Projectivity of analytic Hilbert quotients

Dissertation

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Introduction

For a finite-dimensional regular representation \( \rho : G \to GL_C(V) \) of a complex-reductive Lie group \( G \), it is a classical result due to Hilbert that the algebra \( C[V]^G \) of \( G \)-invariant polynomials is finitely generated. It defines an affine algebraic variety via \( V//G = \text{Spec}(C[V]^G) \) and an \( G \)-invariant surjective affine morphism \( \pi : V \to V//G \). If \( \mathcal{O}_Y \) denotes the structure sheaf of a given algebraic variety \( Y \), the map \( \pi \) fulfills \((\pi_* \mathcal{O}_V)^G = \mathcal{O}_{V//G}\). 

The algebraic variety \( V//G \) parametrises the closed \( G \)-orbits in \( V \). We call \( \pi : V \to V//G \) the algebraic Hilbert quotient of \( V \) by the action of \( G \). More generally, algebraic Hilbert quotients always exist for algebraic actions of complex-reductive Lie groups on affine varieties.

However, given an arbitrary algebraic \( G \)-variety \( X \) for a complex-reductive group \( G \), it is, in general, not possible to find an algebraic Hilbert quotient \( \pi : X \to X//G \) for the action of \( G \) on \( X \). Hence, one looks for open \( G \)-invariant subsets of \( X \) for which an algebraic Hilbert quotient exists. To this end, Mumford introduced the notion of semistability with respect to a linearisation, a lifting of the \( G \)-action to an ample line bundle \( L \) on \( X \), cf. [MFK94]. This yields a Zariski-open subset \( X(L) \) of semistable points for which there exists an algebraic Hilbert quotient \( \pi : X(L) \to X(L)//G \) with quasi-projective \( X(L)//G \).

In the complex-analytic category consider the holomorphic action of a complex-reductive Lie group \( G \) on a Stein space \( X \). In analogy to the affine algebraic situation, there exists a Stein space \( X//G \) and a \( G \)-invariant surjective holomorphic Stein map \( \pi : X \to X//G \) with \((\pi_* \mathcal{H}_X)^G = \mathcal{H}_{X//G}\), see [Sno82] and [Hei91]. Here, \( \mathcal{H}_Y \) denotes the sheaf of holomorphic functions on a given complex space \( Y \). We call \( \pi : X \to X//G \) an analytic Hilbert quotient. In analogy to the algebraic category, the space \( X//G \) parametrises the closed \( G \)-orbits in \( X \). More precisely, it is the quotient of \( X \) by the equivalence relation
\[
x \sim y \text{ if and only if } G \cdot \overline{x} \cap G \cdot \overline{y} \neq \emptyset.
\]

As in the algebraic case, for a general holomorphic \( G \)-space an analytic Hilbert quotient does not necessarily exist. To find \( G \)-invariant open subsets that admit an analytic Hilbert quotient, the role of a linearisation is played by a momentum map for the action of a maximal compact subgroup \( K \) of \( G \). Given a complex Kähler manifold \( X \) with \( K \)-invariant Kähler form \( \omega \), a momentum map is a smooth map \( \mu : X \to \text{Lie}(K)^* \) which is equivariant with respect to the action of \( K \) on \( X \) and the coadjoint representation on \( \text{Lie}(K)^* \), and
whose components $\mu^\xi = \mu(\cdot)(\xi)$ fulfill

$$d\mu^\xi = \iota_{\xi_X} \omega \quad \text{for all } \xi \in \text{Lie}(K).$$

Here, $\iota_{\xi_X}$ denotes contraction with the vector field $\xi_X$ on $X$ that is induced by the $K$-action.

The set of semistable points with respect to $\mu$ and the action of $G$ is defined as

$$X(\mu) := \{ x \in X \mid \overline{G \cdot x} \cap \mu^{-1}(0) \neq \emptyset \}. $$

It is open in $X$, and there exists an analytic Hilbert quotient $\pi : X(\mu) \to X(\mu)//G$ for the action of $G$ on $X(\mu)$.

The two concepts of semistability introduced above coincide in the projective algebraic situation. This is proven using methods developed by Kempf, Ness and Kirwan [Kir84]: if $X$ is a projective algebraic $G$-variety and if $\omega$ is the curvature form of an ample line bundle $L$ to which we can lift the action of $G$, then there exists a momentum map $\mu : X \to \text{Lie}(K)^*$ for the action of a maximal compact subgroup $K$ of $G$ and we have $X(\mu) = X(L)$, see [MFK94]. In particular, it follows that $X(\mu)//G$ is a projective algebraic variety. This has been generalised to non-compact Kähler $G$-manifolds with proper momentum map by Sjamaar [Sja95].

As above, let $G$ be a complex-reductive Lie group and $K$ a maximal compact subgroup of $G$. Let $X$ be an algebraic $G$-variety with a $K$-invariant Kähler structure $\omega$. If there exists a momentum map $\mu : X \to \text{Lie}(K)^*$ with respect to $\omega$, we call $X$ an algebraic Hamiltonian $G$-variety.

Not every Kähler structure on an algebraic variety is necessarily the curvature form of an ample line bundle. Hence, if $X$ is an algebraic Hamiltonian $G$-variety with momentum map $\mu$, the question arises if the quotient $X(\mu)//G$ is an algebraic variety.

The case of smooth projective algebraic varieties has been dealt with by Heinzner and Migliorini:

**Theorem ([HM01]).** Let $G$ be a connected complex-reductive Lie group with maximal compact group $K$. Let $X$ be a smooth projective algebraic Hamiltonian $G$-variety with momentum map $\mu : X \to \text{Lie}(K)^*$. Then, there exists a very ample line bundle $L$ on $X$ such that $X(L) = X(\mu)$. In particular, $X(\mu)//G$ is a projective algebraic variety.

In this monograph we generalise the projectivity results obtained by Kirwan, Sjamaar, and Heinzner/Migliorini in the following directions:

1. we do not restrict to compact or quasi-projective $G$-varieties,
2. we allow $X$ to have a certain type of singularities,
3. we do not require the momentum map to be proper (see Section 2.6 for a discussion).
In Chapter 1 we recall basic results on complex-reductive groups and their actions on complex spaces. The main result of Chapter 2 is

**Theorem 1** (Projectivity Theorem). Let $G$ be a complex-reductive Lie group with maximal compact group $K$. Let $X$ be a $G$-irreducible algebraic Hamiltonian $G$-variety with momentum map $\mu : X \to \text{Lie}(K)^*$. Assume that $X$ has only 1-rational singularities. Then, the analytic Hilbert quotient $X(\mu) \sslash G$ is a projective algebraic variety if and only if $\mu^{-1}(0)$ is compact.

For the proof, we use the following projectivity criterion of Namikawa:

**Theorem ([Nam02]).** Let $Z$ be a Moishezon space with 1-rational singularities. If $Z$ admits a smooth Kähler structure, then it is a projective algebraic variety.

Chapter 2 is organised as follows. Using results of Rosenlicht on the transcendence degree of the field $\mathbb{C}(X)^G$ of invariant rational functions on the algebraic $G$-variety $X$, we first prove that the analytic Hilbert quotient $X(\mu) \sslash G$ is a Moishezon space. As a second step, we prove that $X(\mu) \sslash G$ has only 1-rational singularities (see Section 2.2.2 for the definition). To achieve this, we adapt the proof of the holomorphic Slice Theorem (cf. [Hei91]) to our algebraic situation and then refine Boutot’s result [Bou87] on the rationality of singularities of algebraic Hilbert quotients. Finally, it follows from work of Heinzner, Huckleberry and Loose that the quotient $X(\mu) \sslash G$ carries a continuous Kähler structure. This structure can be regularised to a smooth Kähler structure by results of Varouchas. We conclude that under the assumptions of Theorem 1, the analytic Hilbert quotient $X(\mu) \sslash G$ fulfills Namikawa’s projectivity criterion and hence is a projective algebraic variety.

Note that in the respective setups of Kirwan, Sjamaar, and Heinzner/Migliorini, the set $X(\mu)$ of semistable points is a Zariski-open subset of $X$. For a general complex space $X$ this is no longer true (see Example 2.4.1). Using Theorem 1 and the concept of Chow quotients for algebraic $G$-varieties as developed by Kapranov [Kap93], the following is shown in Section 2.4:

**Theorem 2.** Let $G$ be a complex-reductive Lie group with maximal compact subgroup $K$. Let $X$ be a $G$-irreducible algebraic Hamiltonian $G$-variety with momentum map $\mu : X \to \mathfrak{t}^*$. Assume that $X$ has 1-rational singularities and that $\mu^{-1}(0)$ is compact. Then, $X(\mu)$ is Zariski-open in $X$.

In Section 2.5, we present an example of a smooth algebraic Hamiltonian $G$-variety $X$ such that $X(\mu)$ is not Zariski-open in $X$ and such that the non-compact analytic Hilbert quotient $X(\mu) \sslash G$ cannot be given the structure of an algebraic variety, thus proving that the compactness assumption on $\mu^{-1}(0)$ made in Theorem 1 and Theorem 2 is necessary.

While Chapter 2 is devoted to projectivity and algebraicity results for quotients of algebraic $G$-varieties, we study the converse question in Chapter 3: given a holomorphic $G$-space $X$ which admits a projective algebraic analytic Hilbert quotient, what can be said about $X$ and the group action on it?
Complex-reductive groups and their actions on complex spaces are known to have strong algebraicity properties. Let us recall two of them. First, every complex-reductive Lie group $G$ carries a unique structure of a linear algebraic group and every continuous finite-dimensional representation of $G$ is algebraic with respect to this structure (see Section 1.1.2). Secondly, if a holomorphic $G$-space admits an analytic Hilbert quotient $\pi : X \rightarrow X//G$, every fibre $\pi^{-1}(q)$ carries a unique affine algebraic structure given by the algebra of $G$-finite holomorphic functions on $\pi^{-1}(q)$. More precisely, there exists an affine algebraic $G$-variety $Z$ such that $\pi^{-1}(q)$ is $G$-equivariantly biholomorphic to the complex space $Z^{h}$ associated to $Z$, see [Sno82].

In Chapter 3, we use representation theory relative to an analytic Hilbert quotient and classical vanishing theorems to prove the following generalisation of this second algebraicity result:

**Theorem 3** (Algebraicity Theorem). Let $G$ be a complex-reductive Lie group. Let $X$ be a holomorphic $G$-space such that the analytic Hilbert quotient $\pi : X \rightarrow X//G$ exists and such that $X//G$ is projective algebraic. Then, there exists a uniquely determined quasi-projective $G$-variety $Z$ with algebraic Hilbert quotient $\pi_{Z} : Z \rightarrow Z//G$ such that $X$ is $G$-equivariantly biholomorphic to $Z^{h}$, $X//G$ is biholomorphic to $(Z//G)^{h}$ and such that $\pi : X \rightarrow X//G$ is the analytic map associated to the algebraic quotient map $\pi_{Z} : Z \rightarrow Z//G$.

To arrive at this result, we first prove in Section 3.1 that every holomorphic $G$-space $X$ with projective algebraic Hilbert quotient $X//G$ admits a $G$-equivariant proper holomorphic embedding $\varphi : X \hookrightarrow Y$ into a quasi-projective algebraic $G$-variety $Y$ with algebraic Hilbert quotient $\pi_{Y} : Y \rightarrow Y//G$ in such a way that $\pi_{Y}(\varphi(X)) = Y//G$. In particular, $X//G$ is biregular to $Y//G$. As a second step, in Section 3.2 we prove the following algebraicity result for invariant analytic subsets of $G$-varieties:

**Theorem** (Algebraicity Theorem for invariant analytic subsets). Let $Y$ be an algebraic $G$-variety with algebraic Hilbert quotient $\pi : Y \rightarrow Y//G$. Assume that $Y//G$ is a projective algebraic variety. Let $Z \subset Y$ be a $G$-invariant analytic subset of $Y$ such that $\pi(Z) = Y//G$. Then, $Z$ is an algebraic subvariety of $Y$.

In our situation, we apply this result to the subset $\varphi(X) \subset Y$ and have thus endowed $X$ with an algebraic structure whose uniqueness follows from a second application of the Algebraicity Theorem for invariant analytic subsets.

In Section 3.2.5 we describe how the situation studied in Chapter 3 is naturally a problem of Geometric Invariant Theory.

Finally, we discuss implications of the results proven in Chapter 3 for the situation studied in Chapter 2 and conclude with an indication of open questions and of directions for further study.
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Chapter 1

Preliminaries

In this chapter we review some important properties of complex-reductive Lie groups and their actions on complex spaces. We discuss algebraicity properties both of the groups themselves and of their representations. We introduce the notion of analytic Hilbert quotient for actions of complex-reductive groups on complex spaces and explain how this type of quotient arises in the context of Hamiltonian actions. Moreover, we collect fundamental properties of projective algebraic varieties.

1.1 Complex-reductive Lie groups and their representations

1.1.1 Complexification of Lie groups

Let $G$ be a real Lie group with Lie algebra $\mathfrak{g}$. Associated to $\mathfrak{g}$ we have the complexification $\mathfrak{g}^\mathbb{C} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, which is a complex Lie algebra containing $\mathfrak{g}$ as a real Lie subalgebra. The following definition extends the idea of complexification to the group level.

**Definition 1.1.1.** Let $G$ be a real Lie group. A complex Lie group $G^\mathbb{C}$ together with a continuous group homomorphism $\gamma : G \rightarrow G^\mathbb{C}$ is called a complexification of $G$, if for any continuous group homomorphism $\phi$ from $G$ to a complex Lie group $H$, there exists a unique holomorphic homomorphism $\phi^\mathbb{C}$ from $\mathbb{C}$ to $H$ such that the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\gamma} & G^\mathbb{C} \\
\downarrow{\phi} & & \downarrow{\phi^\mathbb{C}} \\
H & & 
\end{array}
\]

commutes.

A complexification is unique up to holomorphic group isomorphism. Every Lie group carries a unique real analytic structure and the maps $\gamma$ and $\phi$ in the above definition are automatically analytic with respect to this structure.
For a proof of the following result see [Hoc65] and [Hei91].

**Proposition 1.1.2.** For every real Lie group \( G \) there exists a complexification \( \gamma : G \to G^C \).

We note that the map \( \gamma : G \to G^C \) need not be injective (see [Hei91] for an example), although we will see later that injectivity holds if the group \( G \) is compact.

**Definition 1.1.3.** A complex Lie group \( G \) is called **complex-reductive** if it is the complexification of some compact Lie group \( K \).

In the case of compact Lie groups, the complexification can be explicitly constructed as follows (see [Hoc65]):

Let \( V \) be a finite-dimensional complex vector space endowed with an Hermitian inner product \( < \cdot, \cdot > \). We denote the general linear group of \( V \) by \( GL_C(V) \) and the group of linear isometries of \( (V, < \cdot, \cdot >) \) by \( U(V) \).

**Proposition 1.1.4.** Let \( K \) be a compact subgroup of \( U(V) \) and let \( G \) be the smallest complex Lie subgroup of \( GL_C(V) \) containing \( K \). Then, \( G \) is closed in \( GL_C(V) \) and \( G \) together with the inclusion \( \gamma : K \hookrightarrow G \) is the complexification of \( K \). In particular, \( G \) is complex-reductive and contains the compact group \( K \) as a subgroup.

As a consequence of the Theorem of Peter and Weyl, every compact Lie group has a finite-dimensional faithful unitary representation (e.g. see [DK00]). Hence, we obtain

**Corollary 1.1.5.** The complexification \( K^C \) of a compact Lie group \( K \) is isomorphic to a closed Lie subgroup of a general linear group and the homomorphism \( \gamma : K \to K^C \) is injective.

## 1.1.2 Algebraicity properties of complex-reductive Lie groups

In this section we study algebraicity properties of complex-reductive groups and their representations. Although the results presented here are classical (e.g. see [OV90]), we provide proofs for the reader’s convenience and in order to present a first example for the use of holomorphic invariant theory in relation with algebraicity results (see the proof of Theorem 1.1.8).

A representation of a Lie group \( G \) will always mean a finite-dimensional complex continuous representation of \( G \). If \( \rho : G \to V \) is a representation of \( G \), then \( V \) together with the action of \( G \) on \( V \) induced by \( \rho \) is called a \( G \)-module. By an algebraic variety we mean an algebraic variety defined over the field \( \mathbb{C} \) of complex numbers. The structure sheaf of an algebraic variety \( X \) will be denoted by \( \mathcal{O}_X \).
1.1. Complex-reductive Lie groups and their representations

The complex space associated to an algebraic variety

In the following a complex space refers to a reduced complex space with countable topology. For a given complex space X the structure sheaf will be denoted by $\mathcal{H}_X$. Furthermore, an analytic subset $A$ of a given complex space $X$ is a closed subset $A$ of $X$ that is locally given as the common zero set of finitely many holomorphic functions.

To any algebraic variety we may associate, in a canonical way, a complex space as follows:

Any algebraic variety $X$ is as a ringed space locally isomorphic to a subvariety of some $\mathbb{C}^n$. There is a canonical way to associate to an algebraic subvariety $V$ of $\mathbb{C}^n$ a complex space $V^h$ - we give $V$ the Euclidean topology and define its structure sheaf $\mathcal{H}_V$ to be $\mathcal{H}_{\mathbb{C}^n}/\mathcal{I}_V$, where $\mathcal{I}_V$ denotes the ideal sheaf in $\mathcal{H}_{\mathbb{C}^n}$ of the analytic subset $V \subset \mathbb{C}^n$. The holomorphic structure defined in this way on open affine subsets $V$ of an algebraic variety $X$ is canonical and hence agrees on intersections. The topology on $X$ that we obtain by declaring a subset $U \subset X$ to be open if its intersection with every affine open subset is open in the Euclidean topology of this affine subset, is Hausdorff and countable. We call it the Euclidean topology of $X$. Together with the structure sheaf $\mathcal{H}_X$ constructed above, this makes $X$ into a complex space. If it is necessary to distinguish between the algebraic variety $X$ and its associated complex space, we will denote the latter by $X^h$. A subset $U \subset X$ that is open in the Zariski-topology of $X$ will be called Zariski-open. Analogous, a subset $A \subset X$ that is closed in the Zariski-topology of $X$ will be called Zariski-closed.

Any regular map $\phi : X \to Y$ between algebraic varieties $X$ and $Y$ is holomorphic with respect to the holomorphic structures on $X$ and $Y$. Again, if it is necessary to distinguish between the algebraic variety $X$ and its associated complex space, we will denote the latter by $X^h$. A subset $U \subset X$ that is open in the Zariski-topology of $X$ will be called Zariski-open. Analogous, a subset $A \subset X$ that is closed in the Zariski-topology of $X$ will be called Zariski-closed.

As a first comparison result we note:

**Lemma 1.1.6.** Let $X$ be an algebraic variety and $X^h$ the associated complex space. Then, we have $\dim X = \dim X^h$.

**Proof.** Suppose $X$ is irreducible of dimension $m$. Since the dimension is a birational invariant and since every variety is birational to a hypersurface (see [Har77, Chap I, Prop 4.9]), we can assume that $X$ is a hypersurface, i.e. $X = \{f = 0\} \subset \mathbb{C}^{m+1}$ for some $f \in \mathbb{C}[z_0, z_1, \ldots, z_m]$. Hence, we have $\dim X^h = m$. It follows that $X$ and $X^h$ have the same dimension. The same is true for an arbitrary algebraic variety, since it is true for every irreducible component. \qed

**Algebraic structures on complex-reductive Lie groups**

**Definition 1.1.7.** Let $G$ be an algebraic variety endowed with a group structure. If the maps $G \times G \to G, (x, y) \mapsto xy$ and $i : G \to G, x \mapsto x^{-1}$ are regular, we call $G$ an algebraic
group. Two algebraic groups $G$ and $H$ are called isomorphic if there exists an isomorphism of varieties $\phi : G \to H$ which also is an isomorphism of groups. An algebraic group is called linear-algebraic if it is isomorphic to a Zariski-closed subgroup of some $GL(n, \mathbb{C})$.

The following result is well-known. We include a proof for the reader’s convenience.

**Theorem 1.1.8.** Let $L$ be a linear-algebraic group and $K$ a compact subgroup of $L$. Then, $K^C$ is a linear-algebraic subgroup of $L$.

**Proof.** Without loss of generality we can assume that $L$ is equal to the Zariski-closure of $G := K^C$ in $L$ and that $L$ is contained in the group $GL_C(V)$ for some complex finite-dimensional vector space $V$. If $End(V)$ denotes the vector space of complex-linear endomorphism of $V$, we embed the group $GL(V)$ equivariantly into the $GL(V)$-representation space $End(V) \times \mathbb{C}$ via the map $\phi : g \to (g, (\det g)^{-1})$. Here, $GL_C(V)$ acts on $End(V) \times \mathbb{C}$ via $g \cdot (Z, z) = (gZ, (\det g)^{-1}z)$. We have to show that $L = G$.

Assume that there exists an element $g \in L \setminus G$. By Proposition 1.1.4, the group $G$ is closed in $L$, hence it is closed in $End(V) \times \mathbb{C}$. It follows that $G$ and $gG$ are closed disjoint complex submanifolds of the Stein manifold $End(V) \times \mathbb{C}$. Therefore, there exists a holomorphic function $f$ on $End(V) \times \mathbb{C}$ such that $f|_{\phi(G)} \equiv 0$ and $f|_{\phi(gG)} \equiv 1$. By averaging $f$ over the compact group $K$ we obtain a $G$-invariant holomorphic function $\hat{f}$ with the same property. Expanding $\hat{f}$ into a power series around $(0, 0) \in End(V) \times \mathbb{C}$, we see that there exists a $G$-invariant polynomial $p$ on $End(V) \times \mathbb{C}$ such that $p|_{\phi(G)} \equiv 0$ and $p|_{\phi(gG)} \neq 0$. This contradicts the fact that $L$ is the Zariski-closure of $G$. \hfill $\square$

**Corollary 1.1.9.** Let $K_1$ and $K_2$ be compact Lie groups and let the complexifications $K_1^C$ and $K_2^C$ be endowed with any algebraic structure making them into linear-algebraic groups. Then, every holomorphic group homomorphism $\phi : K_1^C \to K_2^C$ is algebraic with respect to the chosen algebraic structures. In particular, the linear-algebraic structure on a complex-reductive group is unique.

**Proof.** The graph $\Gamma_\phi := \{(g, \phi(g)) \mid g \in K_1^C\}$ of $\phi$ in $K_1^C \times K_2^C$ is the smallest complex subgroup of the linear-algebraic group $K_1^C \times K_2^C$ containing the compact Lie group $\{(k, \phi(k)) \mid k \in K_1\}$. Hence, $\Gamma_\phi$ is a closed algebraic subgroup of $K_1^C \times K_2^C$, by Theorem 1.1.8. Application of the following lemma yields the claim. \hfill $\square$

**Lemma 1.1.10.** Let $X$ and $Y$ be two algebraic varieties and let $\phi : X \to Y$ be a holomorphic map. Assume that the graph $\Gamma_\phi$ of $\phi$ is an algebraic subvariety of $X \times Y$. Then, $\phi$ is a regular morphism.

**Proof.** The restriction of the projection $p : X \times Y \to X$ to $\Gamma_\phi$ is a regular one-to-one map of algebraic varieties. It follows from Zariski’s main theorem [Mum99, Chap III, §9] that there exists a rational inverse $q : X \to \Gamma_\phi$ which set-theoretically coincides with $id_X \times \phi : X \to \Gamma_\phi$. However, a rational map between algebraic varieties that induces a holomorphic map of the associated complex spaces is regular, see [Sha94, Chap VIII.3]. \hfill $\square$
From now on, we will endow any given complex-reductive Lie group $G$ with the uniquely determined linear-algebraic group structure, whose existence we have proven above.

Given a linear-algebraic group $G$, a continuous representation $\rho : G \to GL_C(V)$ is called *rational* if $\rho$ is a regular map between the linear-algebraic groups $G$ and $GL_C(V)$.

Corollary 1.1.9 implies that continuous representations of compact groups $K$ extend to rational representations of their complexification $K^C$. More precisely, we have

**Proposition 1.1.11.** Let $K$ be a compact Lie group and let $\gamma : K \to K^C$ be its complexification. Let $\rho : K \to GL_C(V)$ be a continuous representation of $K$. Then, there exists a unique rational representation $\rho^C : K^C \to GL_C(V)$ such that the diagram

$\
to K^C \\
\rho \downarrow \\

\gamma \downarrow \\
GL_C(V)
$

commutes. In particular, every continuous representation of the complex-reductive group $K^C$ is rational.

**Remark 1.1.12.** We have seen that a complex-reductive Lie group is in a natural way a linear-algebraic group. A linear-algebraic group $G$ is called *reductive* if it satisfies one of the following equivalent group-theoretical conditions:

a) the solvable radical of the connected component $G_0$ of the identity of $G$ is isomorphic to $(\mathbb{C}^\times)^k$ for some $k \in \mathbb{N}$,

b) the unipotent radical of $G_0$ is trivial.

A linear algebraic group $G$ is called *linearly-reductive* if each rational representation is completely reducible. It can be shown that a linear algebraic group over $\mathbb{C}$ is reductive if and only if it is linearly-reductive (see [Fog69]). Furthermore, if $G$ is a reductive linear-algebraic group and $K$ is a maximal compact subgroup of $G$, then $G$ together with the inclusion $\gamma : K \hookrightarrow G$ is a complexification of $K$ (see [Hoc65]), hence $G$ is complex-reductive in the sense of Definition 1.1.3.

### 1.2 Quotients for actions of complex-reductive Lie groups

#### 1.2.1 Analytic Hilbert quotients

In this section we recall the basic notation related to actions of Lie groups on complex spaces. We also introduce the notion of an analytic Hilbert quotient, and state its basic properties.
If $G$ is a real Lie group, then a complex $G$-space $Z$ is a complex space with a real-analytic action $\alpha : G \times Z \to Z$ such that all the maps $\alpha_g : Z \to Z, z \mapsto \alpha(g, z)$ are holomorphic. If $G$ is a complex Lie group, a holomorphic $G$-space $Z$ is a complex $G$-space such that the action map $\alpha : G \times Z \to Z$ is holomorphic. A complex $G$-manifold is a complex $G$-space without singular points. In general, the set of singular points of a complex $G$-space $X$ is a $G$-invariant analytic subset of $X$.

Given a continuous action of a Lie group $G$ on a topological space $X$, the quotient of $X$ with respect to the action of $G$ is defined to be the set $X/G := \{ G \cdot x \mid x \in X \}$ of orbits of the $G$-action on $X$ endowed with the quotient topology. Although it is the natural topological space on which $G$-invariant continuous functions on $X$ are defined, it does not have particularly nice properties in general, for example, it often fails to be Hausdorff.

However, when considering a holomorphic action of a complex-reductive groups $G$ on a complex space $X$ it is often possible to find an open subset $U$ of $X$ and a complex space $Y$ that parametrises the closed $G$-orbits in $U$. More precisely, we make the following

**Definition 1.2.1.** Let $G$ be a complex-reductive Lie group and $X$ a holomorphic $G$-space. A complex space $Y$ together with a $G$-invariant surjective holomorphic map $\pi : X \to Y$ is called an analytic Hilbert quotient of $X$ by the action of $G$ if

1. $\pi$ is a locally Stein map,
2. $(\pi_\ast \mathcal{H}_X)^G = \mathcal{H}_Y$ holds.

Here, locally Stein means that there exists an open covering of $Y$ by open Stein subspaces $Q_\alpha$ such that $\pi^{-1}(U_\alpha)$ is a Stein subspace of $X$ for all $\alpha$; by $(\pi_\ast \mathcal{H}_X)^G$ we denote the sheaf $U \mapsto \mathcal{H}_X(\pi^{-1}(U))^G = \{ f \in \mathcal{H}_X(\pi^{-1}(U)) \mid f \text{ $G$-invariant} \}, U$ open in $Y$.

We will see in Proposition 1.2.2 that an analytic Hilbert quotient of a holomorphic $G$-space $X$ is unique up to biholomorphism once it exists. We will denote it by $X//G$.

The next proposition collects fundamental properties of analytic Hilbert quotients. For the proof see [Sno82], [Hei91], and [HMP98].

**Proposition 1.2.2.** Let $G$ be a complex-reductive Lie group and let $X$ be a holomorphic $G$-space with analytic Hilbert quotient $\pi : X \to Y$. Then, the following holds:

1. The analytic Hilbert quotient $Y$ is the categorical quotient of $X$ by the action of $G$ in the category of complex spaces, i.e., given a $G$-invariant holomorphic map $\phi : X \to Z$ into a complex space $Z$, there exists a unique holomorphic map $\overline{\phi} : Y \to Z$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\phi} & Z \\
\pi \downarrow & & \downarrow \\
Y & \xrightarrow{\overline{\phi}} & 
\end{array}
$$

commutes. In particular, $Y$ is unique up to biholomorphism.
2. For every Stein subspace $A$ of $X//G$ the inverse image $\pi^{-1}(A)$ is a Stein subspace of $X$.

3. If $A_j$, $j = 1, 2$ are $G$-invariant analytic subsets of $X$, then $\pi(A_1) \cap \pi(A_2) = \pi(A_1 \cap A_2)$.

4. For a $G$-invariant analytic subset $A$ of $X$, the image $\pi(A)$ in $Y = X//G$ is an analytic subset of $X//G$ and the restriction $\pi|_A : A \to \pi(A)$ is the analytic Hilbert quotient for the action of $G$ on $A$.

It follows that two points $x, x' \in X$ have the same image in $X//G$ if and only if $G \cdot x \cap G \cdot x' \neq \emptyset$. Consequently, for each $q \in X//G$, the fibre $\pi^{-1}(q)$ contains a unique closed $G$-orbit.

**Remark 1.2.3.** Using the theory of momentum maps introduced in Section 1.2.2 it is also possible to show that $X//G$ is the categorical quotient of $X$ with respect to the action of $G$ in the category of topological Hausdorff spaces (see e.g. [HMP98]).

### Coherent analytic $G$-sheaves

If $X$ is a holomorphic $G$-space for a complex-reductive group $G$, we have natural $G$-actions on a number of analytic sheaves on $X$. This section collects the fundamental results related to these actions.

**Definition 1.2.4.** Let $G$ be a Lie group, let $X$ be a complex $G$-space. A sheaf $\pi_\mathcal{F} : \mathcal{F} \to X$ on $X$ is called a $G$-sheaf if there exists a (set-theoretic) $G$-action on $\mathcal{F}$ by sheaf homomorphisms and the diagram

$$
\begin{array}{ccc}
G \times \mathcal{F} & \longrightarrow & \mathcal{F} \\
\id_G \times \pi_\mathcal{F} \downarrow & & \downarrow \pi_\mathcal{F} \\
G \times X & \longrightarrow & X
\end{array}
$$

commutes.

If $\mathcal{F}$ is a $G$-sheaf on $X$, then $G$ also acts on spaces of sections of $\mathcal{F}$. The following definition imposes the appropriate holomorphicity requirement for action on coherent analytic sheaves.

