CONCLUSION AND OUTLOOK

The quasiclassical description of multi-band superconductors with two order parameters presented in this Thesis, allows one to tackle problems involving systems, which can be mapped onto the used model, in nonhomogeneous cases.

The developed approach has been applied to study various issues in a two-band toy model allowing for coexistence of superconducting and spin-density wave order parameters in a certain region of the nesting parameter $\mu$ describing the mismatch of the Fermi surfaces in this model. Thus, in Chapter 3, in a somewhat artificial case of $\mu = 0$ where coexistence is, strictly speaking, not possible, it was shown that, provided the direction of an applied magnetic field coincides with the direction of the magnetization $m$ in the spin-density wave, the Knight shift vanishes at zero temperature if the spin-orbit interactions are absent. If the magnetic field is not collinear with the magnetization vector, the Knight shift is finite, which correlates with the results of a recent paper, where it was shown that the density of states of a superconductor with a spin-density wave also depends on the orientation of an external magnetic field. Moreover, it was demonstrated that near the interface between a superconductor with a spin-density wave and a normal metal, the components of the quasiclassical Green’s function describing both magnetic and superconducting correlations, penetrate into the normal metal over a length of the order $\min\{v_F/\Delta_{sc, sdw}\}$. Using a simple model of an SNS Josephson junction composed of two superconductors with a spin-density wave, the critical Josephson current has been calculated and it has been shown that, in the considered case, it does not depend on the mutual orientation of the magnetization in the spin-density wave.

Nevertheless, in Chapter 4, the two-band model but with a finite $\mu \neq 0$ has been considered. Here, the investigation was focused on spin current $j_{sp}$ in Josephson-like junctions composed of two-band antiferromagnets with either a ferromagnetic or an insulating layer inbetween. It has been shown that the spin current in the former has the form of a Josephson current being dependent on the angle between the magnetization directions in the respective antiferromagnet. It was shown that a ‘short’- and a long-range components of the spin-density wave order parameter can penetrate the ferromagnet resembling the long- and short-range components of the condensate wave function in the case of an SFS Josephson junction with a nonuniform ferromagnetic layer.
The long-range component stems from the spin-density wave component normal to the magnetization direction in the ferromagnet, and the corresponding penetration depth does not depend on the exchange field there, being in this sense analogous to the odd triplet long-range component in an SFS Josephson junction with a nonhomogeneous magnetization. These 'short'- and long-range components interfere giving rise to the spin current, thus making it oscillate in space, see Eq. (4.35) and Fig. 4.3. Moreover, the spin current in AFIAF junctions depends on the angle in completely same way as in AFFAF junctions. The dependence of $j_{sp}$ in this junction on the nesting parameter $\mu$ has been calculated. At $\mu \phi = 0$, an energy gap $\Delta_{sdw}$ opens in the excitation spectrum of the leads with the spin-density wave, see Fig. 4.7. In a certain region of the nesting parameter, the energy gap disappears and the leads of the junction become metals (or semimetals). The spin current exists in both cases and turns to zero at the Néel temperature when $\Delta_{sdw} \rightarrow 0$. Furthermore, the Ferrell–Prange equation for Josephson-like junctions composed of antiferromagnets has been derived, and a solution describing a localized spin current distribution analogously to the fluxon in a tunnel SIS Josephson junction has been found, see Eq. (4.53) and Fig. 4.5. The characteristic length of the fluxon is determined by the barrier transmittance and by the thickness of antiferromagnetic leads. In the case of bulk leads, the thickness should be replaced by a characteristic length, over which the magnetization vector of the spin-density wave restores its favorable direction in the bulk, which is determined by anisotropy effects.

In Chapter 5, subsequently, the Josephson effect in a tunnel Josephson junction composed of two-band superconductors with a spin-density wave has been investigated based on the tunneling Hamiltonian method. Three types of junctions were considered—each a combination of two possible pairings of the order parameter in the superconductor, i.e., the $S_{++}$- and the $S_{+-}$-pairings. The focus of investigation was laid mainly on the case when the superconducting order parameter coexists with a spin-density wave, but for completeness, also the case without the spin-density wave has been analyzed. Even in this case, the Josephson current has an unusual phase dependence if the interband tunneling matrix element has a nonvanishing imaginary part, see Eq. (5.9), i.e., $I_1 = I_0 \sin(\phi + \phi_0)$, thus providing a realization of the so-called $\phi$-junction, in the case when the superconductors have different pairings of the superconducting order parameter. However, the most interesting result is obtained for the case of coexistence of the superconducting and the spin-density wave order parameters (for that, nonideal nesting is necessary in pnictides). In this case, the critical current contains a term that depends on the angle between the magnetization vectors in the spin-density wave in the left and right leads, i.e., $\sim \cos \alpha$. This term allows an identification of the $S_{+-}$-pairing and, on the other hand, a realization of a $\pi$-state of the Josephson junction composed of two-band superconductors with $S_{+-}$-pairing of the order parameter. In the case when the superconductors have different pairings of the superconducting order parameter, there is no angle dependence and the expression for the Josephson current has the same form as in
the case of lacking coexistence, but with different critical current depending on the nesting parameter $\mu$, see Eq. (5.24).

