On Planar Exponential and Logarithm Series

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Introduction

The topics of this thesis are generalizations of classical power series to planar (= non-commutative, non-associative) power series. That means that we are lifting some classical power series into a planar setting.

An isomorphism class of a finite, planar, reduced, and rooted tree represents, by definition, a planar monomial. By \( P \) we denote the set of planar monomials including the empty tree \( 1 \). There is a natural \( k \)-ary grafting operation \( \cdot_k : \mathbb{P}^k \to \mathbb{P} \), for \( k \in \mathbb{N} \geq 2 \). We also have the grafting over a tree \( V \), denoted by \( \cdot_V \). It is defined in the following way:

Let \( T_1, ..., T_m, V \in \mathbb{P} \) and \( |V| = m \), then \( \cdot_V(T_1, ..., T_m) \) is the tree \( T \), where \( T - In(V) \) is a planar rooted forest, which is isomorphic to \( (T_1, ..., T_m) \).

\( T^V := \cdot_V(T, ..., T) \), which means that we replace every leaf \( k \in L(V) \) with the tree \( T \). If we observe a \( K \)-linear combination over \( \mathbb{P} \), then we get a planar polynomial over a field \( K \). We denote the set of all planar polynomials by \( K\{x\} \).

We also have a \( K \)-multi linear extension of the grafting product and consider \( K\{x\} \) as an algebra over an operad \( \mathcal{P} \). \( \mathcal{P} \) is the free operad generated by the system \((\cdot_k)_{k \geq 2}\), see [MSS] and [Hol1]. Here we can take the \( V \)-th power of a polynomial \( f \in K\{x\} \), which is defined as \( f^V := \cdot_V(f, ..., f) \). A planar power series is an infinite formal expression over \( \mathbb{P} \) with coefficients in \( K \). If \( x \) denotes the unique tree with a single vertex, then any planar power series is an infinite sum over \( K \)-multiples of planar products in \( x \). We call the \( K \)-vector space \( K\{\{x\}\} \) together with the system of the \( K \)-multi linear extension of the grafting product the algebra of planar power series. Thus \( K\{x\} \) is a subalgebra of the algebra of planar power series \( K\{\{x\}\} \). On the algebra of planar power series we have a natural topology induced by the order function. The series are continuous with respect to this topology and for a continuous power series \( f \in K\{\{x\}\} \) we can consider the \( V \)-th power \( f^V \), which is defined in the same way as for polynomials. Furthermore, there is a canonical, continuous \( K \)-algebra-homomorphism \( \eta : K\{\{x\}\} \to K[[x]] \), which maps a tree \( T \in \mathbb{P} \) with \( |T| = n \) onto \( x^n \). This map is the projection of the lifted power series onto the classical algebra.

One of the main results is concerned with planar exponential series. We consider a field \( K \) with \( \text{char}(K) = 0 \) and define these series with respect to all planar rooted trees. We investigate the properties, which these series have, and compare them to the classical case. The recursion formulas for the computations of the coefficients are presented and proved. We state the following

**Proposition**  Let \( V \in \mathbb{P} \) and \( k = |V| \) with \( k \in \mathbb{N}^{\geq 2} \). Then there is a unique
power series $\exp_V(x) \in K\{x\}$, such that

(i) $\text{ord}(\exp_V(x) - (1 + x)) \geq 2$,

(ii) $(\exp_V(x))^V = \exp_V(kx)$.

Moreover $\frac{d}{dx} (\exp_V(x)) = \exp_V(x)$.

$\exp_V(x)$ is called the planar exponential series with respect to $V$.

Then for the coefficient $a_V(T)$ of $\exp_V(x)$ we get the recursion formula

$$a_V(x) = 1$$

and

$$a_V(T) = \sum_S \frac{\binom{V}{S}}{kn - k} \cdot a_V(T - S),$$

where $S \in O(T), S \neq x, 2 \leq m = |S|, n = |T|$ and $a_V(T - S) = \prod_{i=1}^{m} a_V(T_i)$, if $T - S = (T_1, ..., T_m)$.