**Definition 1.2.5.** Let $G$ be a complex Lie group and let $\mathcal{F}$ be a coherent analytic sheaf on $X$ that also is a $G$-sheaf. We call $\mathcal{F}$ a coherent analytic $G$-sheaf if for all open sets $N \subset G$ and $U_1, U_2 \subset X$ with $U_2 \subset \bigcap_{g \in N} g \cdot U_1$ the map

$$
\Phi : N \times \mathcal{F}(U_1) \to \mathcal{F}(U_2), \quad (g, f) \mapsto (g \cdot f)|_{U_2}
$$

is holomorphic with respect to the canonical Fréchet topology on $\mathcal{F}(U_j)$, $j = 1, 2$ (see [Akh95]).

**Remark 1.2.6.** If $X$ is a holomorphic $G$-space, then $\mathcal{H}_X$ is a coherent analytic $G$-sheaf on $X$. Furthermore, sheaves of germs of sections of holomorphic vector bundles $B$ over $X$...
are coherent analytic $G$-sheaves if the action of $G$ on $X$ can be lifted to an action of $G$ on $B$ by bundle automorphisms (see [Akh95]). In this case, we call the action of $G$ on $B$ a \textit{linearisation} of the action of $G$ on $X$.

Let $G$ be a complex-reductive Lie group. Consider a holomorphic $G$-space $X$ with analytic Hilbert quotient $\pi : X \to X//G$. For a coherent analytic $G$-sheaf $\pi_*\mathcal{F} : \mathcal{F} \to X$, we denote by $(\pi_*\mathcal{F})^G$ the sheaf of invariants on $X//G$, i.e., for an open set $Q \subset X//G$, we have

$$(\pi_*\mathcal{F})^G(Q) := \mathcal{F}(\pi^{-1}(Q))^G = \{\sigma \in \mathcal{F}(\pi^{-1}(Q)) \mid \sigma(g \cdot x) = g \cdot \sigma(x)\}.$$  

\textbf{Remark 1.2.7.} If $K$ is a maximal compact subgroup of $G$, then $\mathcal{F}$ is a $K$-sheaf and we have $(\pi_*\mathcal{F})^G = (\pi_*\mathcal{F})^K$.

The sheaf $(\pi_*\mathcal{F})^G$ is in a natural way a sheaf of $\mathcal{H}_{X/G}$-modules. It follows that $\pi(\cdot)^G$ is a functor from the category of sheaves of $\mathcal{H}_X$-modules on $X$ to the category of $\mathcal{H}_{X/G}$-modules on $X//G$. In fact, it is an exact functor:

\textbf{Lemma 1.2.8.} For every exact sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ of coherent analytic $G$-sheaves the induced sequence $(\pi_*\mathcal{F}_1)^G \to (\pi_*\mathcal{F}_2)^G \to (\pi_*\mathcal{F}_3)^G$ is likewise exact.

\textbf{Lemma 1.2.8} is a crucial point in the proof of the following result (see [Rob86] and [HH99]).

\textbf{Theorem 1.2.9 (Coherence theorem).} The sheaf $(\pi_*\mathcal{F})^G$ is a coherent analytic sheaf of $\mathcal{H}_{X/G}$-modules on $X//G$.

\textbf{Invariant theory and algebraic Hilbert quotients}

In this section we will introduce a basic class of examples of analytic Hilbert quotients originating from algebraic actions of complex-reductive Lie groups.

Let $G$ be a complex-reductive group endowed with its natural linear-algebraic structure. An \textit{algebraic $G$-variety} is an algebraic variety $X$ together with an action of $G$ on $X$ such that the action map $G \times X \to X$ is regular.

Basic examples of algebraic $G$-varieties are representation spaces $V$ of rational $G$-representations $\rho : G \to GL_C(V)$ of a complex-reductive group $G$.

Let $X$ be an affine algebraic $G$-variety. The action of $G$ on $X$ induces an action of $G$ on the coordinate ring $\mathcal{C}[X]$ of $X$. A classical theorem of Hilbert (see e.g. [MFK94]) says that the algebra of invariants $\mathcal{C}[X]^G$ is finitely-generated, hence it is the coordinate ring of an affine algebraic variety, which we will denote by $X//G$. The inclusion $\mathcal{C}[X]^G \hookrightarrow \mathcal{C}[X]$ induces a map $\pi : X \to X//G$ which is an algebraic Hilbert quotient in the sense of the following

\textbf{Definition 1.2.10.} Let $G$ be a complex-reductive group and let $X$ be an algebraic $G$-
variety. An algebraic variety $Y$ together with a $G$-invariant surjective regular map $\pi : X \to Y$ is called an algebraic Hilbert quotient of $X$ by the action of $G$ if

1. $\pi$ is affine,
2. $(\pi_\ast \mathcal{O}_X)^G = \mathcal{O}_Y$ holds.

Here, affine means that $\pi^{-1}(U)$ is affine for every affine open subset $U$ of $Y$, and $(\pi_\ast \mathcal{O}_X)^G$ is the sheaf $U \mapsto \mathcal{O}_X(\pi^{-1}(U))^G$, $U$ open in $Y$.

Again, an algebraic Hilbert quotient for the action of a complex-reductive group on an algebraic variety is unique once it exists, and we will denote it by $X//G$. Furthermore, algebraic Hilbert quotients have the properties stated in Proposition 1.2.2 in the category of algebraic varieties. For later reference, we state the following criterion for the existence of an algebraic Hilbert quotient.

Lemma 1.2.11 ([BBS97]). Let $X$ be an algebraic $G$-variety. Assume that there exists a $G$-invariant affine map to an algebraic variety $Z$. Then, the algebraic Hilbert quotient $X//G$ exists.

Let $X$ be an algebraic $G$-variety. Then, $X^h$ is in a natural way a holomorphic $G$-space. That algebraic Hilbert quotients are examples of analytic Hilbert quotients is the content of the following

Proposition 1.2.12 ([Lun76]). Let $G$ be a complex-reductive Lie group, and let $X$ be an algebraic $G$-variety. Assume that the algebraic Hilbert quotient $\pi : X \to X//G$ for the action of $G$ on $X$ exists. Then $\pi^h : X^h \to (X//G)^h$ is an analytic Hilbert quotient for the action of $G$ on $X^h$.

As we have seen, in the case of affine varieties it is possible to define an algebraic Hilbert quotient for the action of $G$ on the entire variety. Due to the requirement that the quotient map be an affine map, this will not be possible in the case of projective algebraic $G$-varieties. However, it is possible to find open subsets such that algebraic Hilbert quotients exist:

Let $\rho : G \to GL_C(V)$ be a rational representation of the complex-reductive group $G$. We set

$$\mathcal{N}(V) := \{v \in V \mid f(v) = 0 \text{ for all } f \in C[V]^G\}$$

and call $\mathcal{N}(V)$ the nullcone of the representation space $V$. If $p : V \setminus \{0\} \to \mathbb{P}(V)$ denotes the tautological map (which is an algebraic Hilbert quotient for the $\mathbb{C}^*$-action on $V \setminus \{0\}$ by multiplication), and if $X$ is a $G$-invariant subvariety of $\mathbb{P}(V)$, we call the Zariski-open subset $X(V) := X \setminus p(\mathcal{N}(V) \setminus \{0\})$ the set of semistable points with respect to $V$. Then, the algebraic Hilbert quotient $\pi : X(V) \to X(V)//G$ exists and $X(V)//G$ is a projective algebraic variety. See [MFK94] and Section 3.2.5 for more information on this construction.
1.2.2 Momentum map quotients

Let $K$ be a compact Lie group and $G = K^\mathbb{C}$ its complexification. Up to this point all examples of holomorphic $G$-spaces with analytic Hilbert quotient were constructed from algebraic $G$-varieties. In this section we study holomorphic $G$-spaces $X$ such that the action of $K$ on $X$ is Hamiltonian with respect to a $K$-invariant Kähler structure on $X$. We will introduce a notion of semistability related to momentum maps and we will state the fundamental results about the existence of analytic Hilbert quotients for the action of $G$ on sets of semistable points. For this we have to introduce some notation.

Kähler structures on complex spaces

Let $X$ be a complex space. A continuous function $\rho : X \to \mathbb{R}$ is called plurisubharmonic, if for every holomorphic map $\varphi$ from the unit disc $D$ in $\mathbb{C}$ to $X$, the pullback $\varphi^*(\rho)$ is subharmonic on $D$, i.e., for each $0 < r < 1$ the mean value inequality

$$\varphi^*(\rho)(0) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi^*(\rho)(re^{i\theta}) \, d\theta$$

holds. A perturbation of a continuous function $\rho : X \to \mathbb{R}$ at a point $x \in X$ is a function $\rho + f$, where $f$ is smooth and defined in some neighbourhood $U$ of $x$. The function $\rho$ is said to be strictly plurisubharmonic if for every perturbation $\rho + f$ there exist $\epsilon > 0$ and a neighbourhood $V \subset U$ of $x$ such that $\rho + \epsilon f$ is plurisubharmonic on $V$.

If $X$ and $\rho : X \to \mathbb{R}$ are smooth, then $\rho$ is strictly plurisubharmonic if and only if its Levi form $\omega = \frac{i}{2} \partial \bar{\partial} \rho$ is positive definite, i.e., if $\omega$ is a Kähler form on $X$.

A Kähler structure on a complex space $X$ is given by an open cover $(U_j)$ of $X$ and a family of strictly plurisubharmonic functions $\rho_j : U_j \to \mathbb{R}$ such that the differences $\rho_j - \rho_k$ are pluriharmonic on $U_{jk} := U_j \cap U_k$ in the sense that there exists a holomorphic function $f_{jk} \in \mathcal{H}(U_{jk})$ with $\rho_j - \rho_k = \text{Re}(f_{jk})$. Two Kähler structures $(U_j, \rho_j)$ and $(\tilde{U}_k, \tilde{\rho}_k)$ are considered equal if there exists a common refinement $(V_l)$ of $(U_j)$ and $(\tilde{U}_k)$ such that $\rho_j|_{V_l} - \tilde{\rho}_k|_{V_l}$ is pluriharmonic for every $l, j$, and $k$. A Kähler structure $\omega = \{\rho_j\}$ is called smooth if all the $\rho_j$’s can be chosen as smooth functions.

Here, a function $f : X \to \mathbb{R}$ on a complex space $X$ is called smooth if there exists a covering $\{U_\alpha\}_{\alpha \in A}$ of $X$ and biholomorphic maps $\varphi_\alpha : U_\alpha \to V_\alpha$, where $V_\alpha$ is an analytic subset of an open subset $W_\alpha \subset \mathbb{C}^n$, in such a way that there exist smooth functions $f_\alpha : W_\alpha \to \mathbb{R}$ with $f_\alpha|_{V_\alpha} \circ \varphi_\alpha = f|_{U_\alpha}$ for all $\alpha \in A$.

For smooth $X$, a smooth Kähler structure is the same as a smooth Kähler form which is given locally by $\omega = \frac{i}{2} \partial \bar{\partial} \rho_j$, and in this case we will not distinguish between $\omega = \{\rho_j\}$ as defined above and the associated Kähler form.
Momentum maps and analytic Hilbert quotients

Let $K$ be a Lie group with Lie algebra $\mathfrak{k}$. Let $X$ be a complex $K$-space. A Kähler structure $\omega = \{\rho_j\}_j$ is called $K$-invariant, if the Kähler structure that is given by $k^*\omega = \{k^*(\rho_j) : k^{-1}(U_j) \to \mathbb{R}\}_j$ is equal to $\omega$ in the sense introduced above. We always assume $\omega$ to be smooth in the following. A momentum map with respect to an invariant Kähler structure is a smooth map $\mu : X \to \mathfrak{k}^*$ which is $K$-equivariant with respect to the coadjoint representation of $K$ on $\mathfrak{k}^*$ such that for every $K$-stable complex submanifold $Y$ of $X$ and for every $\xi \in \mathfrak{k}$, we have

$$d\mu^\xi = \iota_{\xi_X} \omega_Y.$$ 

Here, $\iota_{\xi_X}$ denotes contraction with the vector field $\xi_X$ on $X$ that is induced by the $K$-action, $\omega_Y$ denotes the Kähler form induced on $Y$, and the function $\mu^\xi : X \to \mathbb{R}$ is given by $\mu^\xi(x) = \mu(x)(\xi)$. We call the action of $K$ on a complex $K$-space with $K$-invariant Kähler structure $\omega$ Hamiltonian if the $K$-action admits a momentum map with respect to $\omega$.

Let $K$ denote a compact Lie group and $G$ its complexification. Let $X$ be a holomorphic $G$-space with $K$-invariant Kähler structure $\omega$. We call $X$ a Hamiltonian $G$-space, if the $K$-action is Hamiltonian with respect to $\omega$. Given a Hamiltonian $G$-space with momentum map $\mu : X \to \mathfrak{k}^*$, we set $\mathcal{M} := \mu^{-1}(0)$ and

$$X(\mu) := \{x \in X \mid \overline{G \cdot x} \cap \mathcal{M} \neq \emptyset\}.$$ 

We call $X(\mu)$ the set of semistable points with respect to $\mu$ and the $G$-action. In general, if $X$ is a holomorphic $G$-space and $A$ is a subset of $X$, then we set $S_G(A) := \{x \in X \mid \overline{G \cdot x} \cap A \neq \emptyset\}$. With this definition, we have $X(\mu) = S_G(\mathcal{M})$.

The main results about Hamiltonian $G$-spaces needed in the following are collected in

**Theorem 1.2.13** ([HL94], [HHL94]). Let $G$ be the complexification of the compact Lie group $K$. Let $X$ be a Hamiltonian $G$-space with momentum map $\mu : X \to \mathfrak{k}^*$ and zero fibre $\mathcal{M} = \mu^{-1}(0)$. Let $\pi_K : \mathcal{M} \to \mathcal{M}/K$ be the quotient map for the action of $K$ on $\mathcal{M}$ and let $X(\mu)$ be the set of semistable points.

Then, $X(\mu)$ is open in $X$, the analytic Hilbert quotient $\pi : X(\mu) \to X(\mu)//G$ of $X(\mu)$ by the action of $G$ exists, and in the following commutative diagram

$$\begin{array}{ccc}
\mathcal{M} & \longrightarrow & X(\mu) \\
\pi_K \downarrow & & \downarrow \pi \\
\mathcal{M}/K & \longrightarrow & X(\mu)//G
\end{array}$$

the map $\iota$ is a homeomorphism. Furthermore, the space $X(\mu)//G$ carries a (continuous) Kähler structure.

**Example 1.2.14.** Let $K$ be a compact Lie group with Lie algebra $\mathfrak{k}$ and let $G = K^C$ be its complexification. Let $\rho : G \to GL_C(V)$ be a representation of $G$ such that the action of $K$
leaves a Hermitian inner product \( \langle \cdot, \cdot \rangle \) on \( V \) invariant. Endow \( \mathbb{P}(V) \) with the Fubini-Study form induced by \( \langle \cdot, \cdot \rangle \). Let \( X \) be a \( G \)-stable subvariety of \( \mathbb{P}(V) \). Then, \( X^h \) is Kähler with \( K \)-invariant Kähler structure \( \omega \) given by the restriction of the Fubini-Study form to \( X \). In fact, \( X^h \) is a Hamiltonian \( G \)-space and a momentum map with respect to \( \omega \) is given by

\[
\mu^h([v]) = 2i \frac{\langle \rho_* (\xi)v, v \rangle}{\langle v, v \rangle} \quad \forall \xi \in \mathfrak{t}, \forall [v] \in X.
\]

Here, \( \rho_* : \text{Lie}(G) \to \text{End}_\mathbb{C}(V) \) is the Lie algebra homomorphism induced by \( \rho \). The set of semistable points \( X^h(\mu) \) with respect to \( \mu \) coincides with the set of semistable points \( X(V) \) with respect to \( V \) and is therefore Zariski-open in \( X \). It follows that the analytic Hilbert quotient \( X^h(\mu)/G \) is the complex space associated to the projective algebraic variety \( X(V)/G \).

1.3 Projective algebraic varieties

1.3.1 The Cartan-Serre Theorem

**Definition 1.3.1.** A line bundle \( L \) on a compact complex space \( X \) is called **very ample** if there exists a closed embedding \( \phi : X \to \mathbb{P}(\mathbb{C})^n \) into some projective space \( \mathbb{P}(\mathbb{C})^n \) such that

\[
L = \phi^*(H),
\]

where \( H \) denotes the hyperplane bundle on \( \mathbb{P}(\mathbb{C})^n \). A line bundle \( L \) on \( X \) is called **ample** if \( L^\otimes m \) is very ample for some \( m \in \mathbb{N} \).

A compact complex space \( X \) is called **projective algebraic** if there exists a closed holomorphic embedding of \( X \) into some projective space \( \mathbb{P}(\mathbb{C})^n \). By the definition of ampleness a compact complex space \( X \) admits an ample line bundle if and only if it is projective algebraic.

**Remark 1.3.2.** The coherent algebraic locally free sheaf on \( \mathbb{P}(\mathbb{C})^n \) corresponding to \( H \) will be denoted by \( \mathcal{O}_{\mathbb{P}(\mathbb{C})^n}(1) \). If \( p : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}(\mathbb{C})^n \) denotes the canonical projection, the sections of \( \mathcal{O}_{\mathbb{P}(\mathbb{C})^n}(1) \) over an open subset \( U \) of \( \mathbb{P}(\mathbb{C})^n \) are given by

\[
\mathcal{O}_{\mathbb{P}(\mathbb{C})^n}(1)(U) = \{ f \in \mathcal{O}_{\mathbb{C}^{N+1}}(p^{-1}(U)) \mid f(t \cdot z) = t \cdot f(z) \text{ for all } z \in U \}.
\]

The first fundamental fact is that ampleness can be characterised cohomologically (see [Gra62]):

**Theorem 1.3.3 (Cartan-Serre Theorem).** Let \( L \) be a line bundle on a compact complex space \( X \) and \( \mathcal{L} \) the associated locally free sheaf. Then, the following are equivalent:

1. \( L \) is ample.
1.3 Projective algebraic varieties

2. Given any coherent analytic sheaf $\mathcal{F}$ on $X$, there exists a positive integer $m_0 = m_0(\mathcal{F})$ with the property that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^m) = 0 \text{ for all } i > 0, m \geq m_0.$$  

The assertion "1. $\Rightarrow$ 2." is often referred to as Serre’s vanishing theorem.

1.3.2 Serre’s GAGA-Theorems

An algebraic variety that admits a closed algebraic embedding into some projective space $\mathbb{P}^n(\mathbb{C})$ is called a projective algebraic variety. Projective algebraic varieties enjoy a number of remarkable properties which we collect in this subsection. Proofs of the result stated here can be found in [Ser56].

Let $X$ be an algebraic variety and let $X^h$ be the associated complex space. As before, let $\mathcal{O}_X$ and $\mathcal{H}_X$ denote the structure sheaves of $X$ and $X^h$, respectively. Recall that in order to distinguish between the Euclidean and the Zariski-topology of $X$, we refer to subsets of $X$ that are open in the Zariski-topology as Zariski-open. By an algebraic sheaf on $X$ we mean a sheaf of $\mathcal{O}_X$-modules on $X$. Analogously, an analytic sheaf on $X^h$ is a sheaf of $\mathcal{H}_X$-modules on $X^h$.

If $\mathcal{F}$ is any sheaf on $X$, let $\mathcal{F}'$ be the following sheaf on $X^h$: for $U \subset X$ open we set

$$\mathcal{F}'(U) := \lim_\rightarrow \{ \mathcal{F}(W) \mid W \text{ Zariski-open in } X \text{ containing } U \}.$$  

If $\mathcal{F}$ is any algebraic sheaf on the algebraic variety $X$, then we define a corresponding analytic sheaf $\mathcal{F}^h$ on $X^h$ by

$$\mathcal{F}^h := \mathcal{H}_X \otimes_{\mathcal{O}_X} \mathcal{F}'. $$

If an analytic sheaf $\mathcal{G}$ on the complex space $X^h$ is of the form $\mathcal{F}^h$ for some algebraic sheaf $\mathcal{F}$, we call $\mathcal{G}$ algebraically induced. The following theorem summarises the basic properties of the functor $\mathcal{F} \mapsto \mathcal{F}^h$.

**Theorem 1.3.4.** Let $X$ be an algebraic variety. Then, $\mathcal{F} \mapsto \mathcal{F}^h$ is a faithful exact functor from the category of sheaves of $\mathcal{O}_X$-modules on $X$ to the category of sheaves of $\mathcal{H}_X$-modules on $X^h$, i.e., for each exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

of sheaves of $\mathcal{O}_X$-modules on $X$, the sequence

$$0 \to \mathcal{F}_1^h \to \mathcal{F}_2^h \to \mathcal{F}_3^h \to 0$$

of sheaves of $\mathcal{H}_X$-modules is also exact, and if $\mathcal{F}$ is non-zero, then $\mathcal{F}^h$ is non-zero.

**Corollary 1.3.5.** Let $X$ be an algebraic variety. Then, the functor $\mathcal{F} \mapsto \mathcal{F}^h$ takes
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1. $\mathcal{O}_X$ to $\mathcal{H}_X$.

2. coherent algebraic sheaves to coherent analytic sheaves.

3. the ideal sheaf in $\mathcal{O}_X$ of an algebraic subvariety $A \subset X$ to the ideal sheaf in $\mathcal{H}_X$ of the corresponding analytic subset $A^h \subset X^h$.

If $X$ is an algebraic variety, and $\mathcal{F}$ is a sheaf on $X$, then $\mathcal{F}$ and $\mathcal{F}^h$ have the same global sections and in fact the same cohomology. This follows from the fact that a flabby resolution of $\mathcal{F}^h$ yields a flabby resolution of $\mathcal{F}$ when restricted to Zariski-open sets in $X$. So, we have $H^p(X, \mathcal{F}) = H^p(X^h, \mathcal{F}^h)$ for each $p \in \mathbb{N}$. The sheaf morphism $\mathcal{F}^h \to \mathcal{F}^h, m \mapsto 1 \otimes m$ induces a morphism $H^p(X^h, \mathcal{F}^h) \to H^p(X^h, \mathcal{F}^h)$ for each $p \in \mathbb{N}$. Combining these two morphisms we obtain a map $\theta : H^p(X, \mathcal{F}) \to H^p(X^h, \mathcal{F}^h)$.

If the algebraic variety $X$ under consideration is projective algebraic, there is a close connection between properties of $\mathcal{F}$ and $\mathcal{F}^h$.

Theorem 1.3.6 (GAGA). Let $X$ be a projective algebraic variety, and let $\mathcal{F}$ and $\mathcal{G}$ coherent algebraic sheaves on $X$. Then, the following holds:

1. The natural map $H^p(X, \mathcal{F}) \to H^p(X^h, \mathcal{F}^h)$ is an isomorphism for every $p$.

2. Every morphism of analytic sheaves $\mathcal{F}^h \to \mathcal{G}^h$ is induced by a morphism $\mathcal{F} \to \mathcal{G}$ of algebraic sheaves.

3. Every coherent analytic sheaf on the complex space $X^h$ is algebraically induced by a coherent algebraic sheaf $\mathcal{F}$ on $X$. Furthermore, $\mathcal{F}$ is unique up to isomorphism.

4. The category of algebraic vector bundles on $X$ is equivalent to the category of holomorphic vector bundles on $X^h$ under the correspondence $(\cdot)^h$.

5. Every analytic subset of $X^h$ is an algebraic subvariety of $X$.

6. If $Y$ is an algebraic variety, then every holomorphic map from $X^h$ to $Y^h$ is induced by a regular map from $X$ to $Y$.

Note that part 5 and 6 of the preceding theorem imply that a projective algebraic complex space has a unique algebraic structure that makes it into a projective algebraic variety.

1.3.3 Projective morphisms

A morphism $f : Y \to X$ of algebraic varieties defines a morphism $f^h : Y^h \to X^h$ of the associated complex spaces. Given a sheaf $\mathcal{F}$ on $Y$, we consider two image-sheaves associated to $f$ and $f^h$: $R^1f_*\mathcal{F}$ and $R^1f^h_*\mathcal{F}^h$. Here, $R^1f_*\mathcal{F}$ is the sheaf associated to the
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pre-sheaf $U \mapsto H^1(f^{-1}(U), \mathcal{F})$ and $R^1f^h_*\mathcal{F}^h$ is defined analogously. Generalising the map $\theta$ introduced in Section 1.3.2, we obtain a morphism of analytic sheaves

$$\theta : (R^1f_*\mathcal{F})^h \to R^1f^h_*\mathcal{F}^h.$$ 

We will discuss properties of $\theta$ in the special case of projective morphisms, which are defined as follows:

**Definition 1.3.7.** Let $X$ be an affine algebraic variety and $Y$ an algebraic variety. A morphism $f : Y \to X$ is called **projective** if there exists an $n \in \mathbb{N}$ and a closed embedding $\phi : Y \hookrightarrow X \times \mathbb{P}_n(\mathbb{C})$ such that the following diagram commutes

$$
\begin{array}{ccc}
Y & \xrightarrow{\phi} & X \times \mathbb{P}_n(\mathbb{C}) \\
\downarrow{f} & & \downarrow{\text{pr}_1} \\
& & X.
\end{array}
$$

Let $f : Y \to X$ be a projective morphism and let $\mathcal{F}$ be a coherent algebraic sheaf on $Y$. Then, $R^1f_*\mathcal{F}$ is a coherent algebraic sheaf on $X$ (see [Har77, Chap II, Cor 5.20]). Since the associated map $f^h : Y^h \to X^h$ is proper, Grauert’s direct image theorem [GR84, Chap 10] implies that $R^1f^h_*\mathcal{F}^h$ is a coherent analytic sheaf on $X$.

Generalising part 1.) of Theorem 1.3.6, the following is proven in [Gro71, Chap XII]:

**Theorem 1.3.8.** Let $f : Y \to X$ be a projective morphism from an algebraic variety $Y$ to an affine variety $X$, and let $\mathcal{F}$ be a coherent algebraic sheaf on $Y$. Then, the natural map $\theta : (R^1f_*\mathcal{F})^h \to R^1f^h_*\mathcal{F}^h$ is an isomorphism of analytic sheaves over $X^h$. 
Chapter 2

Projectivity of momentum map quotients

In this chapter we prove the main projectivity result for compact momentum map quotients of algebraic varieties. Furthermore, we discuss algebraicity properties of sets of unstable points.

Definition 2.0.9. Let $K$ be a compact Lie group and let $G = K^\mathbb{C}$ be its complexification. Let $X$ be an algebraic $G$-variety such that $X^h$ together with a $K$-invariant Kähler structure and a momentum map $\mu : X^h \to \mathfrak{k}^*$ is a Hamiltonian $G$-space. Then, we call $X$ an algebraic Hamiltonian $G$-variety.

As we have seen in Example 1.2.14, if $V$ is a rational $G$-representation, $G$-invariant sub-varieties of $\mathbb{P}(V)$ provide examples of algebraic Hamiltonian $G$-varieties. Furthermore, in this case, the analytic Hilbert quotient $X(\mu) // G$ is the complex space associated to a projective algebraic variety. That this holds even if the Kähler structure under consideration is not induced by an embedding into some projective space and the momentum map is not induced by a $G$-representation is the content of

Theorem 1 (Projectivity Theorem). Let $K$ be a compact Lie group and $G = K^\mathbb{C}$ its complexification. Let $X$ be a $G$-irreducible algebraic Hamiltonian $G$-variety with momentum map $\mu : X \to \mathfrak{k}^*$. Assume that $X$ has only 1-rational singularities. Then, the analytic Hilbert quotient $X(\mu) // G$ is projective algebraic if and only if $\mu^{-1}(0)$ is compact.

Remark 2.0.10. Note that in view of the fundamental diagram (1.1), the momentum zero fibre $\mu^{-1}(0)$ is compact if and only if the analytic Hilbert quotient $X(\mu) // G$ is compact. It follows that compactness of $\mu^{-1}(0)$ is necessary for projectivity of the quotient $X(\mu) // G$.

The proof uses the following projectivity criterion of Namikawa:

Theorem 2.0.11 ([Nam02]). Let $Z$ be a Moishezon complex space with 1-rational singularities.
If $Z$ admits a smooth Kähler structure, then it is projective algebraic.

**Remark 2.0.12.** This result improves the classical result of Moishezon [Moi66] on the projectivity of Kähler Moishezon manifolds. It does not continue to hold if the assumption on the singularities of $Z$ is dropped (see [Moi75] for an example).

This chapter is organised as follows:

In the first section we recall basic facts about meromorphic function fields of compact complex spaces and we discuss Moishezon spaces. We then prove that compact momentum map quotients of algebraic Hamiltonian $G$-varieties are Moishezon. In the second section we introduce the notion of 1-rational singularity and show that under the assumptions of Theorem 1 the analytic Hilbert quotient $X(\mu)/G$ has 1-rational singularities. In the third section, we use a result of Varouchas to show that momentum map quotients admit smooth Kähler structures and prove Theorem 1.

As an application of Theorem 1, we prove in Section 2.4 that the set of unstable points $X \setminus X(\mu)$ is an algebraic subset of $X$. In Section 2.5 we show that the compactness assumption on $\mu^{-1}(0)$ made in Theorem 1 is essential for algebraicity of the analytic Hilbert quotient $X(\mu)/G$. Furthermore, we discuss proper momentum maps in Section 2.6.

### 2.1 Function fields of momentum map quotients

In this section we study the function fields of momentum map quotients. First, we recall the definition and basic properties of the function field of an irreducible complex space and introduce the class of Moishezon spaces. Then, we consider invariant meromorphic and rational functions on algebraic $G$-varieties and prove that compact momentum map quotients of algebraic Hamiltonian $G$-varieties are Moishezon.

#### 2.1.1 Meromorphic functions

Let $X$ be a complex space and $\mathcal{O}_X$ its structure sheaf. For an open subset $U \subset X$ let $D(U)$ be the subset of zero-divisors of $\mathcal{O}_X(U)$, i.e., the set of those holomorphic functions that vanish on some irreducible component of $U$. Let

$$D(U) = \left\{ \frac{f}{g} \mid f \in \mathcal{O}_X(U), g \in \mathcal{O}_X(U) \setminus D(U) \right\} / \sim$$

be the quotient ring of $\mathcal{O}_X(U)$ with respect to $D(U)$. Here, the equivalence relation $\sim$ is given by

$$\frac{f}{g} \sim \frac{f'}{g'} \text{ if and only if } fg' = f'g.$$
If \( V \subset U \) is an inclusion of open sets, the restriction map \( r^U_V : \mathcal{H}(U) \to \mathcal{H}(V) \) sends \( \mathcal{H}(U) \setminus D(U) \) into \( \mathcal{H}(V) \setminus D(V) \) and thus induces a homomorphism of rings
\[
r^U_V : \mathcal{D}(U) \to \mathcal{D}(V).
\]
In this way, we obtain a presheaf \( \mathcal{D} \) on \( X \). The sheaf \( \mathcal{M}_X \) associated to this presheaf is called the sheaf of germs of meromorphic functions on \( X \). The ring \( H^0(X, \mathcal{M}_X) = \mathcal{M}_X(X) \) is called the ring of meromorphic functions on \( X \). If \( X \) is irreducible, then \( \mathcal{M}_X(X) \) is a field, called the function field of \( X \).