Motivated by the diversity of effects due to the interplay of the order parameters in case of coexistence, in Chapter 6, a time-dependent equation has been derived, Eq. (6.59), which describes the relaxation and the spatial behavior of the amplitude of the spin-density wave in the vicinity of the magnetic quantum critical point, where a second order phase transition takes place. Only the case of a fixed orientation of the magnetization in the spin-density wave has been considered neglecting the rotational mode. At this point, the amplitude of the spin-density wave order parameter turns to zero while the superconducting order parameter remains finite. The derived time-dependent equation is valid at low temperatures when the number of quasiparticles is low, and they do not affect essentially the dynamics of the order parameters. Therefore, the conditions $T \ll \Delta_{sc}$ and $m \ll \Delta_{sc}$ should be satisfied. However, in the stationary case, the Ginzburg–Landau-like equation (6.44) is valid at arbitrary temperatures. The coefficients in this equation are expressed in terms of microscopic characteristics of the system, i.e., the nesting parameter $\mu = \mu_0 + \mu_\phi \cos(2\phi)$, the superconducting order parameter at the quantum critical point $\Delta_c$, and the Fermi velocity $v_F$. Moreover, regions of a second-order phase transition in the $\mu_0 - \mu_\phi$ plane have been found, in agreement with results of Vavilov et al. who analyzed the free energy in the system. The derived equation is valid if the influence of fluctuations can be neglected. This requirement imposes a restriction on temperatures, $\delta \mu \ll T \ll \delta \mu (E_F/T_m)^2$, where the left inequality means that quantum fluctuations can be neglected and the right is similar to the Ginzburg–Levanjuk criterion for the applicability of the Ginzburg–Landau equation to superconductors. On the basis of the derived equation (6.44), a solution that describes a domain wall in the stationary case has been obtained. The width of the domain wall diverges when $\mu$ approaches a critical value, defined by a criticality line, see Eq. (6.34) and Fig. 6.2. The superconducting order parameter increases in the center of the domain wall as compared to its value far away from the center, i.e., a local enhancement of superconductivity has been found in the coexistence regime. Using the time-dependent equation (6.59), the stability of a state with a uniform spin-density wave has been studied in the vicinity of the magnetic quantum critical point where the magnetic order parameter vanishes. It was shown that a homogeneous commensurate spin-density wave is stable on the upper part of the curve defined by Eq. (6.34), which describes a boundary of the region of coexistence of superconducting and magnetic order parameters, see Fig. 6.4. Below this region the homogeneous spin-density wave is unstable against perturbations with a finite wave vector propagating along the direction parallel to the major axis of the elliptic Fermi surface. This instability leads to the appearance of incommensurate spin-density wave or of an inhomogeneous spin-density wave similar to a soliton lattice discussed by Lev Petrovich Gor’kov and Gregory Bentsionovich Teitel’baum. In Fig. 6.1, the region of a homogeneous stable spin-density wave coexisting with superconductivity, and
the region of an inhomogeneous spin-density wave (hatched) state are shown.

On top, in Chapter 7, a quasi-one-dimensional superconducting structure with another order, i.e., a charge-density wave, has been considered. Because of similarity of the obtained expressions, it was possible to conclude that the physics in those systems can be described using the developed approach. Especially, the obtained self-consistency equations allow for coexistence of superconductivity and charge-density wave. A possible route to investigate the dynamics of the corresponding order parameters has been provided.

There are various directions, whereto further research might be performed utilizing the approach developed in this Thesis. Some details remained shall be clarified. For instance, it is of interest how the Knight shift behaves at a finite nesting parameter—the doubts regarding this issue shall be ruled out. In addition, the vicinity of the superconducting quantum critical point shall be studied thus enabling to determine the second boundary of the region in the $\mu_0-\mu_0$ plane where both order parameters coexist.

Having used the simplified two-band toy model in the considerations, a transition to more complex structures shall follow, especially important in order to bring the model closer to the description of real materials like pnictides which have a complex band structure of four and sometimes five bands. The extension to more bands can offer a possibility to capture the physics of the so-called nematic phase observed in pnictides. Nevertheless, in cases where the considered model can be applied to these materials, the obtained results are valid and it would be interesting to test them experimentally. Of particular interest for applications are the results of Chapter 5—possible realization of a $\pi$- and even of a $\phi$-junction in pnictides. The obtained result shall be crosschecked using the approach based on Bogoliubov–de Gennes equation.\textsuperscript{154,155}

Furthermore, a broad variety of problems with time-dependence can be studied, including the behavior of the spin-density wave order parameter in a time dependent external magnetic field, and the route suggested in Chapter 7. In particular, in the latter case it shall be clarified, if the model investigated there can be applied to cuprate superconductors.

All in all, the vast field of (high-$T_c$) superconductivity offers a multiplicity of interesting problems to be tackled and I’d like to conclude with words of the great physicist Vitaly Lazarevich Ginzburg, even though he probably misspelled the expression ‘condensed matter physics’ as ‘astrophysics’ apparently for the sake of simplicity: “In the past century, and even nowadays, one could encounter the opinion that in physics nearly everything had been done. There allegedly are only dim ‘cloudlets’ in the sky or theory, which will soon be eliminated to give rise to the ‘theory of everything’. I consider these views as some kind of blindness. The entire history of physics, as well as the state of present-day physics and, in particular, astrophysics, testifies to the opposite. In my view we are facing a boundless sea of unresolved problems.”