We also deform these series by weighting the leaves of the planar rooted trees $V$ by $\lambda \in K^n$, where $n = \sharp(L(V)) = |V|$, thus we get an other power series with deformed functional equations. The main result is, that for all planar deformed exponential series $\exp_{(V,\lambda)}(x)$ we also get the derivative property $\frac{d}{dx} (\exp_{(V,\lambda)}(x)) = \exp_{(V,\lambda)}(x)$. By using the exponential series we can define the hyperbolic sine and cosine as well as the ordinary sine and cosine. There we get even and odd power series with same derivative properties like in the classical case. See also [Con].

We define the composition of two power series by substituting $g(x)$ for $x$ in a power series $f(x)$ if the order of $g(x)$ is greater than 0. If $\text{ord}(g) = 1$ then $f(g)$ is an automorphism. We can find the compositional inverse to the exponential series $\exp_V(x)$ and denote it by $\log_V(1 + x)$, which we call the planar logarithm series with respect to $V$. We state the

**Proposition**  Let $V \in \mathcal{P}$. Then there is a unique power series $\log_V(1 + x) \in K\{x\}$, such that

(i) $\log_V(\exp_V(x)) = x$ and $\exp_V(\log_V(1 + x)) = 1 + x$.

(ii) $\log_V((1 + x)^V) = k \cdot \log_V(1 + x)$, for $k = |V|$.

From the second property, we get a functional equation for these series and a recursion formula for the coefficients $c_V(T)$ of $\log_V(1 + x)$, such that

$$c_V(x) = 1$$
\[ c_V(T) = \frac{\sum_{S \in O(T)} \left( c_V(S) \cdot \prod_{i=1}^{m} \left( \frac{V}{T_i} \right) \right)}{k - k^n}, \]

where \( n = |T|, m = |S| \) and \( T - S = (T_1, ..., T_m) \).

There is a unique derivation such that \((1 + x) \frac{d}{dx} (x) = 1 + x\). For a power series \( g \in K\{x\} \) with \( g' = 1 + g \) it holds

\[
\left( (1 + x) \frac{d}{dx} \right) (f)(g(x)) = f'(g(x)) + \left( x \frac{d}{dx} \right) (f)(g(x)).
\]

We call this the special planar chain rule.

We find out that for the logarithm we have to use this special planar chain rule to determine a derivative property. Then we get

\[
\left( (1 + x) \frac{d}{dx} \right) (\log V(1 + x)) = 1, \quad \forall \ V \in \mathbb{P}.
\]

There is also a unique power series \( f(x) \), such that

\[
(f(x))^V = 1 + x,
\]

which we denote by \( \sqrt{1+x} \) and call the \( V \)-th root series of \( 1 + x \). It follows that \( \sqrt{1+x} = exp_V(\frac{1}{k} \cdot \log V(1 + x)) \) as in the classical case. For the derivative property we use again the special planar chain rule and get the result that

\[
\left( (1 + x) \frac{d}{dx} \right) (\sqrt{1+x}) = \frac{1}{k} \cdot \sqrt{1+x}.
\]

The following theorem gives us the recursion formula for the coefficients.

**Theorem** Let \( V \in \mathbb{P} \) and \( k = |V| \). Then for the coefficients \( b_V(T) \) of \( \sqrt{1+x} \) it is true that

\[
b_V(x) = \frac{1}{k}
\]

and

\[
b_V(T) = \frac{-\sum_{S} \binom{V}{S} \cdot \prod_{i=1}^{m} b_V(T_i)}{k},
\]

where \( 2 \leq m = |S|, T - S = (T_1, ..., T_m) \) is a planar rooted forest and \( \binom{V}{S} \) the planar binomial coefficient.

If \( S \notin P(V) \cap O(T) \), then the summand is equal to 0.
We state that planar root series are planar algebraic in a similar sense as in the classical theory. There is a planar polynomial $F(x, y) = y^V - (1 + x)$ in $K\{x, y\}$, such that $F(x, \sqrt{1 + x}) = 0$.