Let \( f \in \mathcal{M}_X(X) \) be a meromorphic function on \( X \). Introduce the sheaf of denominators of \( f \) as the sheaf \( \mathcal{D}(f) \) with stalks
\[
\mathcal{D}(f)_x = \{ v_x \in \mathcal{H}_x : v_x f_x \in \mathcal{H}_x \}.
\]
We define the polar set of \( f \) to be the zero set of this sheaf and denote it by \( P_f \). It is a nowhere dense analytic subset of \( X \), see [GR84, Chap 6, §3]

**Remark 2.1.1.** The polar set \( P_f \) of a meromorphic function \( f \in \mathcal{M}_X(X) \) is the smallest subset of \( X \) such that \( f \) is holomorphic on \( X \setminus P_f \). We set \( \text{dom}(f) := X \setminus P_f \) and call \( \text{dom}(f) \) the domain of definition of \( f \).

For \( f \in \mathcal{M}_X(X) \) we define
\[
\Gamma^0_f := \{(x, [f(x) : 1]) \in X \times \mathbb{P}^1(\mathbb{C}) \mid x \in \text{dom}(f)\}.
\]
The point set closure of \( \Gamma^0_f \) in \( X \times \mathbb{P}^1 \) will be called the graph of the meromorphic function \( f \) on \( X \). We denote it by \( \Gamma_f \).

**Definition 2.1.2.** Let \( X \) be a complex space. An analytic set \( \Gamma \subset X \times \mathbb{P}^1 \) together with the canonical projection \( \sigma : \Gamma \to X \) is called a holomorphic graph at \( p \in X \) if there exists an open neighbourhood \( U \) of \( p \) such that
1. \( \sigma|_{\sigma^{-1}(U)} : \sigma^{-1}(U) \to U \) is biholomorphic,
2. \( \sigma^{-1}(U) \cap (U \times \{[1 : 0]\}) = \emptyset \).

If \( f \in \mathcal{H}_X(X) \) is a holomorphic function on \( X \), then its graph \( \Gamma_f \subset X \times \mathbb{C} \subset X \times \mathbb{P}^1 \) is a holomorphic graph at all points \( p \) of \( X \).

**Definition 2.1.3.** An analytic subset \( \Gamma \subset X \times \mathbb{P}^1 \) with the canonical projection \( \sigma : \Gamma \to X \) is called a meromorphic graph over \( X \) if there exists an analytic subset \( A \subset X \) such that
1. \( A \) and \( \sigma^{-1}(A) \) are nowhere dense,
2. \( \Gamma \) is a holomorphic graph outside \( A \).

**Proposition 2.1.4.** The graph \( \Gamma_f \) of a meromorphic function on a complex space is a meromorphic graph.
Conversely, it is a natural idea to try to define a meromorphic function by a meromorphic graph. The following proposition shows that this is indeed possible and that there is a bijection

\[
\begin{cases}
\text{meromorphic functions on } X \\
\text{meromorphic graphs over } X
\end{cases}
\longleftrightarrow
\begin{cases}
\text{meromorphic functions on } X \\
\text{meromorphic graphs over } X
\end{cases}
\]

**Proposition 2.1.5.** Let \( \Gamma \hookrightarrow X \times \mathbb{P}_1 \) be a meromorphic graph over \( X \). Then there exists a uniquely determined meromorphic function \( f \in \mathcal{M}_X(X) \) such that \( \Gamma = \Gamma_f \).

The proofs of Propositions 2.1.4 and 2.1.5 can be found in [Fis75] and [Fis76].

We note that a surjective holomorphic map \( \varphi : X \to Y \) between complex spaces introduces a map \( \varphi^* : \mathcal{M}_Y(Y) \to \mathcal{M}_X(X) \) between the corresponding rings of meromorphic functions.

If \( X \) is an algebraic variety, then we denote by \( C(X) \) the algebra of rational functions on \( X \). Every \( f \in C(X) \) induces a meromorphic function on \( X \). As a consequence of the GAGA results of Section 1.3.2, we obtain

**Proposition 2.1.6.** Let \( X \) be a projective algebraic variety. Then, every meromorphic function on \( X^h \) is induced by a rational function of \( X \).

**Proof.** Let \( f \in \mathcal{M}_X(X) \) be a meromorphic function on \( X \). Then, by Theorem 1.3.6, the graph \( \Gamma_f \) is an algebraic subset of the projective algebraic variety \( X \times \mathbb{P}_1 \), and there exists a dense Zariski-open subset \( U \subset X \) such that \( \sigma|_{\sigma^{-1}(U)} : \sigma^{-1}(U) \to U \) defines a regular function on \( U \). Hence, \( f \) is rational. \( \square \)

**Transcendence degree of function fields**

Let us recall some terminology of field theory. Let \( K \subset L \) be a field extension. It is called **finite** if \( \dim_K L < \infty \). It is called **finitely generated** if there exist finitely many elements \( l_1, \ldots, l_m \in L \) such that \( L = K(l_1, \ldots, l_m) \). Here, \( K(l_1, \ldots, l_m) \) denotes the smallest subfield of \( L \) containing \( K \) as well as \( l_1, \ldots, l_m \in L \). A set of elements \( l_1, \ldots, l_m \) of \( L \) is called **algebraically dependent** over \( K \) if there exists a non-identically zero polynomial \( P \in K[Y_1, \ldots, Y_m] \) such that

\[
P(l_1, \ldots, l_m) = 0 \in L.
\]

Otherwise, \( l_1, \ldots, l_m \) are called **algebraically independent**. The **transcendence degree** of \( L \) over \( K \) is the maximal number of algebraically independent elements of \( L \). We denote it by \( \text{trdeg}_K L \). If \( K \subset L \subset M \) are field extensions, then we have \( \text{trdeg}_K L \leq \text{trdeg}_K M \).

Using this terminology, we have
Theorem 2.1.7. Let $X$ be a compact irreducible complex space. Then $\mathcal{M}_X(X)$ is a finitely generated field extension of $\mathbb{C}$ and we have

$$\text{trdeg}_\mathbb{C} \mathcal{M}_X(X) \leq \dim X.$$  

Furthermore, if $\{f_1, \ldots, f_k\}$ is a maximal algebraically independent set of $\mathcal{M}_X(X)$, then the field extension $\mathbb{C}(f_1, \ldots, f_k) \subset \mathcal{M}_X(X)$ is finite.

Early versions of Theorem 2.1.7 are due to Siegel [Sie55] and Thimm [Thi54]. A modern proof can be found in Chapter 6, §6 of [GR84].

Example 2.1.8. Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$ given by $k \cdot v = 2^k v$. This action is proper and free, hence the quotient $X := \mathbb{C}^2 \setminus \{0\} / \mathbb{Z}$ is a 2-dimensional complex manifold, called a Hopf surface. We claim that $\text{trdeg}_\mathbb{C} \mathcal{M}_X(X) = 1 < \dim X$.

Indeed, if $f \in \mathcal{M}_X(X)$, by Levi’s extension theorem for meromorphic functions, see [GR84, Chap 9, §5], the pullback $\pi^*(f)$ under the the quotient map $\pi : \mathbb{C}^2 \setminus \{0\} \to X$ extends to a $\mathbb{Z}$-invariant meromorphic function $F$ on $\mathbb{C}^2$ (see section 2.1.2). We can write $F = \frac{g}{h}$ for some relatively prime $g, h \in \mathcal{M}_{\mathbb{C}^2}(\mathbb{C}^2)$. Given $z_0 \in \mathbb{C}^2 \setminus \{0\}$, we define two holomorphic functions $F_{z_0}, \tilde{F}_{z_0} \in \mathcal{M}_{\mathbb{C}}(\mathbb{C})$ by setting $F_{z_0}(w) = g(z_0)h(w \cdot z_0)$ and $\tilde{F}_{z_0}(w) = g(w \cdot z_0)h(z_0)$. The sequence $\{2^{-k}\}_{k \in \mathbb{N}}$ converges to zero in $\mathbb{C}$ and we have $F_{z_0}(2^{-k}) = \tilde{F}_{z_0}(2^{-k})$ for all $k \in \mathbb{N}$, due to the $\mathbb{Z}$-invariance of $F$. Hence, $F_{z_0}$ coincides with $\tilde{F}_{z_0}$ and $F$ is $\mathbb{C}^*$-invariant. It follows that $g$ and $h$ are homogeneous (of the same degree), hence polynomial. Consequently, $F$ defines a rational function on $\mathbb{P}1$. Conversely, every quotient of two homogeneous polynomials on $\mathbb{C}^2$ of the same degree induces a meromorphic function on $X$. Therefore, $\mathcal{M}_X(X)$ is isomorphic to $\mathbb{C}(\mathbb{P}1) \cong \mathbb{C}(x)$, and hence, we have $\text{trdeg}_\mathbb{C} \mathcal{M}_X(X) = 1$, as claimed.

Definition 2.1.9. An irreducible compact complex space $X$ is called Moishezon if

$$\text{trdeg}_\mathbb{C} \mathcal{M}_X(X) = \dim X.$$  

Example 2.1.10. Let $X$ be an irreducible projective algebraic variety. Then, every meromorphic function on the complex space $X^h$ is a rational function and hence, we have $\mathcal{M}_X(X^h) = \mathbb{C}(X)$. Furthermore, since $X$ is an irreducible algebraic variety, $\text{trdeg}_\mathbb{C} \mathbb{C}(X) = \dim X$ holds, see [Har77, Chap I, Thm 3.2 and Cor 4.5]. Hence, Lemma 1.1.6 implies that the complex space $X^h$ is Moishezon.

2.1.2 Invariant meromorphic functions

We consider algebraic varieties and complex spaces with algebraic and holomorphic group actions, respectively, and we will investigate induced actions on spaces of rational and meromorphic functions.
Group actions and meromorphic functions

**Definition 2.1.11.** Let $G$ be a Lie group. A complex $G$-space $X$ is called $G$-irreducible if it cannot be non-trivially decomposed as the union $X = X_1 \cup X_2$ of two $G$-invariant analytic subsets $X_1, X_2$ of $X$.

A complex space $X$ is $G$-irreducible if and only if $G$ acts transitively on the set of irreducible components of $X$. In general, every complex $G$-space $X$ has a decomposition $X = \bigcup X_j$ into $G$-irreducible components $X_j$ which is unique up to the numbering of the components. Analogously, if $G$ is an algebraic group, we define the notion of $G$-irreducible algebraic $G$-variety and $G$-irreducible component of an algebraic $G$-variety.

Let $X$ be a $G$-irreducible complex $G$-space. Consider the graph $\Gamma_f \subset X \times \mathbb{P}^1$ of a meromorphic function $f \in \mathcal{M}_X(X)$. The group $G$ acts on $X \times \mathbb{P}^1$ by the $G$-action on the first factor. Given $g \in G$, we define a new meromorphic graph $\Gamma_g \cdot f := g \cdot \Gamma_f \subset X \times \mathbb{P}^1$.

This defines a meromorphic function $g \cdot f$ on $X$ by Proposition 2.1.5. In this way we obtain a group action on $\mathcal{M}_X(X)$ by algebra automorphisms. A meromorphic function $f \in \mathcal{M}_X(X)$ is $G$-invariant if and only if its graph $\Gamma_f$ is a $G$-invariant subset of $X \times \mathbb{P}^1$.

Now let $G$ be an algebraic group and let $X$ be a $G$-irreducible algebraic $G$-variety. Then, we have $C(X) \subset \mathcal{M}_X(X)$. Furthermore, this inclusion is compatible with the natural group action of $G$ on $C(X)$ which is given as follows:

Let $f \in C(X)$ be a rational function on $X$ and let $\text{dom}(f)$ be its domain of definition. Then, $f$ is regular on $\text{dom}(f)$ and hence yields a regular function on $g \cdot \text{dom}(f)$ by $(g \cdot f)(x) = f(g^{-1} \cdot x)$. The function $g \cdot f$ extends to a rational function $\hat{f}$ on $X$ and we set $g \cdot f := \hat{f} \in C(X)$.

Invariant rational functions and geometric quotients

**Definition 2.1.12.** Let $G$ be an algebraic group and let $X$ be an algebraic $G$-variety. A geometric quotient for the $G$-action on $X$ is a surjective regular map $p : X \to Y$ having the following properties:

1. For all $x \in X$, we have $p^{-1}(p(x)) = G \cdot x$. I.e., the fibres of $p$ are the orbits of the $G$-action.

2. The algebraic variety $Y$ has the quotient Zariski-topology with respect to $p$.

3. For every Zariski-open set $V \subset Y$, the comorphism $p^* : \mathcal{O}_Y(V) \to \mathcal{O}_X(p^{-1}(V))^G$ is an isomorphism, i.e., we have $\mathcal{O}_Y = (p^* \mathcal{O}_X)^G$.

**Remark 2.1.13.** It follows from the definition that a geometric quotient for the $G$-action on an algebraic variety $X$ is unique, once it exists. We will denote it by $X / G$. 

Chapter 2. Projectivity of momentum map quotients
Let $G$ be an algebraic group and $X$ a $G$-irreducible algebraic $G$-variety. We define $m := \max_{x \in X} \{ \dim G \cdot x \}$. We call an orbit $G \cdot x$ in $X$ generic if $\dim G \cdot x = m$. This terminology is justified by

**Lemma 2.1.14.** The set $X_{\text{reg}} := \{ x \in X \mid \dim G \cdot x = m \}$ is Zariski-open and dense in $X$.

**Proof.** If $G$ is connected, the lemma is classical and can for example be found in [PV94]. For the general case, let $G^0$ be the connected component of the identity of $G$. We have $m = \max_{x \in X} \{ \dim G^0 \cdot x \}$ and $X_{\text{reg}} = \{ x \in X \mid \dim G^0 \cdot x = m \}$. The decomposition $X = \bigcup_{j=1}^m X_j$ is $G_0$-stable, and we have $X_{\text{reg}} = G \cdot (X_j)_{\text{reg}}$ for all $j$, where $(X_j)_{\text{reg}}$ is the set of generic $G_0$-orbits in $X_j$, and hence, the claim follows from the classical case. \[ \square \]

It follows from the theorem on the dimension of the fibres (see [Mum99, Chap I, §8]) that any open set $U$ of $X$ such that a geometric quotient $U/G$ exists has to be a subset of $X_{\text{reg}}$. The following result of Rosenlicht (see [Ros56] and [Ros63]) shows that there always exists a $G$-invariant open subset $U$ of $X$ with a geometric quotient $U/G$ which is determined by the field of invariant rational functions $C(X)^G$.

**Theorem 2.1.15.** Let $X$ be a $G$-irreducible algebraic $G$-variety. Then $X$ contains a Zariski-open and dense $G$-invariant subset $U_R \subset X_{\text{reg}} \subset X$ such that the geometric quotient $p : U_R \to U_R/G$ exists. The variety $U_R/G$ is isomorphic to the field $C(X)^G$ of $G$-invariant rational functions on $X$. In particular, we have

$$\trdeg_C(C(X)^G) = \dim X - m,$$

where $m = \max_{x \in X} \{ \dim G \cdot x \}$.

### 2.1.3 Function fields of momentum map quotients

Let $K$ be a compact Lie group and let $G = K^C$ its complexification. Let $X$ be a $G$-irreducible algebraic Hamiltonian $G$-variety with momentum map $\mu : X \to \mathfrak{t}^*$. Let $Q = X(\mu) / G$ denote the analytic Hilbert quotient of the set of semistable points with respect to $\mu$. The $G$-irreducibility of $X$ implies that $X(\mu)$ is either empty or $G$-irreducible and dense in $X$ (see [HH96]). It follows that the complex space $Q$ is irreducible.

We will show that, if $\mu^{-1}(0)$ is compact, the momentum map quotient $Q$ is a Moishezon space. For this we will show that sufficiently many invariant rational functions on $X$ descend to the quotient $Q$. The following example shows that in general, not every invariant rational function is obtained by pullback from the quotient.

**Example 2.1.16.** Consider the action of $G := C^* = (S^1)^C$ on $X := C^2$ that is given by scalar multiplication. The action of $S^1$ is Hamiltonian with respect to the standard Kähler form on $C^2$ and, after identification of $\text{Lie}(S^1)^*$ with $\mathbb{R}$, a momentum map is given by $v \mapsto |v|^2$. We have $\mu^{-1}(0) = \{0\}$ and $X(\mu) = C^2$. The analytic Hilbert quotient $X(\mu) / G$
is a point. It follows that every rational function on $X(\mu) // G$ is constant. The rational function $f(z, w) = \frac{z}{w}$ on $X$ is $C^*$-invariant. However, it is not the pullback of a rational function from $X(\mu) // G = \{pt\}$ via the quotient map $\pi : C^2 \to \{pt\}$, since it is non-constant and has poles along the hypersurface $\{w = 0\}$.

**The case** $X_{\text{reg}} \cap \mu^{-1}(0) \neq \emptyset$

In this section, we assume that the intersection $X_{\text{reg}} \cap \mu^{-1}(0)$ is non-empty. Under this assumption we will prove (see Proposition 2.1.20) that compact momentum map quotients of algebraic Hamiltonian $G$-varieties are Moishezon.

As a first observation, we have

**Lemma 2.1.17.** The set $\pi(X_{\text{reg}} \cap \mu^{-1}(0))$ is an analytically Zariski-open dense subset of the analytic Hilbert quotient $Q$. We set $X^s := \pi^{-1}(\pi(X_{\text{reg}} \cap \mu^{-1}(0)))$. Then, $X^s$ is analytically Zariski-open and dense in $X(\mu)$, we have $X^s = G \cdot \mu^{-1}(0)$, and the restriction

$$\pi|_{X^s} : X^s \to \pi(X^s) \subset Q$$

is a geometric quotient for the $G$-action on $X^s$, i.e., the fibres of $\pi|_{X^s}$ are orbits of the $G$-action.

**Proof.** Let $X_{\text{reg}}^c$ be the complement of $X_{\text{reg}}$ in $X$. This is algebraic in $X$ and $G$-invariant, hence its intersection with $X(\mu)$ is analytic in $X(\mu)$ and still $G$-invariant. Since

$$\pi(X_{\text{reg}}^c \cap \mu^{-1}(0)) = \pi(X_{\text{reg}}^c \cap X(\mu))$$

holds, the latter being the image of a $G$-invariant analytic set in $X(\mu)$, $\pi(X_{\text{reg}} \cap \mu^{-1}(0))$ is analytically Zariski-open in $Q$. Since $Q$ is irreducible, $\pi(X_{\text{reg}} \cap \mu^{-1}(0))$ is also dense in $Q$. It follows that $X^s$ is analytically Zariski-open and dense in $X(\mu)$.

Let $x \in X^s$. Then, the closure in $X(\mu)$ of the $G$-orbit through $x$ contains a unique closed $G$-orbit, say $G \cdot y \subset \overline{C \cdot x}$, for some $y \in X_{\text{reg}} \cap \mu^{-1}(0)$. Furthermore, since the action of $G$ on $X$ is algebraic, $G \cdot x$ is Zariski-open in $\overline{G \cdot x}$. Since $y \in X_{\text{reg}}$, this implies that $G \cdot x = G \cdot y$ and that $G \cdot x$ is closed in $X(\mu)$. Therefore, the restriction of $\pi$ to $X^s$ is a geometric quotient.

We will now consider invariant meromorphic functions on $X(\mu)$ and we will show that these can be pushed down to meromorphic functions on the analytic Hilbert quotient $Q$.

The map $\pi \times \text{id}_{\mathbb{P}_1} : X(\mu) \times \mathbb{P}_1 \to Q \times \mathbb{P}_1$ is an analytic Hilbert quotient for the action of $G$ on $X(\mu) \times \mathbb{P}_1$ that is induced by the action of $G$ on the first factor. Let $f \in \mathcal{M}_X(X(\mu))^G$ be a $G$-invariant meromorphic function on $X$ and let $\Gamma_f \subset X(\mu) \times \mathbb{P}_1$ its graph. Since $f$ is $G$-invariant, $\Gamma_f$ is a $G$-invariant analytic subset of $X(\mu) \times \mathbb{P}_1$. Hence, the image $\hat{\Gamma} := (\pi \times \text{id}_{\mathbb{P}_1})(\Gamma_f)$ is an analytic subset of $Q \times \mathbb{P}_1$. We set $\Phi = (\pi \times \text{id}_{\mathbb{P}_1})|_{\hat{\Gamma}} : \Gamma_f \to \hat{\Gamma} \equiv \Gamma_f // G$.

The following is the crucial technical result of this section.
Proposition 2.1.18. Let \( f \in \mathcal{M}_X(X(\mu))^G \). Then, the analytic set \( \hat{\Gamma} \subset Q \times \mathbb{P}_1 \) is a meromorphic graph. It defines a meromorphic function \( \hat{f} \) on \( Q \) with \( f = \pi^*(\hat{f}) \).

Proof. We summarise our setup in the following commutative diagram:

\[
\begin{array}{ccc}
X(\mu) & \xrightarrow{\tilde{pr}_1} & \mathbb{P}_1 \\
\downarrow & & \downarrow \\
X(\mu) \times \mathbb{P}_1 & \xrightarrow{\Gamma_f} & \mathbb{P}_1 \\
\downarrow{\pi \times \text{id}_{\mathbb{P}_1}} & & \downarrow{\pi} \\
Q \times \mathbb{P}_1 & \xrightarrow{\tilde{\pi}} & \mathbb{P}_1 \\
\downarrow{pr_1} & & \downarrow{pr_1} \\
Q & \xrightarrow{\pi} & \pi(V) \\
\end{array}
\]

(2.1)

We notice that the polar set \( P_f \) of \( f \) is \( G \)-invariant, since \( f \in \mathcal{M}_X(X(\mu))^G \). Let \( V := X^s \setminus P_f \). Since \( X^s \) is Zariski-open and dense in \( X(\mu) \) and since \( P_f \) is a nowhere dense analytic subset of \( X(\mu) \), \( V \) is analytically Zariski-open and dense in \( X(\mu) \). Since \( \pi|_{X^s} : X^s \to \pi(X^s) \) is a geometric quotient, and since \( P_f \) is \( G \)-invariant, \( V \) is \( \pi \)-saturated. Thus, \( \pi(V) \) is analytically Zariski-open and dense in \( Q \). The restriction of Diagram (2.1) to \( \pi(V) \) contains the following commutative diagram as a subdiagram:

\[
\begin{array}{ccc}
\Gamma_f|_V & \xrightarrow{\Phi} & \hat{\Gamma}|_{\pi(V)} \\
\downarrow{\tilde{pr}_1} & & \downarrow{pr_1} \\
V & \xrightarrow{\pi} & \pi(V). \\
\end{array}
\]

The subset \( \pi(V) \subset Q \) is isomorphic to \( V//G \) and we have \( f \in \mathcal{M}_X(V)^G \cong \mathcal{M}_Q(\pi(V)) \). Therefore, there exists a unique holomorphic function \( \hat{f} \in \mathcal{M}_Q(\pi(V)) \) such that \( f = \hat{f} \circ \pi \).

We have

\[
\Gamma_f = \hat{\Gamma}|_{\pi(V)} \subset \pi(V) \times \mathbb{C}.
\]

It follows that \( \hat{\Gamma} \) is a holomorphic graph over \( \pi(V) \).

The set \( A := X(\mu) \setminus V \) is \( \pi \)-saturated and analytic in \( X(\mu) \). Hence, \( \pi(A) = Q \setminus \pi(V) \) is a nowhere dense analytic subset of \( Q \). It remains to show that \( \text{pr}_1^{-1}(\pi(A)) \) is nowhere dense in \( \hat{\Gamma} \). We have \( \Phi^{-1}(\text{pr}_1^{-1}(\pi(A))) = \tilde{\text{pr}}_1^{-1}(A) \) by Diagram (2.1). Since \( \Gamma_f \) is a meromorphic graph over \( X(\mu) \) and since \( A \) is nowhere dense in \( X(\mu) \), the preimage \( \tilde{\text{pr}}_1^{-1}(A) \) is nowhere dense in \( \Gamma_f \). It follows that \( \text{pr}_1^{-1}(\pi(A)) \) is nowhere dense in \( \hat{\Gamma} \) since its preimage under \( \Phi \) has this property.

Hence, we have shown that \( \hat{\Gamma} \) is a meromorphic graph in \( Q \times \mathbb{P}_1 \). By Proposition 2.1.5, it defines a meromorphic function \( \hat{f} \) on \( Q \). It follows from the construction that \( f = \pi^*(\hat{f}) \) holds. \( \Box \)
As a next step, we compute the dimension of the analytic Hilbert quotient $X(\mu)///G$.

**Lemma 2.1.19.** We have $\dim Q = \dim X - m$, where $m = \max_{x \in X} \{ \dim G \cdot x \}$.

**Proof.** Since $X^s \subset X_{\text{reg}}$ and the map $\pi|_{X^s} : X^s \to \pi(X^s)$ is a geometric quotient, we have $\dim \pi^{-1}(q) = m$ for every $q \in \pi(X^s)$. The set $\pi(X^s)$ is Zariski-open and dense in $Q$. Hence, it follows that $\dim Q = \dim X - m$ by the theorem on the fibre dimension of holomorphic maps, see [Rem57, Satz 15].

**Proposition 2.1.20.** Let $K$ be a compact Lie group and $G = K^C$ its complexification. Let $X$ be a $G$-irreducible algebraic Hamiltonian $G$-variety with momentum map $\mu : X \to \mathfrak{k}^*$. Let $Q = X(\mu)///G$ denote the analytic Hilbert quotient. Assume that $X_{\text{reg}} \cap \mu^{-1}(0) \neq \emptyset$. Then, $\text{trdeg}_C \mathcal{M}_Q(Q) \geq \dim Q$. In particular, if $Q$ is compact, then it is Moishezon.

**Proof.** We have a necessarily injective field homomorphism $C(X)^G \hookrightarrow \mathcal{M}_X(X(\mu))^G$. Furthermore, by the procedure of Proposition 2.1.18, we have defined a field homomorphism $\mathcal{M}_X(X(\mu))^G \hookrightarrow \mathcal{M}_Q(Q)$. Hence, we get the following chain of inequalities

$$
\dim Q = \dim X - m \quad \text{by Lemma 2.1.19}
= \text{trdeg}_C C(X)^G \quad \text{by Rosenlicht’s theorem}
\leq \text{trdeg}_C \mathcal{M}_X(X(\mu))^G
\leq \text{trdeg}_C \mathcal{M}_Q(Q) \quad \text{by Prop. 2.1.18}.
$$

**The general case**

We now drop the assumption $X_{\text{reg}} \cap \mu^{-1}(0) \neq \emptyset$.

**Theorem 2.1.21.** Let $K$ be a compact Lie group and $G = K^C$ its complexification. Let $X$ be a $G$-irreducible algebraic Hamiltonian $G$-variety with momentum map $\mu : X \to \mathfrak{k}^*$. Let $Q = X(\mu)///G$ denote the analytic Hilbert quotient. Then, the following inequality holds:

$$
\text{trdeg}_C \mathcal{M}_Q(Q) \geq \dim Q.
$$

In particular, if $Q$ is compact, it is Moishezon.

**Proof.** We prove the claim by induction over the dimension of $X$. If $\dim X = 0$, $G$-irreducibility implies that $X$ is $G$-homogeneous. It follows that $Q$ is a point and therefore Moishezon.

Now let $\dim X = n + 1$ and assume that the claim is true for all varieties of dimension less than or equal to $n$. If $X_{\text{reg}} \cap \mu^{-1}(0) \neq \emptyset$, then Proposition 2.1.20 applies and we are done. So we can assume that $X_{\text{reg}} \cap \mu^{-1}(0) = \emptyset$. Consider the algebraic subvariety $Y := X \setminus X_{\text{reg}}$ of $X$. We have $\mu^{-1}(0) \subset Y$ and therefore $\pi(Y(\mu)) = \pi(\mu^{-1}(0)) = Q$. Let
2.2 Singularities of momentum map quotients

Let \( Y = \bigcup_{j=1}^m Y_j \) be the decomposition of \( Y \) into \( G \)-irreducible components. Then, we have \( Y(\mu) = \bigcup_{j=1}^m Y_j(\mu) \cap Y_j(\mu) \) and \( Q = \bigcup_{j=1}^m \pi(Y_j(\mu)) \). Since \( Q \) is irreducible and each \( \pi(Y_j(\mu)) \) is analytic in \( Q \), there exists a \( j_0 \in \{1, \ldots, m\} \) such that \( \pi(Y_{j_0}(\mu)) = Q \). It follows that \( Y_{j_0}(\mu)/G \cong Q \). Since \( \dim Y_{j_0} < \dim X \), induction applies and the claim is proven.

2.2 Singularities of momentum map quotients

In this section we study the singularities of momentum map quotients of algebraic Hamiltonian \( G \)-varieties. We introduce the notion of 1-rational singularities and we show that the class of varieties with this type of singularities is stable under the operation of taking algebraic Hilbert quotients. Afterwards, we apply this result in our study of momentum map quotients.

2.2.1 Singularities and resolution of singularities

If \( X \) is an algebraic variety or complex space we denote by \( \text{tdim}_x X \) the dimension of the Zariski tangent space at \( x \in X \). I.e., if \( m_x \) denotes the maximal ideal in \( \mathcal{O}_{X,x} \), then \( \text{tdim}_x X := \dim_{\mathbb{C}} \frac{m_x}{m_x^2} \). A point \( x \) in an algebraic variety or complex space \( X \) is called \textit{singular} if \( \text{tdim}_x X > \dim X \). We denote the singular set of an algebraic variety or a complex space \( X \) by \( X_{\text{sing}} \). A variety or complex space \( X \) is called \textit{smooth} if \( X_{\text{sing}} = \emptyset \).

A crucial tool in our study is the following

\textbf{Proposition 2.2.1} ([Ser56]). Let \( X \) be the germ at 0 of an algebraic subvariety of \( \mathbb{C}^n \) and let \( X^h \) be the corresponding germ of a holomorphic subvariety. Let \( I(X) \) and \( I(X^h) \) be the ideal of \( X \) and \( X^h \) in \( \mathcal{O}_{\mathbb{C}^n,0} \) and \( \mathcal{H}_{\mathbb{C}^n,0} \), respectively. Then, \( I(X^h) = \mathcal{H}_{\mathbb{C}^n,0} \cdot I(X) \).

\textbf{Proposition 2.2.2.} Let \( X \) be an algebraic variety and \( X^h \) the associated complex space. Then, \( \text{tdim}_x X = \text{tdim}_x X^h \). In particular, the set of singular points \( X_{\text{sing}} \) of \( X \) agrees with the set of singular points \( (X^h)_{\text{sing}} \) of \( X^h \).

\textbf{Proof.} The statement is local, hence we can assume that \( X \) is an affine algebraic subvariety of \( \mathbb{C}^n \). Let \( x \in X \) be any point. Let \( f_1, \ldots, f_m \) be a set of generators for \( I(X_x) \) over \( \mathcal{O}_{\mathbb{C}^n,x} \), then \( f_1, \ldots, f_m \) also generate \( I(X^h_x) \) over \( \mathcal{H}_{\mathbb{C}^n,x} \) by Proposition 2.2.1. Consequently, if \( J_F(x) \) denotes the Jacobian matrix at \( x \) of the map \( F : X \to \mathbb{C}^m \) with coordinate functions \( f_j \), we have

\[ \text{tdim}_x X = n - \text{rk} J_F(x) = \text{tdim}_x X^h \]

by the Jacobi criterion in algebraic geometry (see [Mum99, Chap III, §4]) and in complex analytic geometry (see [GR84, Chap 6, §1]).
A point \( x \in X \) is by definition singular if \( \text{tdim}_x X > \dim_x X \). The same is true for \( X^h \). The discussion above shows that \( \text{tdim}_x X = \text{tdim}_x X^h \), and we know (Lemma 1.1.6) that \( \dim X = \dim X^h \). Therefore, the set of singular points of \( X \) coincides with the set of singular points of \( X^h \).

