotropy does not intersect $i\alpha_+^*$.

### 8.2 Action of $SU(2, 2)$ on $SL_4(\mathbb{C})/B$

We want to give an example where the image is more complicated. We start with the compact connected group $U = SU(4)$. Its complexification is the group $U^C = SL_4(\mathbb{C})$ and we choose the real form $G = SU(2, 2)$ which is given by the involution

$$\sigma : U^C \rightarrow U^C, \quad g \mapsto Jg^tJ, \text{ with } J := \text{diag}(1, 1, -1, -1).$$

Then we have a Cartan decomposition of $G$ of the form

$$SU(2, 2) = S(U(2) \times U(2)) \cdot \exp(p),$$

where $p$ is given by

$$p := \left\{ \begin{pmatrix} 0 & R \\ R^t & 0 \end{pmatrix} \in \mathfrak{s}l_4(\mathbb{C}) \right\}$$

We choose $a$ to be the set of those elements in $p$, for which the matrix $R$ has only entries outside the diagonal. Then $\mathfrak{z}_t(a)$ is abelian and we get the following maximal torus

$$\mathfrak{s}_u = \left\{ H = \begin{pmatrix} it_1 & 0 & 0 & it_2 \\ 0 & -it_1 & it_3 & 0 \\ 0 & it_3 & -it_1 & 0 \\ it_2 & 0 & 0 & it_1 \end{pmatrix} \in u \text{ with } t_j \in \mathbb{R} \right\}$$

of $u = \mathfrak{su}(4)$. Let $e_j : \mathfrak{i}\mathfrak{s}_u \rightarrow \mathbb{R}$ denote the projection of the matrix $iH$ onto the entry $t_j$. Then the restrictions of the simple roots in the set of positive roots of $\Delta(u^C, \mathfrak{s}_u \oplus i\mathfrak{s}_u)$ to
$i\mathfrak{s}_u$ are given by the functionals $\alpha_1 = -2t_1 + t_2 - t_3$, $\alpha_2 = 2t_3$ and $\alpha_3 = 2t_1 + t_2 - t_3$. The dual elements are then given by

$$\alpha_1^* = \begin{pmatrix} -\frac{1}{4} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{4} \end{pmatrix}, \alpha_2^* = \begin{pmatrix} -\frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \end{pmatrix} \quad \text{and} \quad \alpha_3^* = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Let $Z$ be the complex flag manifold $SL_4(\mathbb{C})/B$, where $B$ denotes the Borel subgroup corresponding to the Borel subalgebra

$$\mathfrak{b} = \mathfrak{s}_u \oplus i\mathfrak{s}_u \oplus \bigoplus_{\delta \in \Delta^+} \mathfrak{u}_\delta.$$

We can again identify $Z$ with the coadjoint orbit $U \cdot \xi$ where $\xi$ is an arbitrary element in the cone

$$C_B = \{ \lambda \in \mathfrak{s}_u \mid \lambda = ic_1Z_1 + ic_2Z_2 + ic_3Z_3, \ c_j \in \mathbb{R}^+ \}. $$

Choosing $\xi = 6i \cdot \alpha_1 + 2i \cdot \alpha_2 + 8i \cdot \alpha_3$ we get

$$\xi^* = \begin{pmatrix} 0 & 0 & 0 & 7i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 7i & 0 & 0 & 0 \end{pmatrix} \in i\mathfrak{a}_+.$$

Then $\mu_{ip}(Z) \cap i\mathfrak{a}_+$ looks as follows.

The marked points again represent the set $B_{ip}$. The lines in the interior of $\mu_{ip}(Z) \cap i\mathfrak{a}_+$ are images of the strata $Z_\alpha$ where the corresponding subtorus $T_\alpha$ is 1-dimensional. The point $\beta$ is the image of a fixed point of $T = \exp(i\mathfrak{a})$ and the darker part of $\mu_{ip}(X) \cap i\mathfrak{a}_+^*$ marks the image of the stratum $Z_{\beta}^{ss}(+)$. 
References


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