Another interesting result in this work is that we can define a generic exponential and logarithm series not only for the coronas but for other admissible sequences of planar rooted trees. Some of the admissible sequences are coronas, the binary left and right combs and the sequence $V = (V_q)_{q \geq 0}$ of the form $V_r = x \cdot x^{r-1}$, $r \in \mathbb{N}$.

In the case that we have such a sequence $V$ we denote the generic exponential series by $\exp_V(q, x)$, which is an element in $K(q)\{x\}$. For a fixed $k \in \mathbb{N} \geq 2$ we get the equation $\exp_V(k, x) = \exp_{V_k}(x)$ and it is possible to compute the limit of this generic series. Analogously for the admissible sequences we get a generic logarithm series with the property $\log_V(k, 1 + x) = \log_{V_k}(1 + x)$ for a fixed $k \in \mathbb{N} \geq 2$.

Furthermore, we find another possibility to present planar rooted trees of height 2 and 3 and to compute the planar binomial coefficient of these trees.

In the past Lazard in [Laz] already regarded generalizations of the classical exponential series $\exp(x)$ and logarithm series $\log(x)$ to non-commutative, non-associative and other types of algebras. Drensky, Gerritzens in [DG] and Gerritzens in [Ger1] also consider power series in one non-commutative, non-associative variable. They have already shown that there is a unique planar exponential series $\exp_V(x)$ if $V$ is a corona and for this case there is also a unique planar logarithm $\log_V(k + x)$, for $k \in \mathbb{N} \geq 2$.

In Chapter 1 we collect definitions and theorems for trees and define the planar binomial coefficient and a new multiplication on the set of planar trees.

In Sections 1.1-1.3 we recall definitions and theorems, together with some examples, for rooted trees, planar rooted trees and forests, open and closed subtrees, reduction and contraction of trees onto leaves sets.

In Sections 1.4 and 1.5 the planar binomial coefficients and the $\lambda$-deformation of the same are introduced. The planar binomial coefficient is the number of contractions of a tree $V$ onto an other tree $T$. Some relations and examples are given.

In Section 1.6 a new $\ast$-multiplication on $\mathbb{P}$ is defined and some properties of $\mathbb{P}$ with this multiplication and inverse trees are given.

In Chapter 2 we define polynomials and power series. At first the classical and then the planar ones.

In Sections 2.1-2.3 some information about classical algebras of polynomials and power series are collected. We briefly recall some definitions and theorems of these algebras and of the derivation in this case.

In Sections 2.4-2.5 we introduce the planar algebras of polynomials and power
series and show the connection between these and the associated classical algebra by using the map $\eta$. Even and odd series are also defined.

Chapter 3 deals with substitution homomorphisms, the special planar chain rule and the $k$-th derivative.

We show in Sections 3.1 and 3.2 some properties and formulas of the planar substitution homomorphisms, see also [Ger4]. These formulas and theorems are used in the following Chapters to compute the inverse series. In Sections 3.3 and 3.4 the special planar chain rule and the $k$-th derivative are given. For the derivations of the planar logarithm series and the planar roots the chain rule will be used. By the $k$-th derivative some new coherence of the planar binomial coefficients are shown.

In Chapter 4 we state some properties of the planar exponential and logarithm series as well as the $\lambda$-deformations of them. Furthermore we define the planar hyperbolic sine and cosine as well as the ordinary planar sine and cosine. In Sections 4.1-4.4 the planar exponential series, planar logarithm series and the $\lambda$-deformations of these are introduced. The recursion formulas for the computation of the coefficients are given and proved. Furthermore the derivation properties, relations and examples of these series are described. The planar exponential series and the planar logarithm series are inverse to each other, but the $\lambda$-deformations are not. The derivation properties remain true in the case of the exponential series, but the planar $\lambda$-deformed logarithm series does not show any visible derivation property. We also show that there are some classes of planar rooted trees which have the same planar exponential and logarithm series, for all members of this class. These classes are of the form $V, V_n \in \mathbb{P}, V_1 = V$ and $V_{n+1} = V \ast V_n$, for $n \in \mathbb{N} \geq 1$.