**Definition 2.2.3.** Let \( X \) be an algebraic variety. A *resolution* of \( X \) is a proper birational surjective morphism \( f : Y \to X \) from a smooth algebraic variety \( Y \) to \( X \).

In order to introduce the corresponding concept in the analytic category, we recall

**Definition 2.2.4.** A holomorphic map \( f : Y \to X \) between complex spaces \( X \) and \( Y \) is called a *proper modification*, if

1. \( f \) is proper and surjective,
2. there exists a nowhere dense analytic subset \( S \subset X \), called the *center of the modification*, such that \( f^{-1}(S) \) is nowhere dense in \( Y \) and such that \( f \) induces a biholomorphic mapping of \( Y \setminus f^{-1}(S) \) onto \( X \setminus S \).

**Definition 2.2.5.** Let \( X \) be a complex space. A *resolution* of \( X \) is a proper modification \( f : Y \to X \), where \( Y \) is a complex manifold.

If \( X \) is an algebraic variety or a complex space, by a theorem of Hironaka ([Hir64] and [Hir77]) there exists a resolution \( f : \tilde{X} \to X \) of \( X \) such that the restriction \( f : f^{-1}(X \setminus X_{\text{sing}}) \to X \setminus X_{\text{sing}} \) is an isomorphism. See also [BM91] and [EH02] for later improvements and simplifications of Hironaka’s proof.

The next lemma follows immediately from the definitions.

**Lemma 2.2.6.** Let \( f : Y \to X \) be a resolution of an algebraic variety \( X \). Then, \( f^h : Y^h \to X^h \) is a resolution of the associated complex space \( X^h \).

### 2.2.2 1-rational singularities

In the following we will not allow the varieties under discussion to have arbitrary singularities, but we will restrict to a special class of singularities introduced below.

**Definition 2.2.7.** An algebraic variety or complex space \( X \) is said to have *1-rational singularities*, if the following two conditions are fulfilled:

1. \( X \) is normal,
2. for every resolution \( f : \tilde{X} \to X \) of \( X \), we have \( R^1 f_* \mathcal{O}_{\tilde{X}} = 0 \) or \( R^1 f_* \mathcal{H}_{\tilde{X}} = 0 \), respectively, where \( R^1 f_* \mathcal{O}_{\tilde{X}} \) and \( R^1 f_* \mathcal{H}_{\tilde{X}} \) are defined as in Section 1.3.3.

**Proposition 2.2.8.** Let \( X \) be a normal algebraic variety or a normal complex space. If there exists one resolution \( f_0 : X_0 \to X \) of \( X \) such that \( R^1 (f_0)_* \mathcal{O}_{X_0} = 0 \) or \( R^1 (f_0)_* \mathcal{H}_{X_0} = 0 \), respectively,
then $X$ has 1-rational singularities.

Proof. We give the proof for algebraic varieties. Let $f_1 : X_1 \to X$ be a second resolution of $X$. By [Hir64], there exits a smooth algebraic variety $Z$ and resolutions $g_0 : Z \to X_0$ and $g_1 : Z \to X_1$ such that the following diagram commutes

\[
\begin{array}{ccc}
Z & \xrightarrow{g_0} & X_0 \\
\downarrow & & \downarrow f_0 \\
X_1 & \xrightarrow{g_1} & X
\end{array}
\]

For $j = 0, 1$ there exists a spectral sequence (see [Wei94]) with lower terms

\[0 \to R^1(f_j)_*((g_j)_*\mathcal{O}_Z) \to R^1(f_j \circ g_j)_*\mathcal{O}_Z \to (f_j)_*(R^1(g_j)_*\mathcal{O}_Z) \to \cdots .\]

Since $g_0$ and $g_1$ are resolutions of smooth algebraic varieties, we have $R^1(g_j)_*\mathcal{O}_Z = 0$ and $(g_j)_*\mathcal{O}_Z = \mathcal{O}_{X_j}$ for $j = 0, 1$ (see [Hir64] and [Uen75]). It follows that

\[0 = R^1(f_0)_*\mathcal{O}_{X_0} \cong R^1(f_0 \circ g_0)_*\mathcal{O}_Z \cong R^1(f_1 \circ g_1)_*\mathcal{O}_Z \cong R^1(f_1)_*\mathcal{O}_{X_1}.\]

Remark 2.2.9. The proof shows that in general, if $f_1 : X_1 \to X$ and $f_2 : X_2 \to X$ are two resolutions of $X$, then there is an isomorphism $R^1(f_1)_*\mathcal{O}_{X_1} \cong R^1(f_2)_*\mathcal{O}_{X_2}$. From this, it follows that having 1-rational singularities is a local property. Furthermore, if $X$ is an algebraic $G$-variety for an algebraic group $G$, and $f : \tilde{X} \to X$ is a resolution of $X$, then the support of $R^1f_*\mathcal{O}_X$ is a $G$-invariant analytic subset of $X$.

As a consequence of Proposition 2.2.8, we obtain

Corollary 2.2.10. Let $X$ be an algebraic variety and $X^h$ the associated complex space. Then, $X$ has 1-rational singularities if and only if $X^h$ has 1-rational singularities.

Proof. First, we prove that $X$ is normal if and only if $X$ is normal: if $x \in X$, the completions of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{H}_{X,x}$ with respect to their respective maximal ideals agree (see [Ser56]). The claim now follows from the fact that a local Noetherian ring is normal if and only if its completion is normal (see [ZS75]).

The claim is local, so we can assume that $X$ is affine. Assume that $X$ has only 1-rational singularities. By [Hir64], there exists a resolution $f : \tilde{X} \to X$ of $X$ such that $f$ is a projective morphism. The map $f^h : \tilde{X}^h \to X^h$ is a resolution of singularities by Lemma 2.2.6. It follows from Proposition 1.3.8 that

\[0 = (R^1f_*\mathcal{O}_X)^h \cong R^1f_*^h\mathcal{H}_h.\]

With the help of Proposition 2.2.8 we see that $X^h$ has 1-rational singularities.
Let us now assume that $X^h$ has 1-rational singularities. Consider a projective resolution $f : \tilde{X} \to X$ of $X$. Then, $f^h : \tilde{X}^h \to X^h$ is a resolution of $X^h$. Since $X^h$ has 1-rational singularities, we have
\[(R^1f_*\mathcal{O}_{\tilde{X}})^h \cong R^1f^h_*\mathcal{H}_{\tilde{X}} = 0.\]
Since the functor $\mathcal{F} \to \mathcal{F}^h$ is faithful (see Theorem 1.3.4), it follows that $R^1f_*\mathcal{O}_{\tilde{X}} = 0$.

Rational singularities

In this section we shortly discuss the relation of the notion "1-rational singularity" to the better known notion of "rational singularity".

**Definition 2.2.11.** An algebraic variety or complex space $X$ is said to have rational singularities, if the following two conditions are fulfilled:

1. $X$ is normal,

2. for every resolution $f : \tilde{X} \to X$ of $X$, we have $R^jf_*\mathcal{O}_{\tilde{X}} = 0$ or $R^jf_*\mathcal{H}_{\tilde{X}} = 0$, respectively, for all $j = 1, \ldots, \dim X$.

**Remark 2.2.12.** Again, the vanishing of the higher direct image sheaves is independent of the chosen resolution. Furthermore, an algebraic variety $X$ has rational singularities if and only if $X^h$ has rational singularities.

Due to a result of Malgrange [Mal57, p. 236] asserting that $R^{\dim X}f_*\mathcal{O}_{\tilde{X}} = 0$ for every resolution $f : \tilde{X} \to X$ of an irreducible variety $X$, an algebraic surface has 1-rational singularities if and only if it has rational singularities. That these notions differ for higher dimensions is illustrated by the following example.

**Example 2.2.13.** Consider the smooth quartic hypersurface
\[Z = \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{P}_3,\]
and let $X$ be the cone over $Z$ in $\mathbb{C}^4$. The variety $X$ has an isolated singularity at the origin. Let $L$ be the total space of the line bundle $\mathcal{O}_Z(-1)$, i.e., the restriction of the dual of the hyperplane bundle of $\mathbb{P}_3$ to $Z$. Then, blowing down the zero section $Z_L \subset L$, and setting $\tilde{X} := L$, we obtain a map $f : \tilde{X} \to X$ which is a resolution of singularities, an isomorphism outside of the origin $0 \in X$ with $f^{-1}(0) = Z_L \cong Z$. We claim that $0 \in X$ is a 1-rational singularity which is not rational.

To see that the origin is a normal point of $X$, we have to check that $Z$ is projectively normal, which in turn is equivalent to $Z$ being a normal variety and the restriction maps $H^0(\mathbb{P}_3, \mathcal{O}_{\mathbb{P}_3}(k)) \to H^0(Z, \mathcal{O}_Z(k))$ being onto for all $k \in \mathbb{N}$ [Har77, Chap II, Ex 5.14]. Since $Z$ is smooth, it remains to check the second condition, which is fulfilled by [Har77, Chap III, Ex 5.5].
To compute \((R^j f_* \mathcal{O}_{\tilde{X}})_0\), we use the Leray spectral sequence
\[
\cdots \rightarrow H^i(X, \mathcal{O}_X) \rightarrow H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \rightarrow H^{i+1}(X, R^j f_* \mathcal{O}_{\tilde{X}}) \rightarrow \cdots
\]
and the fact that \(X\) is affine to show that \(H^j(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^0(X, R^j f_* \mathcal{O}_{\tilde{X}}) = (R^j f_* \mathcal{O}_{\tilde{X}})_0\). Expanding cohomology classes into Taylor series along fibres of \(L\), we get that
\[
H^j(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong \bigoplus_{k \geq 0} H^j(Z, \mathcal{O}_Z(k)).
\]
Hence, we have
\[
(R^j f_* \mathcal{O}_{\tilde{X}})_0 \cong \bigoplus_{k \geq 0} H^j(Z, \mathcal{O}_Z(k)) \quad \text{for all } j \geq 1. \tag{2.2}
\]
It follows from (2.2) and [Har77, Chap III, Ex 5.5] that \((R^1 f_* \mathcal{O}_{\tilde{X}})_0 = 0\), and hence that \(0 \in X\) is a 1-rational singularity. Since the canonical bundle \(K_Z\) of \(Z\) is trivial, it follows from Serre duality that \(H^2(Z, \mathcal{O}_Z(k)) \cong H^0(Z, \mathcal{O}_Z(-k))\). As a consequence, we get
\[
H^2(Z, \mathcal{O}_Z(k)) = \begin{cases} C & \text{for } k = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}
\]
This together with (2.2) implies that \((R^2 f_* \mathcal{O}_{\tilde{X}})_0 = C\) and hence, that \(0 \in X\) is not a rational singularity.

### 2.2.3 Singularities of algebraic Hilbert quotients

Let \(G\) be a complex-reductive Lie group and let \(X\) be an algebraic \(G\)-variety such that the algebraic Hilbert quotient \(X/\!\!/ G\) exists. We want to study the singularities of \(X/\!\!/ G\) relative to the singularities of \(X\).

More precisely, we will prove

**Theorem 2.2.14.** Let \(G\) be a complex-reductive Lie group and let \(X\) be an algebraic \(G\)-variety such that the algebraic Hilbert quotient \(\pi : X \rightarrow X/\!\!/ G\) exists. Assume that \(X\) has 1-rational singularities. Then, \(X/\!\!/ G\) has 1-rational singularities.

**Remark 2.2.15.** In our proof we follow [Bou87], where it is shown that algebraic Hilbert quotients of varieties with rational singularities have rational singularities, and we check that in Boutot’s arguments it is possible to separate the different cohomology dimensions.

Before we start to prove the theorem, we prove two technical lemmas. The first one discusses the relation between cohomology modules of \(X\) and of \(X/\!\!/ G\).

**Lemma 2.2.16.** Let \(X\) be an algebraic \(G\)-variety with algebraic Hilbert quotient \(\pi : X \rightarrow X/\!\!/ G\). Then, the natural map \(\pi^* : H^1(X/\!\!/ G, \mathcal{O}_{X/\!\!/ G}) \rightarrow H^1(X, \mathcal{O}_X)\) is injective.

**Proof.** Let \(\mathcal{U} = \{U_i\}_{i \in I}\) be an affine open covering of \(X/\!\!/ G\). Then, we can compute the cohomology module \(H^1(X/\!\!/ G, \mathcal{O}_{X/\!\!/ G})\) via \(\check{\text{C}}\text{ech}\) cohomology with respect to the covering \(\mathcal{U}\). Since \(\pi\) is an affine map, \(\pi^{-1}(\mathcal{U}) := \{\pi^{-1}(U_i)\}_{i \in I}\) is an affine open covering of \(X\) and
we can compute the cohomology module $H^1(X, \mathcal{O}_X)$ via Čech cohomology with respect to the covering $\pi^{-1}(\mathcal{U})$.

Let $\eta = (\eta_{ij}) \in C^1(\mathcal{U}, \mathcal{O}_{X/\mathcal{G}})$ be a Čech cocycle such that the pullback of the associated cohomology class $[\eta] \in H^1(X/\mathcal{G}, \mathcal{O}_{X/\mathcal{G}})$ fulfills $\pi^*([\eta]) = 0 \in H^1(X, \mathcal{O}_X)$. Then, there exists a cocycle $\nu = (\nu_i) \in C^0(\pi^{-1}(\mathcal{U}), \mathcal{O}_X)$ such that

$$\pi^* (\eta_{ij}) = \nu_i|_{\pi^{-1}(U_{ij})} - \nu_j|_{\pi^{-1}(U_{ij})} \in \mathcal{O}_X(\pi^{-1}(U_{ij})).$$

Averaging $\nu_i \in \mathcal{O}_X(\pi^{-1}(U_{ij}))$ over a maximal compact subgroup $K$ of $G$ we obtain an invariant function $\tilde{\nu}_i \in \mathcal{O}_X(\pi^{-1}(U_{ij}))^G$ and new cocycle $\tilde{\nu} = (\tilde{\nu}_i) \in C^0(\pi^{-1}(\mathcal{U}), \mathcal{O}_X)$ which fulfills

$$\pi^* (\eta_{ij}) = \tilde{\nu}_i|_{\pi^{-1}(U_{ij})} - \tilde{\nu}_j|_{\pi^{-1}(U_{ij})} \in \mathcal{O}_X(\pi^{-1}(U_{ij})).$$

since $\pi^* (\eta_{ij}) \in \mathcal{O}_X(\pi^{-1}(U_{ij}))^G$. For all $i$, there exist a uniquely determined function $\tilde{\nu}_i \in \mathcal{O}_{X/\mathcal{G}}(U_i)$ with $\pi^* (\tilde{\nu}_i) = \tilde{\nu}_i$. Consequently, we have

$$\eta_{ij} = \tilde{\nu}_i|_{U_{ij}} - \tilde{\nu}_j|_{U_{ij}} \in \mathcal{O}_{X/\mathcal{G}}(U_{ij}).$$

Therefore, $[\eta] = 0 \in H^1(X/\mathcal{G}, \mathcal{O}_{X/\mathcal{G}})$ and $\pi^*$ is injective, as claimed. \qed

The second lemma will be used to obtain information about the singularities of an algebraic variety $X$ from information about the singularities of a general hyperplane section $H$ of $X$ and vice versa.

**Lemma 2.2.17.** Let $X$ be a normal affine variety, $\dim X \geq 1$, and let $f : \bar{X} \to X$ be a resolution of singularities. Let $\mathcal{L} \subset \mathcal{O}_X(X)$ be a finite-dimensional subspace, such that the associated linear system is base-point free. If $h \in \mathcal{L}$ is a general element, then the following holds for the corresponding hyperplane section $H \subset X$:

1. The preimage $\bar{H} := f^{-1}(H)$ is smooth and $f|_{\bar{H}} : \bar{H} \to H$ is a resolution of $H$.
2. We have $R^j f_* \mathcal{O}_{\bar{H}} \cong R^j f_* \mathcal{O}_{\bar{X}} \otimes \mathcal{O}_H$ for $j = 0, 1$.

**Proof.** 1.) This follows from Bertini’s theorem (see [Har77, Chap III, Cor 10.9]).

2.) In the exact sequence

$$0 \to \mathcal{O}_{\bar{X}}(\bar{H}) \xrightarrow{m} \mathcal{O}_{\bar{X}} \to \mathcal{O}_{\bar{H}} \to 0, \quad (2.4)$$

the map $m$ is given by multiplication with the equation $h \in \mathcal{O}_X(X)$ that defines $H$ and $\bar{H}$. Pushing forward the short exact sequence $(2.4)$ by $f_*$ yields the long exact sequence

$$0 \to f_* \mathcal{O}_{\bar{X}}(\bar{H}) \xrightarrow{m_0} f_* \mathcal{O}_{\bar{X}} \to f_* \mathcal{O}_{\bar{H}} \to$$

$$\to R^1 f_* \mathcal{O}_{\bar{X}}(\bar{H}) \xrightarrow{m_1} R^1 f_* \mathcal{O}_{\bar{X}} \to R^1 f_* \mathcal{O}_{\bar{H}} \to$$

$$\to R^2 f_* \mathcal{O}_{\bar{X}}(\bar{H}) \xrightarrow{m_2} R^2 f_* \mathcal{O}_{\bar{X}} \to R^2 f_* \mathcal{O}_{\bar{H}} \to \cdots, \quad (2.5)$$
Since $X$ is an affine variety, the exact sequence above is completely determined by the following sequence of finite $\mathcal{O}_X(X)$-modules:

$$
0 \to \Gamma(\tilde{X}, \mathcal{O}_\tilde{X}(-\tilde{H})) \xrightarrow{m_0} \Gamma(\tilde{X}, \mathcal{O}_\tilde{X}) \to \Gamma(\tilde{X}, \mathcal{O}_{\tilde{H}}) \to \\
- \to H^1(\tilde{X}, \mathcal{O}_\tilde{X}(-\tilde{H})) \xrightarrow{m_1} H^1(\tilde{X}, \mathcal{O}_\tilde{X}) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{H}}) \to \\
- \to H^2(\tilde{X}, \mathcal{O}_\tilde{X}(-\tilde{H})) \xrightarrow{m_2} H^2(\tilde{X}, \mathcal{O}_\tilde{X}) \to H^2(\tilde{X}, \mathcal{O}_{\tilde{H}}) \to \\
\cdots \tag{2.6}
$$

The maps $m_j$, $j = 0, 1, 2$ are given by multiplication with the element $h \in \mathcal{O}_X(X)$. We claim that we can choose $h \in \mathcal{L}$ in such a way that $m_1$ and $m_2$ are injective. Indeed, if $R$ is a commutative Noetherian ring with unity, $M$ is a finite $R$-module, and $Z_R(M)$ denotes the set of zerodivisors for $M$ in $R$, we have

$$
Z_R(M) = \bigcup_{P \in \text{Ass} M} P,
$$

where Ass $M$ is the finite set of assassins (or associated primes) of $M$ (see for example [ZS75]). Hence, the set of zerodivisors for $H^1(\tilde{X}, \mathcal{O}_\tilde{X}(-\tilde{H}))$ and $H^2(\tilde{X}, \mathcal{O}_\tilde{X}(-\tilde{H}))$ is a union of finitely many prime ideals $P$ of $\mathcal{O}_X(X)$. Since the linear system associated to $\mathcal{L}$ is base-point free, the general element $h$ of $\mathcal{L}$ lies in $\mathcal{L} \setminus \bigcup P$. For $h \in \mathcal{L} \setminus \bigcup P$, the maps $m_1$ and $m_2$ in the sequences (2.6) and (2.5) are injective.

Since $\mathcal{O}_\tilde{X}(-\tilde{H}) \cong f^*(\mathcal{O}_X(-H))$, the projection formula for locally free sheaves (see [Har77, Chap III, Ex 8.3]) yields

$$
R^jf_*\mathcal{O}_\tilde{X}(-\tilde{H}) \cong R^jf_*\mathcal{O}_\tilde{X} \otimes \mathcal{O}_X(-H).
$$

Furthermore, the image of $m_j$ coincides with the image $\mathcal{B}_j$ of the natural map $R^jf_*\mathcal{O}_\tilde{X} \otimes \mathcal{O}_X(-H) \to R^jf_*\mathcal{O}_\tilde{X}$, $j = 0, 1$. Since, $m_1$ and $m_2$ are injective by the choice of $H$, it follows that

$$
R^jf_*\mathcal{O}_{\tilde{H}} \cong R^jf_*\mathcal{O}_\tilde{X} / \mathcal{B}_j \quad \text{for } j = 0, 1.
$$

Since tensoring with $R^jf_*\mathcal{O}_\tilde{X}$ is right-exact, the exact sequence

$$
0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0
$$

yields $R^jf_*\mathcal{O}_{\tilde{H}} \cong R^jf_*\mathcal{O}_\tilde{X} \otimes \mathcal{O}_H$, as claimed. \hfill \Box

**Remark 2.2.18.** The two previous lemmata also hold for higher-dimensional cohomology groups and image sheaves. However, we presented the proof just for those cases that we will need in the following discussion.

**Proof of Theorem 2.2.14.** Since the claim is local and $\pi$ is an affine map, we may assume that $X//G$ and $X$ are affine.

First, we prove that normality of $X$ implies normality of $X//G$. We have to show that $C[X//G] \cong C[X]^G$ is a normal ring. So let $\alpha \in \text{Quot}(C[X]^G) \subset C(X)^G$ be an element of
the quotient field of $\mathbb{C}[X]^G$ and assume that $\alpha$ fulfills a monic equation

$$\alpha^n + c_1\alpha^{n-1} + \cdots + c_n = 0$$

with coefficients $c_j \in \mathbb{C}[X]^G \subset \mathbb{C}[X]$. Since $\mathbb{C}[X]$ is normal by assumption, it follows that $\alpha \in \mathbb{C}(X)^G \cap \mathbb{C}[X] = \mathbb{C}[X]^G$. Hence, $\mathbb{C}[X]^G$ is normal. As a consequence, we can assume in the following that $X$ is $G$-irreducible.

We prove the claim by induction on $\dim X//G$. For $\dim X//G = 0$ there is nothing to show. For $\dim X//G = 1$, we notice that $X//G$ is smooth, since it is normal. So, let $\dim X//G \geq 2$. Let $\pi : X \to X//G$ denote the quotient map and let $p_X : \tilde{X} \to X$ be a resolution of $X$. First, we prove that a general hyperplane section $H \subset X//G$ has 1-rational singularities. If $H$ is a general hyperplane section in $X//G$, Lemma 2.2.17 applied to $\pi^{-1}(H)$ yields that $p_X|_{\hat{H}} : \hat{H} \to \pi^{-1}(H)$ is a resolution, where $\hat{H} = p_X^{-1}(\pi^{-1}(H))$. Furthermore, we have

$$R^j(p_X)_*\mathcal{O}_{\tilde{X}} \otimes \mathcal{O}_X \mathcal{O}_{\pi^{-1}(H)} = R^j(p_X)_*\mathcal{O}_{\hat{H}} \quad j = 0, 1.$$ 

From the case $j = 0$, it follows that $\pi^{-1}(H)$ is normal, since $f_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ by Zariski’s main theorem (see [Har77, Chap III, Cor 11.4]). Together with the case $j = 1$, this implies that $\pi^{-1}(H)$ has 1-rational singularities. By induction, it follows that $H = \pi^{-1}(H)//G$ has 1-rational singularities.

Let $p : Z \to X//G$ be a resolution of $X//G$. As we have seen above, a general hyperplane section $H$ of $X//G$ has 1-rational singularities and the restriction of $p$ to $\hat{H} := p^{-1}(H)$, $p|_{\hat{H}} : \hat{H} \to H$ is a resolution of $H$. It follows that $\mathcal{O}_H \otimes R^1 p_*\mathcal{O}_Z = R^1 p_*\mathcal{O}_\hat{H} = 0$. Consequently, the support of $R^1 p_*\mathcal{O}_Z$ does not intersect $H$ and hence, $\text{supp}(R^1 p_*\mathcal{O}_Z)$ consists of isolated points. Since the claim is local, we can assume in the following that $R^1 p_*\mathcal{O}_Z$ is supported at a single point $x_0 \in X//G$.

The group $G$ acts on the fibre product $Z \times_{X//G} X$ such that the map $p_X : Z \times_{X//G} X \to X$ is equivariant. One of the $G$-irreducible components $\tilde{X}$ of $Z \times_{X//G} X$ is birational to $X$, and, by passing to a resolution of $\tilde{X}$ if necessary, we can assume that $p_X : \tilde{X} \to X$ is a resolution of $X$. We obtain the following commutative diagram

$$\begin{array}{ccc}
X & \xleftarrow{p_X} & \tilde{X} \\
\pi \downarrow & & \downarrow p_Z \\
X//G & \xleftarrow{p} & Z.
\end{array}$$

Since $R^1 p_*\mathcal{O}_Z$ is supported only at $x_0$, we have $(R^1 p_*\mathcal{O}_Z)_{x_0} = H^0(X//G, R^1 p_*\mathcal{O}_Z)$. Recall that $X//G$ is affine, hence, the Leray spectral sequence

$$0 \to H^1(X//G, \mathcal{O}_{X//G}) \to H^1(Z, \mathcal{O}_Z) \to H^0(X//G, R^1 p_*\mathcal{O}_Z) \to H^2(X//G, \mathcal{O}_{X//G}) \to \cdots$$

implies that it suffices to show that $H^1(Z, \mathcal{O}_Z) = 0$. 


Since \( X \) is affine and has 1-rational singularities, it follows from the Leray spectral sequence
\[
0 \to H^1(X, \mathcal{O}_X) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to H^0(X, R^1 f_* \mathcal{O}_{\tilde{X}}) \to \cdots
\]
that \( H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^1(X, \mathcal{O}_X) = 0 \). Consequently, it suffices to show that there exists an injective map \( H^1(Z, \mathcal{O}_Z) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \).

We introduce the following notation: \( U = X//G \setminus \{ x_0 \} \), \( U' = \pi^{-1}(U) \subset X \), \( \tilde{U} = p_X^{-1}(U') \subset \tilde{X} \), \( V = p^{-1}(U) \subset Z \). We obtain the following commutative diagram of canonical maps
\[
\begin{array}{ccc}
H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \xrightarrow{h_{Z,U}} & H^1(\tilde{U}, \mathcal{O}_U) \\
\uparrow^{h_{Z,X}} & & \uparrow^{h_{U,V}} \\
H^1(Z, \mathcal{O}_Z) & \xleftarrow{h_{Z,V}} & H^1(V, \mathcal{O}_V)
\end{array}
\]
(2.7)

Let \( Y := p^{-1}(x_0) \subset Z \). If \( H^i_Y(Z, \mathcal{O}_Z) \) denotes the local cohomology groups with support in \( Y \), there exists an exact sequence
\[
\cdots \to H^i_Y(Z, \mathcal{O}_Z) \to H^i(Z, \mathcal{O}_Z) \xrightarrow{h_{Z,V}} H^i(V, \mathcal{O}_V) \to \cdots,
\] (2.8)
see [Har77, Chap III, Ex 2.3]. We use the following vanishing result which is originally due to Hartshorne and Ogus [HO74].

**Proposition 2.2.19.** Let \( W \) be an affine algebraic variety, \( p : Z \to W \) a resolution. Let \( w \in W \) and \( Y = p^{-1}(w) \). Then, we have \( H^i_Y(Z, \mathcal{O}_Z) = 0 \) for all \( i < \dim W \).

**Sketch of proof.** Set \( n := \dim W \). As a first step, we compactify \( p \): there exist projective completions \( Z \) and \( \overline{W} \) of \( Z \) and \( W \), respectively, and a resolution \( \overline{p} : Z \to \overline{W} \) such that \( \overline{p}^{-1}(W) = Z \) and \( \overline{p}|_Z = p \). By the Excision Theorem of Local Cohomology (see [Har77, Chap III, Ex 2.3]), we have \( H^i_Y(Z, \mathcal{O}_Z) = H^i_Y(Z, \mathcal{O}_Z) \), since \( Z \) is an open neighbourhood of \( Y \) in \( Z \). Let \( \mathcal{K}_Z \) be the locally free sheaf associated to the canonical bundle of \( Z \). The Formal Duality Theorem (see [Har70]) implies that the dual \( H^i_Y(Z, \mathcal{O}_Z)^* \) of \( H^i_Y(Z, \mathcal{O}_Z) \) is isomorphic to \( (R^{n-j-1}\mathcal{K}_Z)^w \), where \( \sim \) denotes completion with respect to the maximal ideal of \( \mathcal{O}_{\overline{W},w} = \mathcal{O}_{W,w} \). In summary, we have obtained an isomorphism
\[
H^i_Y(Z, \mathcal{O}_Z)^* \cong (R^{n-j-1}\mathcal{K}_Z)^w \quad \text{for all } j = 0, \ldots, n.
\]

By Grauert-Riemenschneider vanishing (see e.g. [Laz04, Chap 4.3.B]), the term on the right hand side equals zero for \( n - j \geq 1 \). This proves the claim.

Since \( \dim X//G \geq 2 \), Proposition 2.2.19 yields \( H^1_Y(Z, \mathcal{O}_Z) = 0 \). As a consequence of (2.8), \( h_{Z,V} \) is injective.

The restriction of \( p \) to \( V = p^{-1}(U) \) is a resolution of \( U \). Since the support of \( R^1 p_* \mathcal{O}_Z \) is concentrated at \( x_0 \), the variety \( U \) has 1-rational singularities, and the Leray spectral
Consequently, we have $h^1(U, \mathcal{O}_U) \xrightarrow{h_{U,V}} H^1(V, \mathcal{O}_V) \to H^0(U, R^1 p_* \mathcal{O}_U) \to \cdots$ yields that $h_{U,V}$ is bijective. Similar arguments show that $h_{U,\overline{U}}$ is bijective. Furthermore, Lemma 2.2.16 implies that $h_{U,U'}$ is injective.

By the considerations above, the map

$$h_{Z,\overline{X}} := h_{U',\overline{U}} \circ h_{U,U'} \circ h_{U,V}$$

is injective. By Diagram (2.7), we have $h_{Z,\overline{X}} = h_{X,\overline{U}} \circ h_{Z,\overline{X}}$, and therefore, $h_{Z,\overline{X}}$ is injective. Consequently, we have $H^1(Z, \mathcal{O}_Z) = 0$, and we have completed the proof of Theorem 2.2.14.

### 2.2.4 Local linearisation by the Slice Theorem

We return to our specific situation: let $X$ be an algebraic $G$-variety such that $X^h$ together with a $K$-invariant Kähler structure and a momentum map $\mu : X^h \to \mathfrak{t}^*$ is a Hamiltonian $G$-space. Our goal is to show that if $X$ has 1-rational singularities, then the analytic Hilbert quotient $X(\mu)//G$ has 1-rational singularities. This section is devoted to the reduction of this problem to the equivalent question for algebraic Hilbert quotient with the help of a slice theorem.

**Lemma 2.2.20.** Let $G$ be a complex reductive Lie group and $V$ a $G$-module. Consider the induced action of $G$ on $\mathbb{P}(V)$. If $[v] \in \mathbb{P}(V)^G$, there exists a $G$-invariant hyperplane $H \subset \mathbb{P}(V)$ not containing $[v]$. The set $W := \mathbb{P}(V) \setminus H$ is $G$-equivariantly isomorphic to the $G$-module $L^1 \otimes L^{-1}$, where $L^1$ is a $G$-stable complementary subspace to $L := \mathbb{C}v \subset V$, and $L^{-1}$ is the dual of the $G$-module $L$.

**Proof.** Since $[v] \in \mathbb{P}(V)^G$, by definition $[G \cdot v] = [v]$, which implies that $L := \mathbb{C}v \subset V$ is a 1-dimensional $G$-invariant subspace in $V$. Since $G$ is reductive, $V$ is completely reducible, and hence there exists a complementary $G$-invariant subspace $L^1 \subset V$. Set $H := \mathbb{P}(L^1) \subset \mathbb{P}(V)$. Then, $H$ is a $G$-invariant hyperplane in $\mathbb{P}(V)$ and $[v] \notin H$.

By construction $V = L \oplus L^1$ as $G$-module and $G$ acts on $L$ by a character $\chi : G \to \mathbb{C}^*$. Choose coordinates $v$ for $L$ and $w$ for $L^1$. Then, $[v : w]$ are homogeneous coordinates for $\mathbb{P}(V)$ and $\mathbb{P}(V) \setminus H = \{[v : w] \in \mathbb{P}(V) \mid v \neq 0\}$. Let $L^{-1}$ be the one-dimensional representation of $G$ with character $\chi^{-1}$. Then, $\phi : W := \mathbb{P}(V) \setminus H \to L^1 \otimes L^{-1}, [v : w] \mapsto \frac{w}{v}$ is a $G$-equivariant isomorphism.

**Lemma 2.2.21.** Let $G$ be a connected complex-reductive Lie group and let $X$ be a normal algebraic $G$-variety. Let $x \in X$ have reductive stabiliser $G_x$ in $G$. Then, there exists an affine $G_x$-invariant open neighbourhood of $x$ in $X$. 
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Proof. Let \( x \in X \). Since \( G \) is connected and \( X \) is normal, by a result of Sumihiro [Sum74], there exists a \( G \)-invariant quasi-projective open neighbourhood \( U \) of \( x \) in \( X \). Furthermore, there exists a \( G \)-linearised ample line bundle on \( U \), i.e., we can assume that \( X = U \) is a \( G \)-invariant locally closed subset of \( \mathbb{P}(V) \), where \( V \) is a \( G \)-module. Clearly, \( V \) is also a \( G_x \)-module and \( x \in \mathbb{P}(V)^{G_x} \). Lemma 2.2.20 implies that there exists a \( G_x \)-invariant affine open neighbourhood \( W \) of \( x \) in \( X \subset \mathbb{P}(V) \). Let \( C := W \setminus X \) and let \( \pi : W \rightarrow W // G_x \) denote the algebraic Hilbert quotient. Then, \( \{x\} \) and \( C \) are disjoint \( G_x \)-invariant algebraic subsets of \( W \) and hence \( \pi(x) \notin \pi(C) \subset W // G_x \). It follows that there exists an affine open neighbourhood \( V \) of \( \pi(x) \) in \( W // G_x \setminus \pi(C) \). Since \( \pi \) is an affine map, the inverse image \( \pi^{-1}(V) \) is a \( G_x \)-invariant affine open subset of \( X \) containing \( x \).

Using the two previous lemmata, we now adapt the proof of the holomorphic Slice Theorem (see [Hei91], [HL94] and [HS07]) to our algebraic situation.

Theorem 2.2.22 (Slice Theorem, algebraic version). Let \( K \) be a connected compact Lie group and \( G = K^C \) its complexification. Let \( X \) be a normal algebraic Hamiltonian \( G \)-variety with momentum map \( \mu : X \rightarrow \mathfrak{t}^* \). Let \( \pi : X(\mu) \rightarrow X(\mu) // G \) be the analytic Hilbert quotient. Then, every point in \( \mu^{-1}(0) \) has a \( \pi \)-saturated open neighbourhood \( U \) in \( X(\mu) \) that is \( G \)-equivariantly biholomorphic to a saturated open subset of an affine \( G \)-variety.

Here, a saturated subset of an affine \( G \)-variety \( Y \) is a subset of \( Y \) that is saturated with respect to the algebraic Hilbert quotient \( \pi_Y : Y \rightarrow Y // G \).

Remark 2.2.23. The holomorphic Slice Theorem asserts that every point in \( \mu^{-1}(0) \) has a \( \pi \)-saturated open neighbourhood \( U \) in \( X(\mu) \) that is \( G \)-equivariantly biholomorphic to a \( G \)-invariant analytic subset \( A \) of a saturated open subset of a \( G \)-module. The crucial point of Theorem 2.2.22 is that in our situation we can choose \( A \) to be affine. If \( X \) is smooth, this follows directly from the holomorphic Slice Theorem, since in this case, \( A \) can be chosen to be a saturated open subset of a \( G \)-module.

Proof of Theorem 2.2.22. As in the proof of Lemma 2.2.21, we can assume that \( X \) is a \( G \)-stable locally closed subvariety of \( \mathbb{P}(V) \), where \( V \) is a \( G \)-module.

Let \( x = [v] \in \mu^{-1}(0) \subset X \subset \mathbb{P}(V) \) and set \( H := G_x \). Since \( x \in \mu^{-1}(0) \), its stabiliser group \( H \) is reductive (see [Hei91]). By Lemma 2.2.20 it follows that as an \( H \)-module, \( V = L \oplus L^\perp \), where \( L = \mathbb{C}v \). Furthermore, there exists an open neighbourhood \( U \) of \( x \) in \( \mathbb{P}(V) \) and an \( H \)-equivariant isomorphism \( \psi : U \rightarrow L^\perp \otimes L^{-1} \) such that \( \psi(x) = 0 \). Let \( B := d\psi(x)(T_xG \cdot x) \). Since \( B \) is an \( H \)-stable linear subspace of \( L^\perp \otimes L^{-1} \), there exists an \( H \)-stable linear subspace \( N \) of \( L^\perp \otimes L^{-1} \) such that \( L^\perp \otimes L^{-1} = B \oplus N \). Then, \( \overline{\psi^{-1}(N)} \subset \mathbb{P}(V) \) is an \( H \)-invariant linear subspace of \( \mathbb{P}(V) \), i.e., there exists an \( H \)-invariant linear subspace \( W \) in \( V \) such that \( \overline{\psi^{-1}(N)} = \mathbb{P}(W) \). We have

\[
T_x(G \cdot x) \oplus T_x(\mathbb{P}(W)) = T_x(\mathbb{P}(V)),
\]

by construction.
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It follows that the map \( \tilde{\varphi} : G \times_H \mathbb{P}(W) \to \mathbb{P}(V), [g, y] \mapsto g \cdot y \) is a local biholomorphism at \([e, x]\). Set \( S := X \cap \mathbb{P}(W) \). Since \( \tilde{\varphi}^{-1}(X) = G \times_H S \) by construction, we see that \( \varphi : G \times_H S \to X, [g, y] \mapsto g \cdot y \) is a local biholomorphism at \([e, x]\). Set \( S := X \cap \mathbb{P}(W) \). Since \( \tilde{\varphi}^{-1}(X) = G \times_H S \) by construction, we see that \( \varphi : G \times_H S \to X, [g, y] \mapsto g \cdot y \) is a local biholomorphism at \([e, x]\). Intersecting \( S \) with a \( G \times_H \)-invariant affine open neighbourhood of \( x \) in \( X \), we may assume that \( S \) is affine. Furthermore, the restriction of \( \varphi \) to \( G \cdot [e, x] \) is biholomorphic onto \( G \cdot x \). It follows from the holomorphic version of Luna’s Fundamental Lemma (see Lemma 14.3 and Remark 14.4 in [HS07]; also see Lemma 3.1.10) that there exists an open \( G \)-stable neighbourhood \( U' \) of \([e, x]\) which is mapped biholomorphically by \( \varphi \) onto the open neighbourhood \( \varphi(U') \) of \( G \cdot x \) in \( X \).

Since the variety \( G \times_H S \) is affine, there exists an algebraic Hilbert quotient \( \pi_S : G \times_H S \to (G \times_H S)\!/G \). By shrinking \( U' \) if necessary, we can assume that it is \( \pi_S \)-saturated. Since \( X(\mu) \) is open in \( X \), we may assume that \( \varphi(U') \subset X(\mu) \). Let \( C := X(\mu) \setminus \varphi(U') \). This is a closed \( G \)-invariant subset of \( X(\mu) \) not containing the closed orbit \( G \cdot x \subset X(\mu) \). Hence, \( \pi(x) \notin \pi(C) \). Replace \( U' \) by \( \tilde{U} := \varphi^{-1}(\pi^{-1}(X(\mu)\!/G \setminus \pi(C))) \subset U' \). Due to the following commutative diagram

\[
\begin{array}{ccc}
U' & \xrightarrow{\varphi} & X(\mu) \\
\pi_S \downarrow & & \downarrow \pi \\
U'\!/G & \xrightarrow{\pi} & X(\mu)\!/G,
\end{array}
\]

\( \tilde{U} \) is \( \pi_S \)-saturated. Hence, \( \varphi(\tilde{U}) \) is a \( \pi \)-saturated open neighbourhood of \( x \) in \( X(\mu) \) and it is \( G \)-equivariantly biholomorphic to a saturated open subset of an affine \( G \)-variety.

\[\square\]

2.2.5 Singularities of momentum map quotients

We are now in the position to prove

**Theorem 2.2.24.** Let \( K \) be a connected compact Lie group and \( G = K^C \) its complexification. Let \( X \) be an irreducible algebraic Hamiltonian \( G \)-variety with momentum map \( \mu : X \to \mathfrak{k}^* \). Assume that \( X \) has \( 1 \)-rational singularities. Then, the analytic Hilbert quotient \( X(\mu)\!/G \) has \( 1 \)-rational singularities.

**Remark 2.2.25.** Theorem 2.2.24 follows from Boutot’s result [Bou87], if we additionally assume that \( X \) is smooth. To prove it in full generality, we use the analysis done in the preceding sections.

**Proof.** As a first step, we prove that every point \( x \in \mu^{-1}(0) \) has a \( \pi \)-saturated open neighbourhood in \( X \) that is \( G \)-equivariantly isomorphic to a saturated open subset of an affine \( G \)-variety with \( 1 \)-rational singularities.
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By the Slice Theorem, Theorem 2.2.22, there exists an open saturated neighbourhood \( U \) of \( x \) in \( X(\mu) \) and a \( G \)-equivariant biholomorphic map \( \varphi : U \to \tilde{U} \) to a saturated open subset \( \tilde{U} \) of an affine \( G \)-variety \( A \). Denote the algebraic Hilbert quotient of \( A \) by \( \pi_A : A \to A/G \). Let \( f : A \to A \) be a resolution of \( A \) and set \( B := \text{supp}(R^1f_*\mathcal{O}_A) \). This is a \( G \)-invariant algebraic subvariety of \( A \). Since \( \tilde{U} \) is biholomorphic to an open subset of \( X^h \) and \( X^h \) has only 1-rational singularities, we have \( \tilde{U} \subset A \setminus B \), cf. Lemma 2.2.10. Since the \( G \)-orbit through \( \varphi(x) \) is closed in \( A \), we have \( \pi(\varphi(x)) \notin \pi(B) \). Therefore, there exists an affine \( \pi_A \)-saturated open neighbourhood \( \tilde{A} \) of \( \varphi(x) \) in \( A \setminus B \) such that \( \tilde{U} \subset \tilde{A} \).

Let \( x \in X(\mu)/G \). By the considerations above, there exists a neighbourhood \( W \) of \( x \in X(\mu)/G \) such that \( \pi^{-1}(W) \) is \( G \)-equivariantly biholomorphic to a saturated subset \( \tilde{U} \subset A \) in an affine \( G \)-variety \( A \) with 1-rational singularities. It follows that \( W \) is biholomorphic to \( \tilde{W} := (\tilde{U}/G)^h \subset (A/G)^h \). By Theorem 2.2.14 and Corollary 2.2.10, \((A//G)^h \) has 1-rational singularities. We conclude that \( \tilde{W} \) and hence \( W \) has 1-rational singularities. \( \Box \)

2.3 Projectivity of momentum map quotients

This section is devoted to the proof of the Projectivity Theorem, Theorem 1, which asserts the projectivity of compact momentum map quotients of algebraic Hamiltonian \( G \)-varieties.

**Theorem 1** (Projectivity Theorem). Let \( K \) be a compact Lie group and \( G = K^C \) its complexification. Let \( X \) be a \( G \)-irreducible algebraic Hamiltonian \( G \)-variety with momentum map \( \mu : X \to \mathfrak{k}^* \). Assume that \( X \) has only 1-rational singularities. Then, the analytic Hilbert quotient \( X(\mu)/G \) is projective algebraic if and only if \( \mu^{-1}(0) \) is compact.

**Proof.** First, let us additionally assume that \( G \) is connected and hence, that \( X \) and \( X(\mu)/G \) are irreducible. The analytic Hilbert quotient \( X(\mu)/G \) carries a continuous Kähler structure by Theorem 1.2.13. Using a theorem of Varouchas [Var89], we conclude that there exists a smooth Kähler structure on \( X(\mu)/G \). Owing to Theorem 2.1.21 and Theorem 2.2.24, \( X(\mu) \) is a Moishezon space with 1-rational singularities. Consequently, Theorem 2.0.11 applies, and \( X(\mu)/G \) is projective algebraic.

In the general case, let \( K_0 \) denote the connected component of the identity in \( K \). Then, the complexification \((K_0)^C \) of \( K_0 \) is equal to the connected component of the identity in \( G \), which we denote \( G_0 \). Since the momentum map \( \mu \) depends only on the infinitesimal action of \( K \), it is also a momentum map for the \( K_0 \)-action, which we will call \( \mu_{K_0} \). We have \( \mu^{-1}(0) = \mu_{K_0}^{-1}(0) \). Since \( G_0 \) is a subset of \( G \), this implies \( X(\mu_{K_0}) = S_{G_0}(\mu_{K_0}^{-1}(0)) \subset S_G(\mu^{-1}(0)) = X(\mu) \). In general, if \( L \) is a closed subgroup of \( K \), if \( \mu_L \) denotes the restriction of the momentum map to \( \text{Lie}(L) \), and if \( X(\mu_L) \) is the set of semistable points with respect to \( \mu_L \) and the action of \( L^C < G \), it is shown in [HH96] that \( X(\mu) \subset X(\mu_L) \). Consequently, we have \( X(\mu) = X(\mu_{K_0}) \).
Since $X$ is assumed to have 1-rational singularities, it is normal, and consequently, its irreducible components are disjoint and $G_0$-stable. If $X = \bigcup_j X_j$ is the decomposition of $X$ into irreducible components, then the decomposition of $X(\mu)$ into irreducible components is given by $X(\mu) = \bigcup_j X_j(\mu_{K_0} \cap X_j)$. Setting $X(\mu)_j := X_j(\mu_{K_0} \cap X_j)$, we see that the restriction of the analytic Hilbert quotient map $\pi_{G_0} : X(\mu) \to X(\mu) // G_0$ to $X(\mu)_j$ is the analytic Hilbert quotient for the action of $G_0$ on $X(\mu)_j$ and that $X(\mu) // G_0 = \bigcup_j \pi_{G_0}(X(\mu)_j) = \bigcup_j X(\mu)_j // G_0$ is the decomposition of $X(\mu) // G_0$ into irreducible components.

Since $G_0$ is a normal subgroup of $G$, the finite group $\Gamma := G / G_0$ of connected components of $G$ acts on $X(\mu) // G_0$, and we have the following commutative diagram

$$
\begin{array}{ccc}
X(\mu) & \xrightarrow{\pi} & X(\mu) // G \\
\downarrow{\pi_{G_0}} & & \downarrow{\pi_{\Gamma}} \\
X(\mu) // G_0, & & \\
\end{array}
$$

where $\pi_{\Gamma} : X(\mu) // G_0 \to X(\mu) // G$ is the analytic Hilbert quotient for the action of the finite group $\Gamma$ on $X(\mu) // G_0$.

Since $X(\mu) // G$ is compact by assumption, $X(\mu) // G_0$ is also compact. Let $V$ be any of its irreducible components. Then, $V = X_j(\mu_{K_0} \cap X_j)$ for some $j$, as we have seen above. By the first part of the proof (the case $G$ connected), it follows that $V$ is projective algebraic. Since the irreducible components of $X(\mu) // G_0$ are disjoint, the projectivity of $X(\mu) // G_0$ follows.

The finite group $\Gamma$ acts algebraically on $X(\mu) // G_0$, by Theorem 1.3.6. It is a classical result that the algebraic Hilbert quotient for the action of a finite group on a projective algebraic variety exists as a projective algebraic variety. Let $Y$ be the algebraic Hilbert quotient for the action of $\Gamma$ on $X(\mu) // G_0$. Since $Y^h$ is an analytic Hilbert quotient for the same action by Proposition 1.2.12, it is biholomorphic to $X(\mu) // G$. Hence, $X(\mu) // G$ is a projective algebraic complex space and the claim is shown.

**Remarks 2.3.1.**

a) It follows from Theorem 2.2.14 applied to the algebraic Hilbert quotient $\pi_{\Gamma} : X(\mu) // G_0 \to X(\mu) // G$, that $X(\mu) // G$ also has 1-rational singularities.

b) The assumption that $\mu^{-1}(0)$ is compact is necessary to obtain an algebraic structure on the analytic Hilbert quotient $X(\mu) // G$, cf. Section 2.5.

c) The assumption that $X$ is $G$-irreducible is not necessary: decompose $X$ into the disjoint union of its $G$-irreducible components to each of which we can apply Theorem 1 to obtain the projectivity of each of the disjoint irreducible components of $X(\mu) // G$.\[\square\]
2.4 Algebraicity of the set of unstable points

If $X$ is a Hamiltonian $G$-space for the complex-reductive group $G = K^C$ with momentum map $\mu : X \to \mathfrak{t}^*$, we have seen in Section 1.2.2 and used it throughout our study that $X(\mu)$ is an open subset of $X$. From the point of view of complex analytic geometry, it is natural to ask if the set of unstable points $X \setminus X(\mu)$ is an analytic subset of $X$. The following example (see [HL94]) shows that this is not true in general.

**Example 2.4.1.** Let $X = \mathbb{C}^2 \setminus \{(x,y) \in \mathbb{C}^2 \mid y = 0, \text{Re}(x) \leq 0\}$ with the standard Kähler form. Let $G = \mathbb{C}^*$ act on $X$ by multiplication in the second variable. The $S^1$-action is Hamiltonian with momentum map $\mu : X \to \text{Lie}(S^1)^* \cong \mathbb{R}$, $(x,y) \mapsto |y|^2$. Hence, the momentum zero fibre is given by $\{(x,y) \in X \mid y = 0\}$. The corresponding set of semistable points $X(\mu) = \{(x,y) \in X \mid \text{Re}(x) > 0\}$ is not Zariski-open. Actually, it is not even a dense subset of $X$.

![Figure 2.1: An example with non-dense $X(\mu)$](image)

It has been shown in [HH96] that for a $G$-irreducible Hamiltonian $G$-variety $X$ with momentum map $\mu$, the set of semistable points $X(\mu)$ is either empty or dense in $X$.

In this section we will use the Projectivity Theorem, Theorem 1, to derive the following algebraicity result for the set of unstable points $X \setminus X(\mu)$ of an algebraic Hamiltonian $G$-variety:

**Theorem 2.** Let $K$ be a compact Lie group and let $G = K^C$ be its complexification. Let $X$ be a $G$-irreducible algebraic Hamiltonian $G$-variety with momentum map $\mu : X \to \mathfrak{t}^*$. Assume that $X$ has 1-rational singularities and that $\mu^{-1}(0)$ is compact. Then, $X(\mu)$ is Zariski-open in $X$.

In the first part of this section we introduce the concept of Chow quotient for projective algebraic $G$-varieties and prove some basic properties that we will subsequently apply in
our study of the set of semistable points. The second part contains the proof of Theorem 2.

2.4.1 The Chow quotient of a projective algebraic $G$-variety

In this section, we recall the main properties of the Chow quotient of a projective algebraic $G$-variety. We will use it to obtain information about the saturation of locally closed $G$-invariant subsets with respect to the $G$-action.

Let $X$ be an irreducible projective algebraic $G$-variety for a connected complex-reductive Lie group $G$. For $m, d \in \mathbb{N}$, let $\mathcal{C}_{m,d}(X)$ be the Chow variety of cycles of dimension $m$ and degree $d$ in $X$. This is a projective algebraic component of the Barlet cycle space (see [Bar75]). For a cycle $C \in \mathcal{C}_{m,d}(X)$ let $|C|$ denotes the support of $C$, i.e., the union of the subvarieties of $X$ that appear in $C$.

**Theorem 2.4.2** ([Kap93]). Let $X$ be an irreducible projective algebraic $G$-variety. Then, there exist natural numbers $m, d \in \mathbb{N}$, a $G$-invariant rational map $\varphi : X \dashrightarrow \mathcal{C}_{m,d}(X)$, and a $G$-invariant Zariski-open subset $U_0 \subset \text{dom}(\varphi)$ such that for all $u \in U_0$, we have $|\varphi(u)| = \overline{G \cdot u}$.

We will denote the image of $\varphi$ in $\mathcal{C}_{m,d}(X)$ by $\mathcal{C}_G(X)$, and we will call $\varphi : X \dashrightarrow \mathcal{C}_G(X)$ the Chow quotient of $X$ by $G$.

For the following discussion recall that for a subset $A$ of a $G$-space $X$, we have defined $S_G(A) := \{x \in X \mid \overline{G \cdot x} \cap A \neq \emptyset\}$.

**Lemma 2.4.3.** Let $A$ be a $G$-invariant irreducible locally closed subset of $X$. Then, $S_G(A) \cap U_0$ is constructible.

**Proof.** We define $I_A := \{C \in \mathcal{C}_G(X) \mid |C| \cap A \neq \emptyset\}$. We claim that $I_A$ is a constructible subset of $\mathcal{C}_G(X)$. To see this, consider the universal family $\mathcal{X}$ over $\mathcal{C}_G(X)$,

$$\mathcal{X} := \{(C, x) \in \mathcal{C}_G(X) \times X \mid x \in |C|\}.$$

This is an algebraic subvariety of $\mathcal{C}_G(X) \times X$. Let $p_x : \mathcal{X} \rightarrow \mathcal{C}_G(X)$ and $p_X : \mathcal{X} \rightarrow X$ be the canonical projection maps. We have $I_A = p_x^{-1}(p_X^{-1}(A))$. Since $A$ is locally closed, $I_A$ is a constructible subset of $\mathcal{C}_G(X)$. It follows that $U_0 \cap \varphi^{-1}(I_A) = \{x \in U_0 \mid \overline{G \cdot x} \cap A \neq \emptyset\}$ is constructible in $U_0$. \hfill $\square$

2.4.2 Algebraicity of the set of unstable points

The following algebraicity result, which uses the projectivity of $X(\mu) // G$ in an essential way, is crucial for the proof of Theorem 2.
2.4. Algebraicity of the set of unstable points

Let $G$ be a connected complex-reductive Lie group and let $X$ be an irreducible algebraic Hamiltonian $G$-variety with momentum map $\mu : X \to \mathfrak{t}^*$. Let $\mathcal{X}_{\text{reg}}$ be the union of generic $G$-orbits in $X$. Assume that $\mathcal{X}_{\text{reg}} \cap \mu^{-1}(0) \neq \emptyset$ and that $X(\mu) // G$ is projective algebraic. Then, $X(\mu)$ contains a Zariski-open $G$-invariant subset $A$ of $X$ consisting of $G$-orbits that are closed in $X(\mu)$.

Proof. Let $U_R$ be a Rosenlicht set in $X$ with geometric quotient $p_R : U_R \to Q_R$ (cf. Theorem 2.1.15). Note that $U_R$ intersects $\mu^{-1}(0)$. As a first step we prove that $Q_R$ is birational to $X(\mu) // G$. The function field of $Q_R$ is isomorphic to $C(X)^G$. It follows from the discussion in Section 2.1.3 that every $f \in C(X)^G$ descends to a meromorphic function on $X(\mu) // G$. Since $X(\mu) // G$ is projective algebraic, every meromorphic function on $X(\mu) // G$ is a rational function by Proposition 2.1.6. Consider the rational map $\varphi : X(\mu) // G \dasharrow Q_R$ that corresponds to the resulting inclusion $C(X)^G \subset C(X(\mu) // G)$. Since elements of $\varphi(C(Q_R)) = C(X)^G \subset C(X(\mu) // G)$ separate orbits in $U_R$, they separate orbits in $X(\mu) \setminus S_G(U_R \cap X(\mu))$. The latter set is analytically Zariski-open and $\pi$-saturated in $X(\mu)$. It is mapped by $\pi$ to a Zariski-open subset of $X(\mu) // G$. Consequently, $\varphi$ is generically one-to-one, hence birational.

Let $\psi = \varphi^{-1} : Q_R \dasharrow X(\mu) // G$. Without loss of generality, we can assume that the set where $\psi$ is an isomorphism onto its image is equal to $Q_R$. Hence, $\psi : Q_R \hookrightarrow X(\mu) // G$ is an open embedding. Setting $A := X(\mu) \setminus S_G(U_R \cap X(\mu)) \subset U_R$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{A} & X(\mu) \\
p|_A \downarrow & & \downarrow \pi \\
p(A) & \xrightarrow{\psi|_p(A)} & X(\mu) // G.
\end{array}
$$

Since $X(\mu) // G$ is projective algebraic, $\pi(A) = \psi(p(A))$ is Zariski-open in $X(\mu) // G$. It follows that $A = p^{-1}(p(A))$ is Zariski-open in $U_R$ and contained in $X(\mu)$.

Remark 2.4.5. It follows from the proof that $\pi : X(\mu) \to X(\mu) // G$ extends to a rational map $\tilde{\pi} : X \dasharrow X(\mu) // G$.

Lemma 2.4.6. Let $G$ be a connected complex-reductive Lie group and let $X$ be an irreducible algebraic Hamiltonian $G$-variety with momentum map $\mu : X \to \mathfrak{t}^*$. Then there exists a $G$-invariant irreducible subvariety $Y$ of $X$ with $\mu^{-1}(0) \subset Y$ and $Y_{\text{reg}} \cap \mu^{-1}(0) \neq \emptyset$.

Proof. The proof is by induction on $\dim X$. If $\dim X = 0$, the claim is trivially true. So, let $\dim X = n + 1$. If $X_{\text{reg}} \cap \mu^{-1}(0) \neq \emptyset$, we are done. Otherwise, we have $\mu^{-1}(0) \subset \tilde{Y}$, where $\tilde{Y}$ is the complement of $X_{\text{reg}}$ in $X$. Since $X(\mu) // G$ is irreducible, there exists an irreducible component $Y$ of $\tilde{Y}$ such that $\mu^{-1}(0) \subset Y$. We have $\dim Y \leq n$, hence induction applies and we are done.

Next, we introduce some terminology. Let $G$ be a connected algebraic group and let
X be an irreducible algebraic G-variety. A Sumihiro neighbourhood of a point \( x \in X \) is a G-invariant, Zariski-open, quasi-projective neighbourhood of \( x \) in \( X \) that can be G-equivariantly embedded as a locally Zariski-closed subset of the projective space associated to a rational G-representation. In a normal irreducible algebraic G-variety, every point has a Sumihiro neighbourhood by [Sum74].

**Lemma 2.4.7.** Let \( G \) be a connected complex-reductive Lie group and let \( X \) be an irreducible algebraic Hamiltonian G-variety with momentum map \( \mu : X \to \mathfrak{t}^* \). Assume that \( X(\mu) // G \) is projective algebraic and that every point \( x \in \mu^{-1}(0) \) has a Sumihiro neighbourhood. Then, \( X(\mu) \) contains a Zariski-open subset of \( X \).

**Proof.** By Lemma 2.4.6, there exists an irreducible G-invariant subvariety \( Y \) of \( X \) with \( \mu^{-1}(0) \subset Y \) and \( Y_{reg} \cap \mu^{-1}(0) \neq \emptyset \). It follows that \( X(\mu) // G = Y(\mu) // G \). By Lemma 2.4.4 there exists a Zariski-open subset \( A \) of \( Y \) which is contained in \( X(\mu) \) and such that \( \pi(A) \) is an open subset of \( X(\mu) // G \). Furthermore, \( A \) consists of orbits which are closed in \( X(\mu) \).

Let \( W \) be a Sumihiro neighbourhood of a point \( x \in A \) and let \( \psi : W \to \mathbb{P}(V) \) be a G-equivariant embedding of \( W \) into the projective space associated to a rational G-representation \( V \). Let \( Z \) be the closure of \( \psi(W) \) in \( \mathbb{P}(V) \). Given a subset of \( Z \) of the form \( U_0 \) as in Theorem 2.4.2, it follows from Lemma 2.4.3 that \( S_G^X(\psi(A \cap W)) \cap U_0 \) is constructible in \( U_0 \). Therefore, \( S_G^X(A \cap W) \cap W \) contains a Zariski-open subset \( \tilde{U} \) of its closure. By construction, \( \pi(A \cap W) \) is open in \( X(\mu) // G \), and hence, \( \pi^{-1}(\pi(A \cap W)) \cap W \) is an open subset of \( X \) that is contained in \( S_G(A \cap W) \cap W \). It follows that \( \tilde{U} \) is Zariski-open in \( X \). \( \square \)

**Proposition 2.4.8.** Let \( G \) be a connected complex-reductive Lie group and let \( X \) be an algebraic Hamiltonian G-variety with momentum map \( \mu : X \to \mathfrak{t}^* \). Assume that \( X(\mu) // G \) is projective algebraic and that every point in \( \mu^{-1}(0) \) has a Sumihiro neighbourhood. Then, \( X(\mu) \) is Zariski-open in \( X \).

**Proof.** The proof is by induction on \( \dim X \). For \( \dim X = 0 \) there is nothing to prove. So let \( \dim X = n + 1 \). Let \( X = \bigcup_j X_j \) be the decomposition of \( X \) into irreducible components. Notice that \( X_j(\mu|_{X_j}) = X_j \cap X(\mu) \) holds for every \( j \) and that \( X_j(\mu|_{X_j}) // G \) is projective algebraic. Let \( V_j \) be the Zariski-open subset of \( X_j \) whose existence is guaranteed by Lemma 2.4.7. The set

\[
V := \bigcup_j \left( V_j \setminus \bigcup_{k \neq j} X_j \right)
\]

is contained in \( X(\mu) \) and it is Zariski-open in \( X \). Setting \( \bar{X} := X \setminus V \), we have \( \dim \bar{X} \leq n \) and \( X \setminus X(\mu) \subset \bar{X} \). Furthermore, \( \bar{X}(\mu) // G = \pi(X(\mu) \cap \bar{X}) \) is an analytic subset of \( X(\mu) // G \), hence it is projective algebraic. Furthermore, the condition on the existence of Sumihiro neighbourhoods of points in \( \mu^{-1}(0) \) is inherited by \( \bar{X} \). By induction, \( X \setminus X(\mu) \) is algebraic in \( \bar{X} \). \( \square \)
2.5. The case of non-compact $\mu^{-1}(0)$

**Proof of Theorem 2.** Recall that the connected component $G_0$ of the identity of $G$ is the complexification of the connected component $K_0$ of the identity in $K$. The set $S_{G_0}(\mu^{-1}_0(0))$ of semistable points with respect to $G_0$ and the restricted momentum map for the $K_0$-action, $\mu_0 : X \to \mathfrak{t} = \text{Lie}(K_0)^*$, coincides with $X(\mu)$, as we have seen in the proof of Theorem 1. If $\pi_{G_0} : X(\mu) \to X(\mu)/G_0$ denotes the analytic Hilbert quotient of $X(\mu)$ by $G_0$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
X(\mu) & \xrightarrow{\pi} & X(\mu)/G \\
\downarrow{\pi_{G_0}} & & \downarrow{\pi_{\Gamma}} \\
X(\mu)/G_0 & & 
\end{array}
$$

where $\pi_{\Gamma}$ is the analytic Hilbert quotient for the action of the finite group $\Gamma := G/G_0$ on $X(\mu)//G_0$. By the Projectivity Theorem, Theorem 1, we know that $X(\mu)//G$ is projective algebraic. Since $\pi_{\Gamma}$ is a finite map, the quotient $X(\mu)/G_0$ is also projective algebraic, see e.g. [Laz04, Prop 1.2.13].

The variety $X$ has 1-rational singularities by assumption; in particular, it is normal. Consequently, by [Sum74], every point in $\mu^{-1}(0)$ has a Sumihiro neighbourhood for the action of $G_0$. Hence, Proposition 2.4.8 applies and $X(\mu)$ is Zariski-open in $X$.

**Remarks 2.4.9.** At this point it is an open question whether the analytic Hilbert quotient $\pi : X(\mu) \to X(\mu)//G$ is an algebraic Hilbert quotient. We refer to Section 3.3 for a discussion of this and related open problems.

### 2.5 The case of non-compact $\mu^{-1}(0)$

The aim of this section is the construction of an example which shows that the compactness assumption on $\mu^{-1}(0)$ that was made in Theorems 1 and 2 is essential for algebraicity of the momentum map quotient $X(\mu)//G$ and of the set $X \setminus X(\mu)$ of unstable points.

First we introduce the following notation: for a smooth $K$-invariant strictly plurisubharmonic function $\rho$ on a complex $K$-manifold $X$, we set

$$
\mu_{\rho}^\mu(x) = \frac{d}{dt} \bigg|_{t=0} \rho(\exp(it\xi) \cdot x) .
$$

This defines a momentum map $\mu_{\rho}$ for the $K$-action on the Kähler manifold $(X, 2i\partial\bar{\partial}\rho)$.

As a first step in the construction of our example, consider the $\mathbb{C}^*$-action on $\mathbb{C}^2$ that is given by $t \cdot (z_1, z_2) = (tz_1, t^{-1}z_2)$. The induced $S^1$-action on $\mathbb{C}^2$ is Hamiltonian with respect to $2i\partial\bar{\partial}(\|\cdot\|^2)$, where $\|\cdot\|^2$ denotes the norm function of the standard Hermitian product on $\mathbb{C}^2$. We compute $\mu_{\|\cdot\|^2}(z_1, z_2) = |z_1|^2 - |z_2|^2 \in \mathbb{R} \cong \text{Lie}(S^1)^*$. We set $\mathcal{M}_{\mathbb{C}^2} = \mu_{\|\cdot\|^2}^{-1}(0)$. 

As a next step, consider $Y := \mathbb{C}^* \times \mathbb{C}$. The holomorphic map $\varphi : Y \to \mathbb{C}^2, (t, w) \mapsto t \cdot (w, 1)$ is a $\mathbb{C}^*$-equivariant isomorphism onto the open subset $U := \mathbb{C}^2 \setminus \{z_2 = 0\}$.

The pullback $\rho_Y := \varphi^*(\| \cdot \|^2)$ is strictly plurisubharmonic and $S^1$-invariant. It is computed to be equal to $\rho(t, w) = |t|^2|w|^2 + |t^{-1}|^2$. The set of semistable points $X(\mu_Y)$ with respect to $\mu_Y := \mu_\rho_Y$ fulfills (cf. [HH96])

$$Y(\mu_Y) = \mathbb{C}^* \cdot \varphi^{-1}(U \cap \mathcal{M}_{\mathbb{C}^2}) = Y \setminus \{(\mathbb{C}^* \times \{0\}\}.$$

We introduce $Z := Y \setminus (\mathbb{C}^* \times \{-1\}) = \mathbb{C}^* \times (\mathbb{C} \setminus \{-1\})$. This is a complex Hamiltonian $\mathbb{C}^*$-manifold with Kähler form $\omega_Z$ and momentum map $\mu_Z$ both induced by $\rho_Z := \rho_Y|_Z$. We have $Z(\mu_Z) = Z \setminus (\mathbb{C}^* \times \{0\})$.

The example

Consider the affine algebraic variety $X := \mathbb{C}^* \times \mathbb{C}$. The pullback $\rho_X := \psi^*(\rho_Z)$ of $\rho_Z$ under the $\mathbb{C}^*$-equivariant unbranched holomorphic covering $\psi : X \to Z, (t, z) \mapsto (t, e^z - 1)$ is a smooth $S^1$-invariant strictly plurisubharmonic function. Explicitly, it is given by $\rho_X(t, z) = |t|^2|e^z - 1|^2 + |t^{-1}|^2$. The associated momentum map for the $S^1$-action is computed to be

$$\mu_X = \mu_{\rho_X} : (t, z) \mapsto |t|^2|e^z - 1|^2 - |t^{-1}|^2.$$

We have $X(\mu_X) = \psi^{-1}(Z(\mu_Z)) = X \setminus (\mathbb{C}^* \times 2\pi i \mathbb{Z})$, and hence, $X(\mu_X)$ it is not algebraically Zariski-open in $X$. 

![Figure 2.2: $\mathbb{C}^*$ acting on $\mathbb{C}^2$](image-url)
We obtain \( X(\mu_X) / \mathbb{C}^* = X(\mu_X) / \mathbb{C}^* = \mathbb{C} \setminus 2\pi i \mathbb{Z} =: Q \) as the analytic Hilbert quotient of \( X(\mu_X) \) by the action of \( \mathbb{C}^* \).

We claim that \( Q \) is not the complex manifold associated to any algebraic variety. Assume the contrary and suppose that \( Q = V^h \) for some smooth algebraic variety \( V \). Let \( U \) be an affine Zariski-open subset of \( V \). Then, there exist finitely many points \( p_1, \ldots, p_N \) in \( Q \) such that \( U^h = Q \setminus \{ p_1, \ldots, p_N \} \). In particular, \( U^h \) is homotopy-equivalent to \( Q \). There exist a regular open embedding \( \varphi : U \hookrightarrow C \) of \( U \) into a smooth projective curve \( C \). Hence, there exist finitely many points \( c_1, \ldots, c_m \) in \( C \) such that \( \varphi(U) = C \setminus \{ c_1, \ldots, c_m \} \). If \( C \) has genus \( g \), it follows that the fundamental group \( \pi_1(U^h) \) is freely generated by \( 2g + m - 1 \) generators, see [Ful95, Chap 17]. Consequently, \( \pi_1(Q) \) is finitely generated, a contradiction.

### 2.6 Proper momentum maps

Results for Hamiltonian group actions on compact symplectic manifolds often continue to hold on non-compact manifolds, if the momentum map is proper (see e.g. [HNP94], [Ler05], [Woo05], [Wei01]), and many natural group actions on non-compact symplectic manifolds admit proper momentum maps (see [HNP94]). Also in the study of Hamiltonian actions of compact groups \( K \) on Kähler \( K^C \)-spaces, the properness assumption is often made in order to transfer results known for the compact case to the non-compact case, see e.g. [Sja95].

Note that Theorem 1 and Theorem 2 apply in particular if the momentum map \( \mu \) under consideration is proper. In this section we discuss this additional assumption on \( \mu \) from the point of view of complex geometry and we show that it may impose strong conditions on the holomorphic \( G \)-space under discussion. More precisely, we prove

**Proposition 2.6.1.** Let \( X \) be a holomorphic \( \mathbb{C}^* \)-principal bundle over a compact complex manifold \( B \). Then there exists a Kähler structure \( \omega \) on \( X \) such that the induced \( S^1 \)-action on \( X \) is Hamiltonian with proper momentum map if and only if \( B \) is Kähler and \( X \) is topologically trivial.

**Remark 2.6.2.** An alternative description of the setup of Proposition 2.6.1 is the following: let \( X \) be a Kähler manifold with a free \( \mathbb{C}^* \)-action such that the induced \( S^1 \)-action is Hamiltonian with proper momentum map. First it follows that \( X = X(\mu) \). The slice theorem then implies that \( \mathbb{C}^* \) acts properly on \( X \) and realises \( X \) as a \( \mathbb{C}^* \)-principal bundle over the compact base manifold \( B = \mu^{-1}(0) / S^1 \).

**Example 2.6.3.** Proposition 2.6.1 implies that there exists no \( S^1 \)-invariant Kähler form \( \omega \) on \( \mathbb{C}^2 \setminus \{ 0 \} \) such that the standard \( S^1 \)-action by multiplication is Hamiltonian with proper momentum map.

**Proof of Proposition 2.6.1.** Let \( X \to B \) be a holomorphic \( \mathbb{C}^* \)-principal bundle over some
compact complex manifold $B$. Assume that there exists an $S^1$-invariant Kähler form with proper momentum map $\mu : X \to \text{Lie}(S^1)^*$ for the $S^1$-action. Then, since $S^1$ acts freely on $X$, the defining equation for $\mu$,

$$d\mu^g = t_{\xi} \omega \quad \text{for all } \xi \in \text{Lie}(S^1),$$

implies that $\mu$ is a submersion. It follows that $\mu(X)$ is open in $\text{Lie}(S^1)^*$. On the other hand, since $\mu$ is proper by assumption, $\mu(X)$ is also closed in $\text{Lie}(S^1)^*$. It follows that $\mu$ is surjective.

From now on, we will identify $\text{Lie}(S^1)^*$ with $\text{Lie}(S^1) \cong i\mathbb{R}$. For every $\xi \in i\mathbb{R}$, the fibre $\mu^{-1}(\xi)$ is an $S^1$-invariant submanifold of $X$. The gradient flow of $\mu$ induces an $S^1$-equivariant diffeomorphism $\mu^{-1}(\xi) \to \mu^{-1}(0)$. It follows that the quotient $\mu^{-1}(\xi)/S^1$ is diffeomorphic to the base manifold $B$. Since $\mu^{-1}(0)$ is the zero fibre for the shifted momentum map $\mu_0 = \mu - \xi$, the restriction of the Kähler form $\omega$ to $\mu^{-1}(\xi)$ induces a Kähler form $\omega_\xi^\mathfrak{k}$ on the complex manifold $B$, see [GS82] and [HHL94]. The Duistermaat-Heckman-Theorem [DH82] yields the following linear relation between the cohomology classes of these differential forms in $H^2(B, \mathbb{R})$:

$$[\omega_\xi] = [\omega_0] - 2\pi i\xi \cdot c_1(X),$$

(2.9)

where $c_1(X)$ is the Chern class in $H^2(B, \mathbb{Z})$ of the $\mathfrak{c}^*$-principal bundle $X \to B$.

Let $K_B$ be the set of all cohomology classes in $H^2(B, \mathbb{R})$ that can be represented by a Kähler form on $B$. We have seen that $[\omega_\xi] \in K_B$ for all $\xi \in \text{Lie}(S^1)$. Since $K_B$ does not contain any lines (see e.g. [Huy05]), equation (2.9) together with surjectivity of $\mu$ implies that $c_1(X) = 0$, i.e., the $\mathfrak{c}^*$-principal bundle $X \to B$ is topologically trivial.

For the following results about topologically trivial $\mathfrak{c}^*$-principal bundles over compact Kähler manifolds we refer to [Huy05] and [KN96]. Let $X \to B$ be a topologically trivial holomorphic $\mathfrak{c}^*$-principal bundle over a compact Kähler manifold. Since $c_1(X) = 0 \in H^2(B, \mathbb{Z})$, the bundle $X$ admits a flat hermitian metric. If $p : \tilde{B} \to B$ is the universal covering of $B$, then the pull-back $p^* (X)$ of $X$ to $\tilde{B}$ is isomorphic to the trivial holomorphic $\mathfrak{c}^*$-bundle. If $\Gamma := \pi_1 (B)$ denotes the fundamental group of $B$, it follows that $X$ is $\mathfrak{c}^*$-equivariantly isomorphic to a quotient of the trivial bundle $\tilde{X} := \tilde{B} \times \mathfrak{c}^*$ by an action of $\Gamma$ of the form

$$\gamma \cdot (p, z) = (\gamma \cdot p, \alpha(\gamma)z).$$

Here, $\Gamma$ acts on $\tilde{B}$ by covering transformations, and $\alpha : \Gamma \to S^1$ is a character. The quotient map $P : \tilde{X} \to \tilde{X}/\Gamma = X$ is a $\mathfrak{c}^*$-equivariant holomorphic covering map.

Let $\omega_B$ be any Kähler form on $B$, let $\rho : \mathfrak{c}^* \to \mathbb{R}$ be defined by $z \mapsto \frac{1}{4} (\ln |z|^2)^2$, and let $\omega_E$ be the Kähler form on $\mathfrak{c}^*$ that is given by

$$\omega_E = 2i\partial\bar{\partial}(\rho) = \frac{i}{|z|^2} \, dz \wedge d\bar{z}. $$
The Kähler form \( \tilde{\omega} = p^*(\omega_B) \oplus \omega_E \) on \( \tilde{B} \times \mathbb{C}^* \) is \((S^1 \times \Gamma)\)-invariant, hence it descends to an \( S^1 \)-invariant Kähler form \( \omega \) on \( X \). If we identify \( \text{Lie}(S^1)^* \) with \( \mathbb{R} \),
\[
\tilde{\mu} : \tilde{B} \times \mathbb{C}^* \to \mathbb{R},
(p, z) \mapsto \ln(|z|^2).
\]
defines an \((S^1 \times \Gamma)\)-invariant momentum map for the action of \( S^1 \) on \( \tilde{B} \times \mathbb{C}^* \) with respect to \( \tilde{\omega} \). It descends to an \( S^1 \)-invariant momentum map \( \mu : X \to \mathbb{R} \) with respect to \( \omega \) for the \( S^1 \)-action on \( X \).

It remains to show that \( \mu \) is proper. Let \( K \) be a compact subset of \( \mathbb{R} \). There exists a compact subset \( F \) in \( \tilde{B} \) such that \( p(F) = B \). Since \( P : \tilde{X} \to X \) is \( \mathbb{C}^* \)-equivariant, it follows that \( P(F \times \mathbb{C}^*) = X \). As \( \tilde{\mu} = \mu \circ P \), we have
\[
\mu^{-1}(K) = P(\tilde{\mu}^{-1}(K) \cap (F \times \mathbb{C}^*)) = P(F \times \tilde{K})
\]
for some compact subset \( \tilde{K} \) of \( \mathbb{C}^* \). It follows that \( \mu^{-1}(K) \) is compact, and hence that \( \mu \) is proper. \( \square \)
Chapter 3

Analytic Hilbert quotients and Geometric Invariant Theory

In this chapter we show that every holomorphic $G$-space whose analytic Hilbert quotient $X//G$ is projective algebraic admits the unique structure of a quasi-projective algebraic $G$-variety with algebraic Hilbert quotient $X//G$. This is done in two steps. In the first section, we prove that we can embed $X$ as a closed analytic subset of a quasi-projective algebraic $G$-variety $Y$ with algebraic Hilbert quotient $Y//G$. In the second section, we show that the analytic subset of $Y$ obtained in this way is in fact an algebraic subvariety. This will define the desired algebraic structure on $X$, which is shown to be unique in Section 3.2.4. Afterwards, we relate the situation studied in this chapter to Geometric Invariant Theory and to the setup of Chapter 2. Unless explicitly stated, in this chapter $G$ always denotes a complex-reductive Lie group.

3.1 Embedding $G$-spaces into $G$-vector bundles over $X//G$

3.1.1 The basic construction

Let $X$ be a holomorphic $G$-space such that the analytic Hilbert quotient $\pi : X \to X//G$ exists and such that $X//G$ is a projective algebraic complex space.

In this section we construct the basic spaces into which we will later embed our given complex space $X$.

Let $L$ be holomorphic line bundle on the quotient $X//G$. By Theorem 1.3.6, $L$ is an algebraic line bundle, i.e., there exists a covering $\{U_a\}_a$ of $X//G$ by Zariski-open subsets $U_a$ of $X//G$ such that $L|_{U_a}$ is algebraically trivial, and a collection $\{\mathcal{G}_{a\beta}\}_{a,\beta}$ of regular transition functions on the intersections $U_{a\beta} := U_a \cap U_\beta$. We will denote the corresponding locally free sheaf by $\mathcal{L}$. Let $V$ be a $G$-module.
Given these data, we construct an algebraic vector bundle over \( X//G \) as follows: consider the vector bundle \( \mathcal{V} \) of rank equal to \( \dim V \) that is trivial on \( U_a \) and has transition functions \( g_{a\beta} \cdot \text{Id}_V : U_{a\beta} \to GL_C(V) \). The bundle \( \mathcal{V} \) is isomorphic to the vector bundle \( V \otimes L \). Since \( g_{a\beta}(U_{a\beta}) \text{Id}_V \) is contained in the center of \( GL_C(V) \), the group actions

\[
G \times (U_a \times V) \to U_a \times V
\]

\[
g \cdot (z, v) = (z, g \cdot v)
\]

glue together to give a global action

\[
G \times \mathcal{V} \to \mathcal{V}.
\]

By Proposition 1.1.11, the action of \( G \) on \( V \) is algebraic. It follows that \( \mathcal{V} \) is an algebraic \( G \)-variety. Furthermore, it follows from the construction that \( \mathcal{V} \) is an algebraic \( G \)-vector bundle over the \( G \)-space \( X//G \) (with the trivial \( G \)-action) in the sense of the following

**Definition 3.1.1.** Let \( Z \) be an algebraic \( G \)-variety and let \( p : \mathcal{V} \to Z \) be an algebraic vector bundle over \( Z \). Then, \( Z \) is called a \( G \)-vector bundle for the algebraic group \( G \) if there exists an algebraic action of \( G \) on \( \mathcal{V} \) by vector bundle automorphisms such that the following diagram commutes

\[
\begin{array}{ccc}
G \times \mathcal{V} & \longrightarrow & \mathcal{V} \\
\text{id}_G \times p & \downarrow & p \\
G \times Z & \longrightarrow & Z.
\end{array}
\]

The pullback \( \pi^*L \) is a line bundle on \( X \). The action of \( G \) trivially lifts to an action of \( G \) on \( \pi^*L \) by bundle automorphisms; over \( \pi^{-1}(U_a) \) the action is given by \( (x, w) \mapsto (g \cdot x, w) \). This action and the diagonal \( G \) action on \( X \times V \) yield an action of \( G \) on the vector bundle \( V \otimes \pi^*L \) by bundle automorphisms making the bundle projection equivariant. More precisely, the bundle \( V \otimes \pi^*L \) trivialises over \( \{ \pi^{-1}(U_a) \} \) via an isomorphism \( \phi_a : V \otimes \pi^*L \to \pi^{-1}(U_a) \times V \) and on \( (V \otimes \pi^*L)|_{\pi^{-1}(U_a)} \cong \pi^{-1}(U_a) \times V \), the action is given by

\[
G \times \left( \pi^{-1}(U_a) \times V \right) \to \pi^{-1}(U_a) \times V, \quad g \cdot (x, v) = (g \cdot x, g \cdot v).
\]

It follows that the sheaf \( V \otimes \pi^*\mathcal{L} \) of germs of sections of \( V \otimes \pi^*L \) is a coherent analytic \( G \)-sheaf on \( X \). We are going to use sections of \( \pi_* (V \otimes \pi^*\mathcal{L})^G \) to construct the maps that we will use later on for the embedding of our given holomorphic \( G \)-space \( X \).

**Lemma 3.1.2.** Let \( X \) be a complex \( G \)-space such that the analytic Hilbert quotient \( X//G \) exists as a projective algebraic complex space. Let \( V \) be a \( G \)-module. Let \( L \) be an algebraic line bundle on \( X//G \) and \( \mathcal{L} \) the associated locally free sheaf. Then, an element \( s \in H^0(X//G, \pi_*(V \otimes \pi^*\mathcal{L})^G) \) yields a \( G \)-equivariant holomorphic map \( \sigma : X \to \mathcal{V} \) into the algebraic \( G \)-vector bundle \( \mathcal{V} = V \otimes L \) over \( X//G \) such that the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{\sigma} & \mathcal{V} \\
\downarrow\pi & & \downarrow \pi \\
X//G & & 
\end{array}
\]
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Proof. Let $s$ be an element of $H^0(X//G, \pi_* (V \otimes \pi^* L)^G)$. Then, the restriction $s|_{\pi^{-1}(U_a)}$ yields a map $s_a : \pi^{-1}(U_a) \to V$ via

$$
\phi_a \circ s|_{\pi^{-1}(U_a)} : \pi^{-1}(U_a) \to \pi^{-1}(U_a) \times V
\ x \mapsto (x, s_a(x)).
$$

Since $s : X \to V \otimes \pi^* L$ was assumed to be $G$-equivariant, each of the $s_a$ is $G$-equivariant. Furthermore, the $s_a$’s satisfy the following compatibility condition on $\pi^{-1}(U_{a\beta})$:

$$
s_a(x) = g_{a\beta}(\pi(x)) \cdot s_\beta(x)
$$

(3.1)

This yields a collection of holomorphic maps

$$
\sigma_a : \pi^{-1}(U_a) \to U_{a\beta} \times V
\ x \mapsto (\pi(x), s_a(x)).
$$

Considering the diagram

\[
\begin{array}{ccc}
V|_{\pi^{-1}(U_{a\beta})} & \xrightarrow{\phi_a} & U_{a\beta} \times V \\
\downarrow{\phi_a} & & \downarrow{\phi_{a\beta}} \\
\pi^{-1}(U_{a\beta}) & = & \pi^{-1}(U_{a\beta}) \\
\end{array}
\]

and recalling that the $s_a$ fulfill (3.1), we see that the $\sigma_a$ glue together to give a global holomorphic map $\sigma : X \to V$.

We claim that $\sigma$ is $G$-equivariant with respect to the action of $G$ on $X$ and the action of $G$ on $V$ defined above: indeed, since $\pi^{-1}(U_a)$ is $G$-invariant, it suffices to check that this is true on $\pi^{-1}(U_a)$: we compute

$$
\phi_a(\sigma(g \cdot x)) = (\pi(g \cdot x), s_a(g \cdot x)) = (\pi(x), g s_a(x)) = \phi_a(g \cdot \sigma(x)).
$$

Algebraic Hilbert quotients of algebraic $G$-vector bundles

Let $Z$ be an algebraic variety endowed with the trivial $G$-action. If $V$ is an algebraic $G$-vector bundle over $Z$, the group $G$ acts linearly on each fibre. In fact, it follows from Luna’s Slice Theorem [Lun73] that all representations $V_x$ of $G$ occurring in $V$ are equivalent (see for example [Kra89]). Since $V$ is locally trivial in the Zariski-topology of $Z$, there exists an covering $\{U_a\}$ of $X//G$ by affine sets over which $V$ is trivial. The preimage of $U_a$ under the vector bundle projection is an affine open subset of $V$. It follows (see [Mum99, Chap 3, §1]) that the vector bundle projection is affine. By Lemma 1.2.11, $V$ admits an algebraic Hilbert quotient $V//G$.

The following lemma gives a more detailed description of this quotient:
Lemma 3.1.3. Let $V$ be an algebraic $G$-vector bundle over an algebraic variety $Z$ having the $G$-module $V$ as fibre. Then, the algebraic Hilbert quotient $V \sslash G$ exists as a Zariski-locally trivial fibre bundle over $Z$ with typical fibre $V \sslash G$.

Proof. Let $\{U_\alpha\}$ be an open covering of $Z$ such that $V|_{U_\alpha}$ is trivial and let $\{g_{\alpha\beta}\}_{\alpha,\beta}$ be the corresponding set of transition functions. Since $G$ acts on $V$, the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to GL(V)$ take values in a subgroup $H < GL(V)$ that commutes with the image of $G$ in $GL(V)$. It follows that there is an action of $H$ on the algebraic Hilbert quotient $V \sslash G$ such that the quotient map $\pi_V : V \to V \sslash G$ is $H$-equivariant. Consider the fibre bundle $Q$ over $X \sslash G$ with typical fibre $V \sslash G$ that is trivial over $U_\alpha$ and whose transition functions are given by the action $\phi : H \times V \sslash G \to V \sslash G$ of $H$ on $V \sslash G$. More precisely, the transition functions are

$$U_{\alpha\beta} \times V \sslash G \to U_{\alpha\beta} \times V \sslash G \quad \quad (z, w) \mapsto (z, \phi(g_{\alpha\beta}(z), w)).$$

We claim that $Q \cong V \sslash G$: indeed, consider the following collection of maps

$$\pi_\alpha : U_\alpha \times V \to U_\alpha \times V \sslash G \quad \quad (x, v) \mapsto (x, \pi_V(x)).$$

By the definition of the transition maps for $Q$, we have

$$\pi_V(g_{\alpha\beta}(x) \cdot v) = \phi(g_{\alpha\beta}(x), \pi_V(v)) \quad (3.2)$$

since $\pi_V : V \to V \sslash G$ is $H$-equivariant. However, (3.2) is exactly the compatibility condition for maps between fibre bundles. Hence, the $\pi_\alpha$ glue together to a global map of fibre bundles $\pi_V : V \to Q$. Since $\pi_V$ is locally given by $\pi_\alpha$, and since $\pi_V : V \to V \sslash G$ is an algebraic Hilbert quotient, it follows that $\pi_V$ is $G$-invariant and that $(\pi_V)_* = \pi^*_V$. Since $\pi_V$ is affine, we conclude that $\pi_V$ is an affine map.

3.1.2 Extending maps from $\pi^{-1}(\pi(x))$ to $X$

Recall that we consider a holomorphic $G$-space $X$ such that the analytic Hilbert quotient $X \sslash G$ exists as a projective algebraic complex space. We will use Serre vanishing to extend $G$-equivariant holomorphic maps that are defined on fibres of $\pi$ and take values in $G$-modules $V$ to $G$-equivariant holomorphic maps from $X$ into $G$-vector bundles over $X \sslash G$.

First we note the following consequence of the classical projection formula:

Lemma 3.1.4 (Equivariant projection formula). Let $X$ be a holomorphic $G$-space such that the analytic Hilbert quotient $\pi : X \to X \sslash G$ exists. Let $V$ be a $G$-module, let $\mathcal{F}$ be the coherent analytic $G$-sheaf of germs of maps from $X$ to $V$, and let $\mathcal{L}$ be the locally free sheaf associated to a holomorphic line bundle $L$ on $X \sslash G$. Then, we have

$$(\pi_* \mathcal{F})^G \otimes \mathcal{L} \cong (\pi_*(V \otimes \pi^* \mathcal{L}))^G.$$
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Proof. First we note that the natural map $v : \pi_*\mathcal{F} \otimes \mathcal{L} \to \pi_*(V \otimes \pi^*\mathcal{L})$ is $G$-equivariant. Since $\mathcal{F}$ is the locally free sheaf associated to the vector bundle $V \times X$, it follows from the classical projection formula [Har77, Chap II, Ex 5.1] that $v$ is an isomorphism of analytic $G$-sheaves over $X//G$. Since $G$ acts trivially on $L$ and $X//G$, we conclude that $(\pi_*\mathcal{F} \otimes \mathcal{L})^G$ is isomorphic to $(\pi_*\mathcal{F})^G \otimes \mathcal{L}$.

From now on, we fix an ample line bundle $L$ on $X//G$. Serre vanishing allows us to extend holomorphic maps from the fibres of the quotient map $\pi$ to the whole space $X$ after twisting with an appropriate power of $L$. More precisely, we have

**Proposition 3.1.5.** Let $A = \{q_1, \ldots, q_N\}$ be a finite set of points in $X//G$. Let $F := \pi^{-1}(A)$ be the corresponding collection of fibres of $\pi$. Assume that there exists an equivariant holomorphic map $\varphi$ of a $\pi$-saturated open neighbourhood $U$ of $F$ into a $G$-module $V$ that is an immersion at every point in $F$. Then, there exists $m \in \mathbb{N}$ and an equivariant holomorphic map $\Phi : X \to V(m)$ into the algebraic $G$-vector bundle $V(m) := V \otimes L^m$. The map $\Phi$ coincides with $\varphi$ on $F$ and is an immersion at every point in $F$.

Proof. Let $\mathcal{F}$ be the sheaf of germs of holomorphic maps from $X$ into $V$. By Theorem 1.2.9, $\mathcal{G} := \pi_*(\mathcal{F})^G$ is a coherent analytic sheaf of $\mathcal{H}_{X//G}$-modules on $X//G$. Let $m_A$ be the ideal sheaf of the analytic set $A = \{q_1, \ldots, q_N\}$ in $\mathcal{H}_{X//G}$. Since $\mathcal{L}$ is locally free, the exact sequence

$$0 \to m_A^2 \mathcal{G} \to \mathcal{G} \to \mathcal{G}/m_A^2 \mathcal{G} \to 0$$

for every $m \in \mathbb{N}$ induces the exact sequence

$$0 \to m_A^2 \mathcal{G} \otimes \mathcal{L}^m \to \mathcal{G} \otimes \mathcal{L}^m \to \mathcal{G}/m_A^2 \mathcal{G} \otimes \mathcal{L}^m \to 0. \quad (3.3)$$

We note that for all $p \in X//G$, we have $(\mathcal{G}/m_A^2 \mathcal{G} \otimes \mathcal{L}^m)_p \cong (\mathcal{G}/m_A^2 \mathcal{G})_p$. Hence, we first get $\text{supp}(\mathcal{G}/m_A^2 \mathcal{G} \otimes \mathcal{L}^m) = A$ and then $\mathcal{G}/m_A^2 \mathcal{G} \otimes \mathcal{L}^m \cong \mathcal{G}/m_A^2 \mathcal{G}$. Furthermore, by the equivariant projection formula (Proposition 3.1.4), the sheaf $\mathcal{G} \otimes \mathcal{L}^m$ is isomorphic to $\pi_*(V \otimes \pi^*\mathcal{L}^m)^G$. Hence, from the exact sequence (3.3) we obtain

$$0 \to m_A^2 \mathcal{G} \otimes \mathcal{L}^m \to \pi_*(V \otimes \pi^*\mathcal{L}^m)^G \to \mathcal{G}/m_A^2 \mathcal{G} \to 0. \quad (3.4)$$

By Serre vanishing (Theorem 1.3.3), there exists an $m_0 \in \mathbb{N}$ such that

$$H^1(X//G, m_A^2 \mathcal{G} \otimes \mathcal{L}^m) = 0 \quad \text{for all } m \geq m_0.$$ 

Then, for $m \geq m_0$, the exact sequence (3.4) induces an exact sequence

$$0 \to H^0(X//G, m_A^2 \mathcal{G} \otimes \mathcal{L}^m) \to H^0(X//G, \pi_*(V \otimes \pi^*\mathcal{L}^m)^G) \to H^0(X//G, \mathcal{G}/m_A^2 \mathcal{G}) \to 0, \quad (3.5)$$

where the map $r$ is given by restriction.

The map $\varphi : U \to V$ defines an element $\varphi_q \in H^0(X//G, \mathcal{G}/m_A^2 \mathcal{G})$, and it follows from the exact sequence (3.5) that there exists an element $\phi \in H^0(X//G, \pi_*(V \otimes \pi^*\mathcal{L}^m)^G)$ such that $r(\phi) = \varphi_q$. 

Lemma 3.1.2 yields a G-equivariant holomorphic map Φ : X → V(m), where V(m) = V ⊗ L⊗m is the algebraic G-vector bundle constructed from V and L⊗m. We note that by construction, Φ|U : U → π(U) × V coincides with π × φ : U → π(U) × V up to second order along F. This shows the claim.

3.1.3 The Embedding Theorem

In this section, we will prove the first main result of this chapter: a holomorphic G-space with projective algebraic quotient X//G admits a G-equivariant embedding into a G-vector bundle V over X//G.

Orbit types and slice types of G-actions

Let G be a complex-reductive Lie group and let X be a holomorphic G-space. Let Hx, Hy be two isotropy subgroups of the action of G on X. We define a preorder on the set of G-orbits in X as follows: we say G · x ≤ G · y if and only if there exists an element g ∈ G such that Int(g)Hy < Hx. Here, Int(g) : G → G is given by Int(g)(h) = ghg⁻¹. There is an equivalence relation on the set of G-orbits in X given as follows: G · x ∼ G · y if and only if there exists an element g ∈ G such that Int(g)Hy = Hx. The equivalence class of an orbit G · x will be denoted by Type(G · x) or Type(G/Hx). We call Type(G/Hx) the orbit type of x. The preorder \( \leq \) defined above for G-orbits induces a preorder on the set of orbit types.

Two representations \( \rho_i : H_i → GL(V_i) \); \( i = 1, 2 \) of two closed complex subgroups \( H_1, H_2 \) of G will be called G-isomorphic, if there exists an element g ∈ G and a linear isomorphism \( L : V_1 → V_2 \), such that

\[
i) \ H_2 = Int(g)H_1, \quad ii) \ L(\rho_1(h) \cdot v) = \rho_2(\text{Int}(g)(h)) \cdot L(v).
\]

This is the case if and only if the associated G-vector bundles \( G ×_{H_i} V_i \) are G-isomorphic. The equivalence class of a representation \( ρ : H → GL(V) \) under this equivalence relation as well as the associated G-isomorphism class of G-vector bundles represented by \( G ×_H V \) will be called the slice type of ρ and will be denoted by \( \text{Type}_G(V, H) \).

A holomorphic G-space X that admits an analytic Hilbert quotient X//G will be called of finite orbit type, if the set \{Type(G · x) | x ∈ X, G · x is closed in X\} is finite. The space X will be called of finite slice type, if the set of slice types \{TypeG(T_x, G_x) | x ∈ X, G · x is closed in X\} is finite. We collect basic properties of the notions that we just introduced:
3.1. Embedding G-spaces into G-vector bundles over $X//G$

**Lemma 3.1.6.** Let $G$ be a complex-reductive Lie group. Then, the following holds:

1. Every $G$-invariant analytic subset of a $G$-module $V$ is of finite orbit type and of finite slice type.

2. Finite slice type implies finite orbit type.

3. Let $X$ be a complex $G$-space with compact analytic Hilbert quotient $X//G$. Then, $X$ is of finite slice type.

**Proof.** The first part is classical, a proof can be found in [Lun73]. The second part follows from the first part by the Slice Theorem (see [Hei88]): if $X$ is a holomorphic $G$-space with analytic Hilbert quotient $X//G$, every point $q \in X//G$ has an open neighbourhood $Q$ such that $\pi^{-1}(Q)$ is $G$-equivariantly isomorphic to a $G$-invariant analytic subset of a saturated open subset in $G \times G$, $N$, where $x$ is a point in the unique closed $G$-orbit in $\pi^{-1}(q)$ and $N$ is a submodule of $T_xX$ complementary to $T_x(G \cdot x)$. Hence, if $X$ is of finite slice type, then the first part of the lemma implies that $X$ is of finite orbit type. For the third part, we use the Slice Theorem again to see that every point $q \in X//G$ has an open neighbourhood $Q$ such that $\pi^{-1}(Q)$ has finite slice type. Since $X//G$ is compact, it can be covered by a finite number of such neighbourhoods. This implies the claim. \qed

As a direct consequence, we obtain

**Corollary 3.1.7.** Let $X$ be a complex $G$-space such that $X//G$ is projective algebraic. Then, $X$ is of finite slice type and of finite orbit type.

**The proof of the Embedding Theorem**

**Lemma 3.1.8.** Let $X$ be a complex $G$-space, let $G \cdot x \subset X$ be a closed orbit and let $\varphi_1 : X \to V_1$ be a $G$-equivariant holomorphic map into a $G$-module $V_1$ such that the restriction $\varphi_1|_{G \cdot x} : G \cdot x \to V_1$ is a proper embedding. Let $\varphi_2 : X \to V_2$ be a $G$-equivariant holomorphic map into a $G$-module $V_2$, and let $\Phi : X \to V_1 \oplus V_2$ be the product map. Then, the orbit $G \cdot \Phi(x)$ is closed in $V_1 \oplus V_2$.

**Proof.** Suppose that there exists an $a \in \overline{G \cdot \Phi(x)} \setminus G \cdot \Phi(x)$. Since the $G$-action on $V_1 \oplus V_2$ is algebraic by Proposition 1.1.11, we have

$$\dim G \cdot a < \dim G \cdot \Phi(x). \quad (3.6)$$

On the other hand, consider the $G$-equivariant projection $p : V_1 \oplus V_2 \to V_1$. We have $p(G \cdot a) \subset \overline{G \cdot \varphi_1(x)} = G \cdot \varphi_1(x)$. It follows that $p(G \cdot a) = G \cdot \varphi_1(x)$. However, since $p$ maps $G \cdot \Phi(x)$ biholomorphically onto $G \cdot \varphi_1(x)$ this implies that $\dim(G \cdot \Phi(x)) = \dim(G \cdot \varphi_1(x)) \leq \dim(G \cdot a)$. This contradicts (3.6), and the claim is shown. \qed

The next result is the crucial technical part of the Embedding Theorem:
Lemma 3.1.9. Let $X$ be a holomorphic $G$-space with analytic Hilbert quotient $\pi : X \to X//G$. Assume that $X//G$ is projective algebraic. Then, for every $d$-dimensional analytic subset $A$ of $X//G$ the following holds:

1. There exists an analytic subset $\tilde{A} \subset A$, $\dim \tilde{A} < d$, an algebraic $G$-vector bundle $V$ over $X//G$, and an equivariant holomorphic map $\Phi : X \to V$ that is an immersion at every point in $\pi^{-1}(A \setminus \tilde{A})$.

2. There exists an analytic subset $\tilde{A} \subset A$, $\dim \tilde{A} < d$, an algebraic $G$-vector bundle $W$ over $X//G$, and an equivariant holomorphic map $\Psi : X \to W$ whose restriction to every closed $G$-orbit in $\pi^{-1}(A \setminus \tilde{A})$ is a proper embedding.

Proof. The proof goes along the lines of the proof of the corresponding result in the Stein case (see [Hei88]). Without loss of generality, we can assume that $A$ is pure-dimensional. Let $A_i, i \in I$ be the irreducible components of $A$.

1.) for every $i \in I$, choose a point $p_i \in A_i \setminus \bigcup_{j \neq i} A_j$. The set $D = \{p_i \mid i \in I\}$ is discrete, hence finite and analytic in $A$. For every $p \in D$, the fibre $\pi^{-1}(p)$ contains exactly one closed orbit $G \cdot x$ and we set the slice type of $p$ to be the slice type of $\rho_x : G_x \to GL(T_x X)$. The set $D$ is a finite union of pairwise disjoint analytic sets $D_a$ such that each $D_a$ contains points of only one particular slice type. Hence, $A$ is the finite union $A = \bigcup_a A(a)$ where $A(a)$ is the union of those irreducible components of $A$ that contain a point of $D_a$.

For fixed $a$, let $G \times_{H_a} T_a$ be a fixed representative of the slice type of $D_a$. Consider a proper equivariant embedding of $G \times_{H_a} T_a$ into a complex $G$-module $V_a$ (the existence of such an embedding is guaranteed by the fact that $G \times_{H_a} T_a$ is an affine $G$-variety). Since $D_a$ is finite, there exists a covering $\{U_{as}\}$ of $D_a$ by open subsets $U_{as}$ of $X$, such that $U_{as} \cap U_{at} = \emptyset$ for $s \neq t$. If we choose the $U_{as}$ small enough, by the holomorphic Slice Theorem, there exists an equivariant holomorphic embedding of $\pi^{-1}(U_{as})$ into a saturated open subset of the $G$-module $V_a$. By Proposition 3.1.5, we get an equivariant holomorphic map $\Phi_a : X \to V_a$ that is an immersion along $\pi^{-1}(D_a)$. We set

$$R_a := \{x \in X \mid \Phi_a(x) \text{ is not an immersion in } x\}.$$

This is a $G$-invariant analytic subset of $X$. Furthermore, we define

$$\tilde{A} := \bigcup_a (\pi(R_a) \cap A(a)).$$

This is an analytic subset of $A$. Since the map $\Phi_a$ is an immersion at every point in $\pi^{-1}(D_a)$, we see that $\dim \tilde{A} < d = \dim A$. The product map

$$\Phi := \bigoplus \Phi_a : X \to \bigoplus \bigoplus_a V_a =: V$$

into the $G$-product of the $V_a$ is an immersion along $\pi^{-1}(A \setminus \tilde{A})$.

2.) For $p \in X//G$, let $\text{Type}(p)$ be the orbit type of the uniquely determined closed orbit in $\pi^{-1}(p)$ and for any subset $Y \subset X//G$, let $\text{Type} Y := \{\text{Type}(p) \mid p \in Y\}$. We note
that due to Corollary 3.1.7, Type$(X//G)$ is finite. For any $Y \subset X//G$, we have Type$(Y \subset X//G)$, which implies that Type$(Y)$ is finite.

Inductively, we define the following finite collection of analytic subsets $E_a \subset A$ with $\bigcup_a E_a = A$: set $I_0 := I$ and for $I_a \subset I$, set $A(a) := \bigcup_{i \in I_a} A_i$. Furthermore, we define

$$
\mathcal{H}_a := \{ \text{Type}(p) \mid \text{Type}(p) \text{ is maximal in Type}(A(a)) \text{ w.r.t. } \leq \}
$$

$$
I_a := \{ j \in I_a \mid \text{Type}\ A_j \cap \mathcal{H}_a \neq \emptyset \}
$$

$$
E_a := \bigcup_{j \in I_a} A_j
$$

and finally, we set $I_{a+1} := I_a \setminus I_a$.

Since Type$(A)$ is finite, this procedure stops after finitely many steps. In $E_a$ we choose a finite set $D_a$ with the following two properties:

(i) Type$(p) \in \mathcal{H}_a$ for all $p \in D_a$

(ii) $D_a \cap A_j \neq \emptyset$ for all $j \in J_a$.

We claim that there exists a G-module $W_a$ such that every orbit type Type$(p) \in \mathcal{H}_a$ is represented by a closed orbit $G \cdot w_p$ in $W_a$. Indeed, any closed orbit $G$-orbit in $X$ carries a natural structure of an affine $G$-variety. Hence, it can be $G$-equivariantly embedded as a closed orbit in a G-module $V$. Since $\mathcal{H}_a \subset$ Type$(A)$ is finite, using Lemma 3.1.8, we see that we can find a G-module $W_a$ with the desired properties.

For every $p \in D_a$, the closed orbit $T(p)$ in $\pi^{-1}(p)$ is equivariantly biholomorphic to $G \cdot w_p \subset W_a$. Hence, we have a G-equivariant holomorphic map from the disjoint union of the $T(p), p \in D_a$ into $W_a$. This map extends to a G-equivariant holomorphic map from a saturated Stein neighbourhood of $\pi^{-1}(D_a)$ into $W_a$ (see [Hei88]). Hence, Proposition 3.1.5 yields a G-equivariant holomorphic map $\Psi_a : X \rightarrow W_a$ into an algebraic G-vector bundle $W_a$ over $X//G$ with the property that $\Psi_a(T(p)) = G \cdot w_p \subset (W_a)_p \cong W_a$ for all $p \in D_a$.

Note that if there exists a G-equivariant map between two $G$-orbits $\varphi : G \cdot x \rightarrow G \cdot y$ in G-spaces, then Type$(G \cdot y) \leq$ Type$(G \cdot x)$. Hence, it follows that for every closed orbit $G \cdot y \subset X$ with $q := \pi(y) \in E_a$, we have

$$
\text{Type}(G \cdot \Psi_a(y)) \leq \text{Type}(G \cdot y) = \text{Type}(q)
$$

$$
\leq \text{Type}(p) = \text{Type}(G \cdot w_p) \quad (*)
$$

for some $p \in \mathcal{H}_a$.

Lemma 3.1.3 implies that the algebraic Hilbert quotient $\pi_{W_a} : W_a \rightarrow W_a//G$ exists.

**Claim:** In $W_a$ there exists a $\pi_{W_a}$-saturated analytic subset $W'_a$ with the following two properties:

(a) $G \cdot \Psi_a(p) \subset W_a \setminus W'_a$ for all $p \in X//G$ with Type$(p) \in \mathcal{H}_a$
(b) If \( w \in \mathcal{W}_a \setminus \mathcal{W}_a' \) and \( \text{Type}(G \cdot w) \leq \text{Type}(G \cdot w_p) \) for some \( p \in D_a \), then we have \( \text{Type}(G \cdot w) = \text{Type}(G \cdot w_p) \) and \( G \cdot w \) is closed in \( \mathcal{W}_a \).

Assuming the claim, we finish the proof of the Lemma as follows: set \( \tilde{A}_a := \pi(\Psi^{-1}(\mathcal{W}_a')) \) and let \( G \cdot \tilde{y} \) be the closed orbit from (\( \ast \)) such that in addition, \( \pi(\tilde{y}) \in E_a \setminus \tilde{A}_a \) holds. Then, (\( \ast \)) shows that \( \text{Type}(G \cdot \Psi_a(\tilde{y})) \leq \text{Type}(G \cdot w_p) \) for some \( p \in D_a \) and \( \Psi_a(\tilde{y}) \in \mathcal{W}_a \setminus \mathcal{W}_a' \). Hence, by part (b) of the claim, it follows that \( \text{Type}(G \cdot \Psi_a(\tilde{y})) = \text{Type}(G \cdot w_p) \) and \( G \cdot \Psi_a(\tilde{y}) \) is closed in \( \mathcal{W}_a \). This implies that \( \Psi_a|_{G \cdot \tilde{y}} \) is a proper embedding. From part (a) it follows that for every \( j \in J_a \), the set \( A_j \) contains a point of \( E_a \setminus \tilde{A}_a \). Hence, \( \dim \tilde{A}_a < d \).

We define \( \tilde{A} := \bigcup_a \tilde{A}_a \) and consider the map

\[
\Psi := \bigoplus_a \Psi_a : X \longrightarrow \bigoplus_a \mathcal{W}_a := \mathcal{W}.
\]

We claim that \( \Psi \) has the desired properties. Indeed, let \( G \cdot x \) be a closed orbit in the set \( \pi^{-1}(A \setminus \tilde{A}) \). Then, \( \pi(x) \in E_a \setminus \tilde{A}_a \) for some \( a \) and the restriction of \( \Psi_a : X \rightarrow \mathcal{W}_a \) to \( G \cdot x \) is a proper embedding. We have \( \Psi_a(G \cdot x) \subset (\mathcal{W}_a)_{\pi(x)} \cong \mathcal{W}_a \). Hence, Lemma 3.1.8 gives the desired result.

It remains to prove the claim. We omit the index \( a \). Since the quotient \( \pi_W : \mathcal{W} \rightarrow \mathcal{W}//G \) is locally the quotient of an affine \( G \)-variety, it follows from Luna’s Slice Theorem [Lun73] that for all \( p \in \mathcal{W}//G \), the set \( (\mathcal{W}//G)^{(p)} := \{ q \in \mathcal{W}//G \mid \text{Type}(q) = \text{Type}(p) \} \) is Zariski-open in its closure. The set

\[
(\mathcal{W}//G)' := \{ q \in \mathcal{W}//G \mid \text{Type}(q) < \mathcal{M} \text{ for some } \mathcal{M} \in D \}
\]

is a finite union

\[
(\mathcal{W}//G)' = \bigcup_{\mathcal{M} \in \mathcal{H}} \{ q \in \mathcal{W}//G \mid \text{Type}(q) < \mathcal{M} \}
\]

and each of the sets \( \{ q \in \mathcal{W}//G \mid \text{Type}(q) < \mathcal{M} \} \) is finite union of sets of the form \( (\mathcal{W}//G)^{(p)} \). This implies that \( (\mathcal{W}//G)' \) is an algebraic subset of \( \mathcal{W}//G \).

Let \( \mathcal{W}' := \pi^{-1}((\mathcal{W}//G)') \). It clearly has property (a), since \( \text{Type}(G \cdot w_p) = \text{Type}(p) \) for all \( p \) such that \( \text{Type}(p) \in \mathcal{H} \). It remains to show property (b): let \( w \in \mathcal{W} \setminus \mathcal{W}' \) such that \( \text{Type}(G \cdot w) \leq \text{Type}(G \cdot w_p) \) for some \( p \) with \( \text{Type}(p) \in \mathcal{H} \). Suppose that \( G \cdot w \) is not closed. Then there exists a unique closed \( G \)-orbit \( G \cdot \tilde{w} \subset G \cdot \tilde{w} \). Since the \( G \)-action on \( \mathcal{W} \) is algebraic, we have \( \text{Type}(G \cdot \tilde{w}) < \text{Type}(G \cdot w) \leq \text{Type}(G \cdot w_p) \). This implies \( G \cdot \tilde{w} \subset \mathcal{W}' \). However, this is a contradiction to the fact that \( \mathcal{W} \setminus \mathcal{W}' \) is \( \pi_W \)-saturated. Hence, \( G \cdot w \) is closed. The definition of \( (\mathcal{W}//G)' \) now implies that \( \text{Type}(G \cdot w) = \text{Type}(G \cdot w_p) \). This shows the claim and completes the proof of the Lemma.

In addition to Lemma 3.1.9, the following version of the Fundamentallemma (see [Hei88] and [Sno82]) is an essential ingredient of the proof of the Embedding Theorem.
Lemma 3.1.10 (Fundamental lemma). Let $X, Y$ be holomorphic $G$-spaces with the property that the analytic Hilbert quotients $\pi_X : X \to X//G$ and $\pi_Y : Y \to Y//G$ exist. Let $x \in X$ be a point such that the $G$-orbit $G \cdot x$ through $x$ is closed. Let $\phi : X \to Y$ be an equivariant holomorphic map with the following properties:

1. $\phi$ is an immersion at $x$, and
2. $\phi|_{G \cdot x} : G \cdot x \to Y$ is a proper embedding.

Then, given an open neighbourhood $U$ of $x$, there exists an open $\pi_X$-saturated neighbourhood $T$ of $x$ in $X$ and a $\pi_Y$-saturated open subset $T'$ of $Y$ such that $\pi_X(T) \subset \pi(U)$, $\phi(T) \subset T'$, and such that $\phi|_T : T \to T'$ is a proper holomorphic embedding.

Theorem 3.1.11 (Embedding Theorem). Let $X$ be a holomorphic $G$-space with analytic Hilbert quotient $X//G$. Assume that $X//G$ is projective algebraic. Then, there exists an equivariant holomorphic embedding $\Phi : X \to V$ of $X$ into an algebraic $G$-vector bundle $V$ over $X//G$.

Proof. Apply Lemma 3.1.9 to $A = X$ to produce an equivariant holomorphic map $\Psi_1 : X \to \mathcal{W}_1$ that is an immersion outside of an analytic set $\tilde{A}$ of dimension less than $\dim X$. Applying the lemma again to $A = \tilde{A}$, we get a second map $\Psi_2 : X \to \mathcal{W}_2$. The product map $\Psi_1 \oplus \Psi_2 : X \to \mathcal{W}_1 \oplus \mathcal{W}_2$ is an immersion outside an analytic set of dimension less than $\dim \tilde{A}$. Hence, after finitely many steps we obtain an equivariant holomorphic immersion $\Psi : X \to \mathcal{W}$ into an algebraic $G$-vector bundle over $X//G$.

Furthermore, by a repeated application of the second part of Lemma 3.1.9 we get an equivariant holomorphic map $\Psi : X \to \tilde{W}$ into a second algebraic $G$-vector bundle $\tilde{W}$ over $X//G$, whose restriction to every closed orbit in $X$ is a proper embedding.

Consider $V := \mathcal{W} \oplus \tilde{W}$ and let $\Phi := \Psi \oplus \Psi : X \to V$ be the product map. Then, $\Phi$ is an immersion and its restriction to every closed orbit in $X$ is a proper embedding. From the Fundamental lemma, Lemma 3.1.10, it follows that the restriction of $\Phi$ to every fibre of $\pi : X \to X//G$ is a proper embedding. Furthermore, by construction, $\Phi$ separates the fibres of $\pi$. Hence, $\Phi$ is an injective immersion; it remains to show that it is proper.

We consider the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\Phi} & V \\
\downarrow{\pi} & & \downarrow{\pi_V} \\
X//G & \xrightarrow{\Phi} & V//G,
\end{array}
$$

where $\Phi$ is the holomorphic map induced by the $G$-invariant holomorphic map $\pi_V \circ \Phi : X \to V//G$. 
Since $X//G$ is compact, $\Phi$ is a proper embedding. Let $(x_n)$ be a sequence in $X$ such that $\Phi(x_n) \to v_0 \in V$. This implies that $\pi_V(\Phi(x_n)) \to \pi_V(v_0)$ in $V//G$. Since $\Phi$ is proper, without loss of generality, $\pi(x_n)$ converges to a point $q_0$ in $X//G$ and $\Phi(q_0) = \pi_V(v_0)$. Hence, given an arbitrary open neighbourhood $Q$ of $q_0$ in $X//G$, without loss of generality we can assume that $x_n \in \pi^{-1}(Q)$ for all $n$. Choose $Q$ to be a neighbourhood of the form $\pi(T)$ as given by the Fundamental lemma, Lemma 3.1.10, and let $T'$ be the corresponding $\pi_V$-saturated neighbourhood of $v_0$ in $V$. It follows that $\Phi(x_n) \in T'$ for all $n \in \mathbb{N}$. Since $\Phi|_T : T \to T'$ is a proper embedding, the convergence of $\Phi(x_n)$ to $v_0$ in $T'$ implies that a subsequence of $(x_n)$ converges to some $x_0$ in $T \subset X$. Hence, $\Phi$ is proper, and the claim is shown.

3.2 Algebraicity of spaces with projective algebraic quotient

We have shown in the previous section that a holomorphic $G$-space whose analytic Hilbert quotient is projective algebraic admits a proper $G$-equivariant holomorphic embedding into an algebraic $G$-variety $V$ with algebraic Hilbert quotient $V//G$. In this section we prove that the image of such an embedding is actually an algebraic subvariety of $V$. In this way we can endow $X$ with an algebraic structure. Finally, we show that this algebraic structure is essentially unique.

3.2.1 GAGA for quotient morphisms

We will show that in the setup of algebraic $G$-varieties with algebraic Hilbert quotient, the “algebraic to holomorphic”-functor is compatible with the quotient map.

Let $Y$ be an algebraic $G$-variety such that the algebraic Hilbert quotient $\pi : Y \to Y//G$ exists.

**Definition 3.2.1.** A $G$-sheaf on an algebraic variety $Y$ is a sheaf $\pi_F : \mathcal{F} \to Y$ on $Y$ with an action $G \times \mathcal{F} \to \mathcal{F}$ such that the projection map $\pi_{\mathcal{F}}$ is equivariant. A coherent algebraic $G$-sheaf on $Y$ is a $G$-sheaf on $Y$ that also is a coherent algebraic sheaf of $\mathcal{O}_Y$-modules such that the induced action of $G$ on the linear space $L_{\mathcal{F}}$ associated to $\mathcal{F}$ (see [Gra62, §3.6]) is algebraic.

**Remark 3.2.2.** A coherent algebraic sheaf on $Y$ is a coherent algebraic $G$-sheaf if and only if a certain cocycle condition is fulfilled (see [MFK94, Chap 1, §3]).

As in the case of analytic Hilbert quotients (see Section 1.2.1), it can be shown that for any coherent algebraic $G$-sheaf $\mathcal{F}$ the $G$-invariant push-forward $(\pi_*\mathcal{F})^G$ is a coherent algebraic sheaf on $Y//G$. If $\mathcal{F}$ is a coherent algebraic $G$-sheaf on $Y$, then $(L_{\mathcal{F}})^h$ is the linear space associated to $\mathcal{F}^h$. It follows that $\mathcal{F}^h$ is a coherent analytic $G$-sheaf.

Recall that the map $\pi^h : Y^h \to (Y//G)^h$ is an analytic Hilbert quotient for the associ-
ated holomorphic $G$-action on $Y^h$ (see Proposition 1.2.12) and hence, $(\pi^h_*, \mathcal{F})^G$ as well as $((\pi_*, \mathcal{F})^G)^h$ is a coherent analytic sheaf on $(Y // G)^h$. In fact, they are isomorphic. This is the content of

**Proposition 3.2.3** (GAGA for quotient morphisms). Let $Y$ be a $G$-variety such that the algebraic Hilbert quotient $\pi : Y \to Y // G$ exists. Let $\mathcal{F}$ be a coherent algebraic $G$-sheaf on $X$. Then, we have an isomorphism

$$( (\pi_*, \mathcal{F})^G )^h \cong (\pi^h_*, \mathcal{F}^h)^G$$

of coherent analytic sheaves on $(Y // G)^h$.

**Proof.** The result is classical (see for example [Nee89]), but we will sketch a proof for the reader’s convenience. We first construct a map of coherent analytic sheaves $\Phi : ( (\pi_*, \mathcal{F})^G )^h \to (\pi^h_*, \mathcal{F}^h)^G$: by definition, see Section 1.3.2, we have

$$( (\pi_*, \mathcal{F})^G )^h = \mathcal{H}_{Y/G} \otimes_{\mathcal{O}_{Y/G}} ( (\pi_*, \mathcal{F})^G )^h.$$ 

So let $s \in ( (\pi_*, \mathcal{F})^G )^h$. Then, there exists an open set $U \subset (Y // G)^h$, and elements $f_j \in \mathcal{H}_{Y/G}(U)$ and $s_j \in ( (\pi_*, \mathcal{F})^G )^h(U)$ such that $s$ is represented by $\sum f_j \otimes s_j$ on $U$. Note that each $s_j$ is the restriction to $\pi^{-1}(U)$ of a $G$-equivariant polynomial map from some Zariski-open set of $Y$ to $V$. Hence,

$$\Phi(s) : \pi^{-1}(U) \to V,$$

$$x \mapsto \sum_j f_j(\pi(x)) \cdot s_j(x)$$

is a $G$-equivariant holomorphic map, i.e., an element of $(\pi^h_*, \mathcal{F}^h)^G(U)$. Mapping $s$ to the germ of $\Phi(s)$ at $y$ yields the desired morphism of sheaves.

We claim that $\Phi$ is an isomorphism. This has to be checked only locally. Since $\pi$ is an affine map, we can assume that $Y$ is a closed $G$-stable subvariety of a $G$-module $W$.

Lemma 1.2.12 implies that the claim is true for the structure sheaf $\mathcal{O}_Y$. The key point is the following equivariant version of Nagata’s idealisation trick (see [Nag62]): there exist an affine $G$-variety $X$ and a closed $G$-equivariant embedding $Y \hookrightarrow X$ such that $\mathcal{F}$ is the ideal sheaf of $Y$ in $X$, i.e., there exists an exact sequence of coherent algebraic $G$-sheaves

$$0 \to \mathcal{F} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$ (3.7)

The $G$-variety $X$ can be constructed as follows: since $Y$ can be assumed to be affine, the coherent algebraic sheaf $\mathcal{F}$ on $Y$ is the localisation $\mathcal{M}$ of the finitely generated $\mathcal{O}_Y(Y)$-module $M = \mathcal{F}(Y)$. The direct sum $\mathcal{O}_Y(Y)^* := \mathcal{O}_Y(Y) \oplus M$ can be made into a ring with the following multiplication rule: $(f, m) \cdot (f', m') = (ff', fm' + f'm)$. The set $M$ is an ideal in $\mathcal{O}_Y(Y)^*$. The ring $\mathcal{O}_Y(Y)^*$ is a finitely generated $C$-algebra without nilpotent elements and the diagonal $G$-action on it is locally finite and rational. Hence, the variety $X := \text{Spec}(\mathcal{O}_Y(Y)^*)$ is an affine $G$-variety. The natural map from $\mathcal{O}_Y(Y)^* = \mathcal{O}_Y(Y) \oplus M$ to $\mathcal{O}_Y(Y)$ has kernel equal to $M$. Furthermore, it induces a closed equivariant embedding
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\[ Y \hookrightarrow X. \] The exact sequence \( 0 \rightarrow M \rightarrow \mathcal{O}_Y(Y)^* \rightarrow \mathcal{O}_Y(Y) \rightarrow 0 \) induces the exact sequence (3.7) of coherent algebraic \( G \)-sheaves by exactness of the localisation functor.

This yields the following commutative diagram with exact rows:

\[
\begin{array}{cccc}
0 & \rightarrow & (\pi_*\mathcal{F})^G & \rightarrow (\pi_*\mathcal{O}_X)^G & \rightarrow (\pi_*\mathcal{O}_Y)^G & \rightarrow 0 \\
\downarrow \Phi & & \cong & & \cong & \\
0 & \rightarrow & (\pi_*\mathcal{F})^h & \rightarrow (\pi_*\mathcal{H}_X)^G & \rightarrow (\pi_*\mathcal{H}_Y)^G & \rightarrow 0
\end{array}
\]

Here, the exactness of the upper row follows from the exactness of the \( \pi_* \cdot \)\(^G\)- and \( \cdot \)^h-functors and exactness of the lower row is provided by Lemma 1.2.8. Therefore, \( \Phi \) is an isomorphism as claimed.

3.2.2 Separation properties of \( K \)-finite holomorphic functions

In this section we will investigate the following situation: let \( Y \) be a holomorphic Stein \( G \)-space and let \( X \subset Y \) be a \( G \)-invariant analytic subset of \( Y \). We will see that \( X \) is the common zero set of a family of (global) \( K \)-finite holomorphic functions.

Continuous representations on Fréchet spaces

**Definition 3.2.4.** Let \( K \) be a Lie group and let \( F \) be a Fréchet space over \( \mathbb{C} \). A continuous representation of \( K \) on \( F \) is a homomorphism \( \rho \) from \( K \) into the group of continuous invertible linear operators on \( F \) such that the map

\[ \rho_f : K \rightarrow F, \ k \mapsto \rho(k)(f) \]

is continuous for every \( f \in F \). A vector \( f \in F \) is called \( K \)-finite if the linear span of its orbit \( K \cdot f \) is a finite-dimensional vector subspace of \( F \). If \( G \) is a complex Lie group, then a representation \( \rho \) of \( G \) on a Fréchet space \( F \) is called holomorphic if for each \( f \in F \), the map \( \rho_f : G \rightarrow F \) is holomorphic as a map from the complex manifold \( G \) into the Fréchet space \( F \) (see [Akh95]).

Let \( K \) be a compact Lie group. If a continuous representation of \( K \) on a Fréchet space \( F \) extends to a holomorphic representation of \( G := K^\mathbb{C} \), then a vector \( f \in F \) is \( K \)-finite if and only if it is \( G \)-finite.

The crucial result about representations of compact groups on Fréchet spaces that we will use is

**Theorem 3.2.5 (Fourier theorem).** Let \( K \) be a compact Lie group and let \( F \) be a Fréchet space with a continuous \( K \)-representation. Then, the subspace of \( K \)-finite vectors is dense in \( F \).

The proof of the Fourier theorem is due to Harish-Chandra [HC66]; also see [Akh95]. We will apply the Fourier Theorem to the following special situation:
3.2. Algebraicity of spaces with projective algebraic quotient

Lineray equivariant maps and $K$-finite holomorphic functions

Let $K$ be a Lie group and $X$ a complex $K$-space. Then we have a continuous representation of $K$ on the Fréchet space $\mathcal{O}_X(X)$. We denote by $\mathcal{F}_K(X)$ the subspace of $K$-finite holomorphic functions in $\mathcal{O}_X(X)$. Let $V$ be a finite-dimensional $K$-invariant linear subspace of $\mathcal{O}_X(X)$ and $V^*$ the dual space. The group $K$ acts linearly on $V^*$ and the holomorphic map $\phi : X \to V^*, \phi(x)(f) = f(x)$ is an equivariant map from $X$ into the $K$-module $V^*$. Conversely, the components of an equivariant holomorphic map from $X$ into a $K$-module are elements of $\mathcal{F}_K(X)$.

Now, let $G = K^\mathbb{C}$ be the complexification of a compact Lie group $K$ and let $X$ be a holomorphic $G$-space. Let $\tilde{K}$ be a complete system of irreducible unitary representations of $K$. For $\rho : K \to GL(V)$ in $\tilde{K}$ consider the continuous projection operator

$$p_\rho : \mathcal{O}_X(X) \to \mathcal{O}_X(X)$$

$$f \mapsto \dim_C(\rho) \cdot \int_K (k \cdot f)(x) \cdot \overline{\chi_\rho(k)} \, dk,$$

where $\overline{\cdot}$ denotes complex conjugation, $\chi_\rho$ is the character corresponding to $\rho$, and the integration is with respect to the Haar measure on $K$. Note that $p_\rho(\mathcal{O}_X(X)) \subset \mathcal{F}_K(X)$, since every irreducible representation of $K$ is finite-dimensional. The Fourier theorem in this context now states that

$$f = \sum_{\rho \in \tilde{K}} p_\rho(f), \quad (3.8)$$

i.e., the series on the right hand side converges in the Fréchet topology of $\mathcal{O}(X)$ to $f$. The right hand side of $(3.8)$ is called the Fourier series of $f$.

The following result is well-known (see e.g. [Hei91]); however, we discuss it here in some detail due to its importance in the proof of the Algebraicity Theorem in the next section.

**Lemma 3.2.6.** Let $Y$ be a holomorphic Stein $G$-space and $X$ a $G$-invariant analytic subset. Then, for each point $y \in Y \setminus X$, there exists a $G$-module $V$ and a $G$-equivariant holomorphic map $\varphi : Y \to V$ such that $\varphi|_X \equiv 0$ and $\varphi(y) \neq 0$.

**Proof.** Let $\mathcal{F}_X$ be the ideal sheaf of $X$ in $\mathcal{O}_Y$. Our first claim is

$$X = \{ y \in Y \mid f(y) = 0 \text{ for all } f \in \mathcal{F}_X(Y) \}.$$ 

The inclusion "$\subset$" is clear, so it remains to show the converse inclusion. So let $y \in Y \setminus X$. Then, since $Y$ is Stein, there exists a holomorphic function $f \in \mathcal{O}_X(X)$ such that $f|_X \equiv 0$ and $f(y) = 1$. This implies that $y$ is not an element of the right hand side.

Let $y \in Y \setminus X$. By the considerations above, there exists an $f \in \mathcal{F}_X(Y)$ such that $f(y) \neq 0$. Consider the Fourier series $(3.8)$ of $f$. Since $f(y) \neq 0$, there exists a $\rho \in \tilde{K}$ such that $p_\rho(f)(y) \neq 0$. Note that due to the definition of $p_\rho(f)$, we have $p_\rho(f) \in \mathcal{F}_X(Y)$. Since $p_\rho(f) \in \mathcal{F}_K(X)$, it defines a $G$-module $V$ and a $G$-equivariant holomorphic map $\varphi : Y \to V$ with $\varphi(X) = 0$ and $\varphi(y) \neq 0$. \(\square\)
**Corollary 3.2.7.** Let $Y$ be a holomorphic Stein $G$-space and $X$ a $G$-invariant analytic subset. Then, we have

$$X = \{ y \in Y \mid f(y) = 0 \text{ for all } f \in \mathcal{F}_K(X) \text{ with } f|_X \equiv 0 \}.$$ 

### 3.2.3 The Algebraicity Theorem for invariant analytic subsets

Let $Y$ be an algebraic $G$-variety with algebraic Hilbert quotient $\pi : Y \to Y//G$. Assume that $Y//G$ is a projective algebraic variety. Let $X$ be a $G$-invariant analytic subset of $Y$ such that $\pi(X) = Y//G$. We will prove that $X$ is an algebraic subvariety of $Y$, and apply this result to the image of the embedding guaranteed by the Embedding Theorem (Theorem 3.1.11).

For a vector bundle $\mathcal{V}$ on a complex space $Z$, let $Z_\mathcal{V} \subset \mathcal{V}$ be the zero section. Fix an ample line bundle $L$ on $Y//G$.

**Lemma 3.2.8.** For every $y \in Y \setminus X$, there exist a $G$-module $V$, a natural number $m \in \mathbb{N}$, and a $G$-equivariant holomorphic map $\Phi : Y \to \mathcal{V}$ into the $G$-vector bundle $\mathcal{V} = V \otimes L^m$ such that $\Phi(X) \subset Z_\mathcal{V}$ and $\Phi(y) \notin Z_\mathcal{V}$.

**Proof.** Let $y \in Y \setminus X$. Since $\pi(X) = Y//G$, the fibre of $\pi$ over $q := \pi(y)$ intersects $X$ non-trivially, see Figure 3.1. Set $A := X \cap \pi^{-1}(q)$.

\[\text{Figure 3.1: The situation of Lemma 3.2.8}\]
Since $\pi^{-1}(q)$ is affine and hence Stein, Lemma 3.2.6 implies that there exists a $G$-module $V$ and a $G$-equivariant holomorphic map $\varphi : \pi^{-1}(q) \to V$ such that $\varphi|_A \equiv 0$ and $\varphi(y) \neq 0$. Consider the sheaf $\mathcal{F}$ of germs of holomorphic maps from $Y$ to $V$ and let $\mathcal{I} \subset \mathcal{F}$ be the subsheaf of those germs that vanish along $X \subset Y$, i.e., $\mathcal{I} = \mathcal{I}_X \cdot \mathcal{F}$, where $\mathcal{I}_X$ denotes the ideal sheaf of $X$ in $\mathcal{K}_Y$. Since $X$ is $G$-invariant, $\mathcal{I}$ is a coherent analytic $G$-sheaf on $Y$. The $G$-invariant push-forward $\mathcal{F} := (\pi_Y^*\mathcal{I})^G$ is a coherent analytic sheaf on $X//G = Y//G$. Let $m_q$ be the ideal sheaf in $\mathcal{K}_{Y//G}$ of the point $q \in Y//G$. Let $m \in \mathbb{N}$ be chosen such that $H^1(Y//G, m_q \cdot \mathcal{F} \otimes \mathcal{L}^\otimes m) = 0$ (Serre vanishing). Then, the exact sequence

$$0 \to m_q \cdot \mathcal{F} \to \mathcal{F} \to \mathcal{F}/m_q \cdot \mathcal{F} \to 0$$

induces an exact sequence on the level of sections

$$0 \to H^0(Y//G, m_q \cdot \mathcal{F} \otimes \mathcal{L}^\otimes m) \to H^0(Y//G, \mathcal{F} \otimes \mathcal{L}^\otimes m) \to H^0(Y//G, \mathcal{F}/m_q \cdot \mathcal{F}) \to 0.$$

The map $\varphi$ gives an element in $H^0(Y//G, \mathcal{F}/m_q \cdot \mathcal{F})$ and hence, there exists a section of $\mathcal{F} \otimes \mathcal{L}^\otimes m$ that coincides with $\varphi$ at $q$. Lemma 3.1.2 now yields a $G$-equivariant holomorphic map $\Phi : Y \to V \otimes \mathcal{L}^\otimes m$ with $\Phi(X) \subset Z_Y$ and $\Phi(y) \notin Z_Y$. \hfill $\square$

In the setup of the previous lemma, if $\Phi(X) \subset Z_Y$, we also say that $\Phi$ vanishes on $X$. Analogously, if $\Phi(y) \in Z_Y$, we also write $\Phi(y) = 0$.

**Lemma 3.2.9.** The holomorphic map $\Phi : Y \to V$ constructed in the previous lemma is algebraic.

**Proof.** We have $\mathcal{I} = \mathcal{I}_X \cdot \mathcal{F} \subset \mathcal{F}$, where $\mathcal{F}$ is the coherent analytic $G$-sheaf of germs of holomorphic maps from $Y^h$ to $V$ and hence $\mathcal{F} \subset (\pi_Y^*\mathcal{F})^G$ due to the exactness of the $\pi_Y^*\cdot^G$-functor. Let $L$ be the ample line bundle introduced above, and let $\mathcal{L}$ be the locally free sheaf associated to $L$. Since tensoring with $\mathcal{L}$ is left-exact, $\Phi$ is induced by an element $\phi \in H^0(Y//G, (\pi_*\mathcal{F})^G \otimes \mathcal{L}^\otimes m)$. Let $\mathcal{A}$ be the coherent algebraic $G$-sheaf of germs of algebraic maps from $Y$ to the $G$-module $V$. Then, $\mathcal{F}$ is isomorphic to $\mathcal{A}^h$ as a coherent analytic sheaf of $\mathcal{K}_Y$-modules. We have

$$(\pi_*\mathcal{A})^G \otimes \mathcal{O}_{Y//G} \mathcal{L}^\otimes m)^h = ((\pi_*\mathcal{A})^G)^h \otimes \mathcal{K}_{Y//G} \mathcal{L}^\otimes m = (\pi_Y^*\mathcal{F})^G \otimes \mathcal{K}_{Y//G} \mathcal{L}^\otimes m,$$

where for the second equality we used GAGA for quotient morphisms, Proposition 3.2.3.

It follows from the classical GAGA Theorem, Theorem 1.3.6, that the holomorphic section $\phi \in H^0((Y//G)^h, (\pi_Y^*\mathcal{F})^G \otimes \mathcal{K}_{Y//G} \mathcal{L}^\otimes m)$ is an element of $H^0(Y//G, (\pi_*\mathcal{A})^G \otimes \mathcal{O}_{Y//G} \mathcal{L}^\otimes m)$ and hence algebraic. \hfill $\square$

We are now in the position to prove the main result of this section:

**Theorem 3.2.10** (Algebraicity Theorem for invariant algebraic subsets). Let $Y$ be an algebraic $G$-variety with algebraic Hilbert quotient $\pi : Y \to Y//G$. Assume that $Y//G$ is a projective algebraic variety. Let $X \subset Y$ be a $G$-invariant analytic subset of $Y$ such that $\pi(X) = Y//G$. Then, $X$ is an algebraic subvariety of $Y$. 
Proof. We set \[ A := \{ \Phi : Y \to V \otimes L^m \mid \Phi \text{ a } G\text{-equivariant algebraic map, } V \text{ a } G\text{-module, } m \in \mathbb{N} \}. \]

It suffices to show that \( X = \{ y \in Y \mid \Phi(y) = 0 \text{ for every } \Phi \in A \text{ vanishing on } X \} := B, \)

since then \( X \) is the intersection of a family of algebraic subvarieties of \( Y \), hence algebraic. Clearly, we have \( X \subset B \). So, let \( y \in Y \setminus X \). Then, using Lemmata 3.2.8 and 3.2.9, there exists a map \( \Phi \in A \) such that \( \Phi \) vanishes on \( X \) and \( \Phi(y) \neq 0 \). Hence, \( y \notin B \) and the claim is shown.

The claims of Lemma 3.2.8 and of Theorem 3.2.10 are no longer true if the quotient is not assumed to be projective:

Example 3.2.11. Consider the action of \( \mathbb{C}^* \) on \( \mathbb{C}^3 = \mathbb{C} \times \mathbb{C}^2 \) that is given by multiplication in the second factor. The analytic Hilbert quotient is given by \( \pi : \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}, (z,w) \mapsto z \).

The holomorphic function \( F : \mathbb{C}^3 \to \mathbb{C}, (u_1,u_2,u_3) \mapsto u_3 - u_2 \cdot e^{u_1} \) is equivariant with respect to the \( \mathbb{C}^*\)-action on \( \mathbb{C}^3 \) and the standard action of \( \mathbb{C}^* \) on \( \mathbb{C} \) by multiplication. Its zero set \( X = \{ F = 0 \} \) is a \( \mathbb{C}^*\)-invariant, non-algebraic, analytic subset of \( \mathbb{C}^3 \) with \( \pi(X) = \mathbb{C}^3//\mathbb{C}^* = \mathbb{C} \).

3.2.4 Proof of the Algebraicity Theorem

Let \( X \) be a holomorphic \( G\)-space such that the analytic Hilbert quotient \( X//G \) exists as a projective algebraic complex space. The discussion in the following section will show that \( X \) is the complex space associated to an essentially unique quasi-projective algebraic \( G\)-variety and that the quotient \( \pi : X \to X//G \) is an algebraic Hilbert quotient.

By the following result, we are in the setup considered in the previous section.

Lemma 3.2.12. Let \( X \) be a holomorphic \( G\)-space such that the analytic Hilbert quotient \( X//G \) exists as a projective algebraic complex space. Then, there exists a quasi-projective \( G\)-variety \( Y \) with algebraic Hilbert quotient \( \pi_Y : Y \to Y//G \) and a proper \( G\)-equivariant holomorphic embedding \( \Phi : X \to Y \) such that \( \pi(\Phi(X)) = Y//G \).

Proof. By the Embedding Theorem, Theorem 3.1.11, there exists a proper holomorphic embedding \( \Phi : X \hookrightarrow \mathcal{V} \) into an algebraic \( G\)-vector bundle over \( X//G \). Furthermore, \( \mathcal{V} \) admits an algebraic Hilbert quotient \( \pi_V : \mathcal{V} \to \mathcal{V}//G \). Since the projection map \( p : \mathcal{V} \to X//G \) is affine, \( \mathcal{V} \) is a quasi-projective algebraic variety, see [Har77, Chap II.7].

Consider the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Phi} & \mathcal{V} \\
\downarrow\pi & & \downarrow\pi_V \\
X//G & \xrightarrow{\overline{\Phi}} & \mathcal{V}//G.
\end{array}
\]
Since $X/G$ is projective, $\Phi$ is algebraic by Theorem 1.3.6, and its image is a compact algebraic subvariety of $V/G$. Set $Y := \pi_Y^{-1}(\Phi(X/G)) \subset V$. The restriction $\pi_Y := \pi_Y|_{Y} : Y \to \pi_Y(Y) \cong X/G$ is an algebraic Hilbert quotient, and $\Phi : X \to Y$ is a proper holomorphic embedding.

**Proposition 3.2.13.** Let $X$ be a holomorphic $G$-space such that the analytic Hilbert quotient $\pi : X \to X/G$ exists and $X/G$ is projective algebraic. Then, there exists an algebraic $G$-variety $X_0$ with algebraic Hilbert quotient $\pi_0 : X_0 \to X_0/G \cong X/G$ such that $X = X_0^h$ and $\pi = \pi_0^h$.

**Proof.** By Lemma 3.2.12, we can assume that $X$ is a $G$-invariant analytic subset of a quasiprojective $G$-variety $Y$ that admits an algebraic Hilbert quotient $\pi_Y : Y \to Y/G$ such that $\pi(X) = Y/G$. Hence, $\pi = \pi_Y|_{X}$. The Algebraicity Theorem for invariant analytic sets, Therorem 3.2.10, implies that $X$ is an algebraic subset of $Y$, hence a quasi-projective algebraic variety. Hence, we can endow $X$ with the structure of an algebraic $G$-variety with algebraic Hilbert quotient $X/G$. Next, we will investigate uniqueness properties of this algebraic structure.

**Proposition 3.2.14.** Let $X$ be a holomorphic $G$-space such that the analytic Hilbert quotient $\pi : X \to X/G$ exists and such that $X/G$ is a projective algebraic variety. Assume that there are two algebraic $G$-varieties $X_1$ and $X_2$ with algebraic Hilbert quotients $\pi_j : X_j \to X_j/G, j = 1, 2$ such that

1. $X = X_j^h$ for $j = 1, 2$,
2. $X/G = (X_j/G)^h$ for $j = 1, 2$, and
3. $\pi = \pi_j^h$ for $j = 1, 2$.

Then, $X_1$ and $X_2$ are isomorphic as algebraic $G$-varieties.

**Proof.** Let $\text{id} : X_1^h = X \to X = X_2^h$ be the identity map. Its graph is equal to

$$\Delta(X) = \{(x, x) \mid x \in X\} \subset X \times X.$$ 

The holomorphic $G$-space $X \times X$ is the analytic space associated to the algebraic $G$-variety $X_1 \times X_2$ and $\Delta(X)$ is a $G$-invariant analytic subset that is $G$-equivariantly biholomorphic to $X$ via $\iota : X \to \Delta(X), x \mapsto (x, x)$. We will show that it is an algebraic subvariety of $X_1 \times X_2$.

First, we notice that $X_1/G$ and $X_2/G$ are isomorphic as algebraic varieties, since they are isomorphic as complex spaces and projective algebraic (see Theorem 1.3.6). Consider the $G$-invariant algebraic map $\pi_1 \times \pi_2 : X_1 \times X_2 \to X/G \times X/G$. We claim that it is affine. Indeed, let $\{A_i\}_{i=1,...,k}$ be an affine covering of $X/G$. Then $\{A_i \times A_j\}_{i,j=1,...,k}$ is an affine
covering of $X//G \times X//G$. The preimage $(\pi_1 \times \pi_2)^{-1}(A_i \times A_j) = \pi^{-1}(A_i) \times \pi^{-1}(A_j)$ is affine, since both $\pi_1$ and $\pi_2$ are affine, and therefore, $\pi_1 \times \pi_2$ is an affine map. It follows from Lemma 1.2.11 that there exists an algebraic Hilbert quotient

$$\Pi : X_1 \times X_2 \to (X_1 \times X_2)//G =: Q$$

for the $G$-action on $X_1 \times X_2$. Consider the following commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & X_1 \times X_2 \\
\pi \downarrow & & \downarrow \Pi \\
X//G & \xrightarrow{i} & Q.
\end{array}
$$

Since $X//G$ is projective, it follows that $\Pi(\Delta(X)) = i(X//G)$ is a compact algebraic subvariety of $Q$ isomorphic to $X//G$. Consider $Y := \Pi^{-1}(\Pi(\Delta(X)))$. This is clearly an algebraic $G$-subvariety of $X_1 \times X_2$. Then, $\pi_Y := \Pi|_Y : Y \to \Pi(Y) = Y//G$ is an algebraic Hilbert quotient, $Y//G \cong X//G$ is projective, and $\Delta(X) \subset Y$ is a $G$-invariant analytic subset such that $\pi_Y(\Delta(X)) = Y//G$. The Algebraicity Theorem for invariant analytic sets, Theorem 3.2.10, implies that $\Delta(X) \subset Y \subset X_1 \times X_2$ is algebraic. Therefore, the graph of $id : X_1 \to X_2$ is an algebraic subset of $X_1 \times X_2$, and the claim follows from Lemma 1.1.10. \[\square\]

We summarise our results in

**Theorem 3** (Algebraicity Theorem). Let $X$ be a holomorphic $G$-space such that the analytic Hilbert quotient $\pi : X \to X//G$ exists and such that $X//G$ is projective algebraic. Then, there exists a uniquely determined quasi-projective $G$-variety $Z$ with algebraic Hilbert quotient $\pi_Z : Z \to Z//G$ such that $X$ is $G$-equivariantly biholomorphic to $Z^h$, such that $X//G$ is biholomorphic to $(Z//G)^h$, and such that $\pi : X \to X//G$ is the analytic map associated to the algebraic quotient map $\pi_Z : Z \to Z//G$.

The claim of Theorem 3 is no longer true, if the quotient is not assumed to be projective, as illustrated by the following

**Example 3.2.15.** Let $A \subset \mathbb{C}^2$ be any non-algebraic, analytic subset of $\mathbb{C}^2$. E.g. let $exp : \mathbb{C} \to \mathbb{C}, z \mapsto \exp z$ be the exponential function and let $A := \Gamma_{\exp}$ be its graph in $\mathbb{C}^2$. If $\varphi : A \to \mathbb{C}^2$ is the inclusion of $A$ into $\mathbb{C}^2$, we consider the space

$$X := \mathbb{C}^2 \cup \varphi \mathbb{C}^2 = (\mathbb{C}^2 \cup \mathbb{C}^2) / \sim,$$

where $x \sim y$, if and only if $y = \varphi(x)$. Let $Y := \mathbb{C}^2 \cup \mathbb{C}^2$ and let $p : Y \to Y / \sim$ be the quotient map. For an open subset $U$ in $Y / \sim$ we define

$$\mathcal{H}_Y / \sim(U) := \{ f \in \mathcal{H}_X(p^{-1}(U)) \mid f \text{ constant on } \sim\text{-equivalence classes}\}.$$ 

It follows from a Theorem of H. Cartan [Car60] that the ringed space $(X, \mathcal{H}_Y / \sim)$ is a complex space, and that the projection map $p : Y \to X$ is holomorphic.
Consider the holomorphic map \( \sigma : Y \rightarrow Y \) which sends a point \( x \) in one of the connected components \( C^2 \) of \( Y \) to the corresponding point in the other connected component of \( Y = C^2 \cup C^2 \). The automorphism \( \sigma \) generates a holomorphic \( Z_2 \)-action on \( Y \) that descends to a holomorphic \( Z_2 \)-action on \( X \) in such a way that \( p : Y \rightarrow X \) is equivariant.

Consider the holomorphic map \( \psi : Y \rightarrow C^2 \) which sends a point \( x \in Y \) to the corresponding point in \( C^2 \). The map \( \psi \) is constant on the equivalence classes and hence descends to a holomorphic map \( \pi : X \rightarrow C^2 \) which is an analytic Hilbert quotient for the \( Z_2 \)-action on \( X \).

We claim that there exists no algebraic structure on \( X \) making \( X \) into an algebraic \( Z_2 \)-variety in such a way that the map \( \pi : X \rightarrow C^2 \) becomes the algebraic Hilbert quotient for the action of \( Z_2 \) on \( X \). Indeed, suppose that this was possible. Then, \( X \) would be an affine variety and the map \( \pi : X \rightarrow C^2 \) would be finite. Therefore, the branch locus \( R_\pi \subset C^2 \) of \( \pi \) would be an algebraic subvariety of \( C^2 \). However, set-theoretically, we have \( R_\pi = A \), a contradiction.

### 3.2.5 Relations to Geometric and Classical Invariant Theory

Let \( G \) be a complex-reductive Lie group and let \( V \) be a \( G \)-module. Then, we have an induced algebraic action of \( G \) on the projective space \( \mathbb{P}(V) \). Let \( \mathcal{N}(V) \) be the null-cone of \( V \) as defined in Section 1.2.1. If \( p : V \setminus \{0\} \rightarrow \mathbb{P}(V) \) denotes the natural projection, we set \( \mathbb{P}(V)(V) := \mathbb{P}(V) \setminus p(\mathcal{N}(V)) \).

As we have seen in Section 1.2.1, the algebraic Hilbert quotient \( \mathbb{P}(V)(V) // G =: Q \) exists. In fact, we have \( Q \cong \text{Proj}(C[V]^G) \), and the quotient map \( \pi : \mathbb{P}(V)(V) \rightarrow Q \) is given by the inclusion of finitely generated graded rings \( C[X]^G \subset C[V] \).

Let \( X \) be an algebraic \( G \)-variety with projective algebraic Hilbert quotient \( X//G \). Let \( \pi : X \rightarrow X//G \) denote the quotient map, and let \( L \) be an ample line bundle on \( X//G \). The pull-back \( \pi^*(L) \) carries a natural \( G \)-action by bundle automorphisms, and therefore \( G \) acts linearly on the vector space \( \Gamma(X, \pi^*(L)) \) of regular sections of \( \pi^*(L) \). Every non-trivial finite dimensional \( G \)-invariant linear subspace \( W \subset \Gamma(X, \pi^*(L)) \) defines a \( G \)-equivariant rational map of \( X \) into \( \mathbb{P}(V) \), where \( V := W^* \). Using compactness of \( X//G \) it can be shown that \( W \) can be chosen in such a way that it defines a \( G \)-equivariant closed embedding \( \psi \) of \( X \) into \( \mathbb{P}(V)(V) \), cf. [MFK94, Chap 1, §5]. It follows that the algebraic Hilbert quotient \( \pi : X \rightarrow X//G \) is given by the restriction of the algebraic Hilbert quotient \( \pi_P : \mathbb{P}(V)(V) \rightarrow Q \) to \( \psi(X) \). In this way, we have reduced the study of projective algebraic Hilbert quotients to the study of invariants of rational \( G \)-representations, i.e., to Classical Invariant Theory.

Now, let \( X \) denote a holomorphic \( G \)-space with analytic Hilbert quotient \( \pi : X \rightarrow X//G \) and assume that \( X//G \) is projective algebraic. Then, it follows from Theorem 3 and from the considerations above that there exists a \( G \)-module \( V \), a \( G \)-invariant subvariety \( Z \subset \)
\[ P(V)(V), \text{ and a } G\text{-equivariant biholomorphic map } \psi : X \to Z. \text{ It follows that } X//G \text{ is isomorphic to the image of } Z \text{ in } Q = \text{Proj}(\mathbb{C}[V]^G) \text{ and that the quotient map } \pi \text{ is given by the restriction of } \pi_{\psi} \text{ to } Z. \text{ Again, this reduces the study of holomorphic } G\text{-spaces with projective algebraic analytic Hilbert quotient to Classical Invariant Theory.} \]

3.3 Algebraicity of sets of semistable points: open problems

We conclude this monograph with the discussion of open questions related to the results proven in Chapters 2 and 3.

Let us return to the setup of Chapter 2. Let \( G \) be the complexification of the compact Lie group \( K \). Let \( X \) be a \( G \)-irreducible algebraic Hamiltonian \( G \)-variety with momentum map \( \mu : X \to \mathfrak{t}^* \). Assume that \( X \) has only 1-rational singularities and that \( \mu^{-1}(0) \) is compact. Then, Theorem 1 and Theorem 2 imply that \( X(\mu)//G \) is projective algebraic and that \( X(\mu) \) is an algebraically Zariski-open subset in \( X \), hence itself an algebraic \( G \)-variety. It now follows from Theorem 3 that there exists a quasi-projective algebraic \( G \)-variety \( Z \) and a \( G \)-equivariant biholomorphic map \( \psi : X(\mu)^h \to Z^h \).

In this situation, presently we do not know the answer to the following question:

\[ \text{Is } \psi \text{ induced by an isomorphism between the algebraic varieties } X(\mu) \text{ and } Z? \quad (\star) \]

Related questions include:

- Is the quotient map \( \pi : X(\mu) \to X(\mu)//G \) algebraic?
- If \( \pi : X(\mu) \to X(\mu)//G \) is algebraic, is it an algebraic Hilbert quotient?

By Proposition 3.2.14, a positive answer to these two questions would imply a positive answer to Question \((\star)\). Conversely, if \( \psi \) is an isomorphism of algebraic varieties, then \( \pi : X(\mu) \to X(\mu)//G \) is an algebraic Hilbert quotient.

In fact, we can answer Question \((\star)\) positively in the following special case:

**Proposition 3.3.1.** Let \( K \) be a compact Lie group and let \( G = K^\mathbb{C} \) be its complexification. Let \( X \) be a \( G \)-irreducible algebraic Hamiltonian \( G \)-variety with momentum map \( \mu : X \to \mathfrak{t}^* \). Assume that \( X \) has only 1-rational singularities and that \( \mu^{-1}(0) \) is compact. Assume in addition that all stabiliser groups of points in \( X(\mu) \) are finite. Then, the map \( \psi \) of Question \((\star)\) is a regular isomorphism.

**Sketch of proof.** Since the stabiliser groups of elements in \( X(\mu) \) are finite, all \( G \)-orbits in \( X(\mu) \) are closed in \( X(\mu) \). The quotient map \( \pi : X(\mu) \to X(\mu)//G \) is rational (cf. Remark 2.4.5). Since \( \pi \) is a priori holomorphic, this implies that it is regular. It follows that
π : X(μ) → X(μ)/G is a geometric quotient in the sense of Definition 2.1.12. The existence of local slices for the action of G on X(μ) implies that G acts properly on X(μ). It then follows from results of Kollár [Kol97] that π is an algebraic Hilbert quotient.

Remark 3.3.2. If X is smooth, the finiteness of the isotropy groups required in Proposition 3.3.1 is equivalent to 0 being a regular value of the momentum map μ.

Recall that if X is smooth and projective, there exists a very ample line bundle L on X such that X(μ) = X(L), see [HM01]. Hence, in the smooth projective algebraic situation, the Kählerian reduction theory coincides with Geometric Invariant Theory.

If U is an open subset in a Q-factorial normal algebraic G-variety X such that the algebraic Hilbert quotient π : U → U//G exists, it follows from a refinement of results of Mumford [MFK94, Chap 1, §4, Conv 1.13] that there exists a line bundle L on X such that U = X(L).

Hence, we have

Proposition 3.3.3. Let K be a compact Lie group and let G = KC be its complexification. Let X be a G-irreducible algebraic Hamiltonian G-variety with momentum map μ : X → ℍ*. Assume that X is Q-factorial, that it has 1-rational singularities and that μ−1(0) is compact. Assume in addition that Question (⋆) has a positive answer. Then, π : X(μ) → X(μ)//G is an algebraic Hilbert quotient, and there exists a line bundle L on X such that X(L) = X(μ).
Bibliography


Bibliography


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