In Sections 4.5 and 4.6 both the planar hyperbolic sine and cosine and the planar sine and cosine are defined. For these series we use the coefficients of the planar exponential series. Some relations and properties are given. One new property in the planar algebra is that the equation

$$\sin_V^2(x) + \cos_V^2(x) = 1$$

is not true, because there are higher terms, which do not vanish. But most other properties remain true.

Chapter 5 contains formulas and examples of planar root series and the planar algebraic power series are also defined in this chapter. We describe the planar root functions in Sections 5.1 and 5.2. We show properties in connection with the planar exponential and logarithm series and give some examples. The recursion formulas for the coefficients are proved. In Section 5.3 the planar algebraic power series are defined and we see that the planar root series
are algebraic. We analyze some other algebraic series and prove the recursion formulas.

In Chapter 6 the main results are the definitions of admissible sequences of rooted trees and \( q \)-functions which help us define the generic exponential and logarithm. The tupel-presentation of rooted trees is also introduced.

In Section 6.1 the \( q \)-polynomials \([T]\) of a rooted tree \( T \) are given and the admissible sequences of rooted trees are defined. Some examples of admissible sequences are presented. The generic exponential and logarithm are defined in Sections 6.2 and 6.3, moreover limits of some examples of these generic power series are computed. Another presentation for trees is described in Section 6.4. We show, that it is possible to present every tree of height 2 or 3 only by using the coronas and that we can then compute the planar binomial coefficient only by using the classical one. From this we can also use this presentation for the planar exponential and logarithm series and check what happens if we let one corona run towards infinity.

In Chapter 7 we consider further aspects and applications of the planar theory. In Section 7.1 we introduce a new map \( \Delta \), which is a co-additive \( K \)-algebra homomorphism and the completion of it \( \hat{\Delta} \). This maps are also regarded by Holtkamp in [Hol1] and by Schiller in [Sch]. We state that if we apply the map \( \hat{\Delta} \) to the planar exponential series we get the equation

\[
\hat{\Delta}(\exp_V(x)) = \exp_V(x) \otimes \exp_V(x), \forall V \in \mathbb{P}.
\]

Furthermore from this equation we get some identities for the coefficients of the exponential series and we define the planar binomial coefficients \( \binom{R}{S,T} \) of the second kind, which are equal to the coefficients of \( R \) in the shuffle of two trees \( S \) and \( T \).

We define left- and right-inverse series with respect to the grafting in Section 7.2. An interesting result is that \( \exp_V(-x) \) is not the inverse of \( \exp_V(x) \), but both the left- and right-inverse of \( \exp_V(x) \) have the same derivative property like \( \exp_V(-x) \), namely if \( f \in \left\{ \exp_V(-x), \left( \frac{1}{\exp_V(x)} \right)_R, \left( \frac{1}{\exp_V(x)} \right)_L \right\} \) then

\[
\frac{d}{dx}(f) = -f.
\]

For these series we can define a planar quotient rule, which we describe and prove in Section 7.3.

In Section 7.4 we introduce the planar Hermite polynomials. We define new sets that we need and show some properties of these polynomials, which are similar to the classical ones. In this case we have three possible definitions of the Hermite polynomials, because we have two variables \( x \) and \( z \) for which we can choose weather they are commutative and associative or not.
The last Section 7.5 describes planar modules $E_\mu$. The elements of these modules are planar power series $f$, which have the property

$$\frac{d}{dx}(f) = \mu \cdot f.$$

We state that the planar exponential series $exp_V(x)$ and $exp_{(V,\lambda)}(x)$ are elements of $E_1$, resp. $exp_V(\mu \cdot x) \in E_\mu$, but the question if these series are a basis remains open.
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Chapter 1

Trees

1.1 Open and closed rooted trees

It is assumed, that the reader is familiar with the terminology of a graph, see [Har] or [Sta2] as a general reference.

Definition 1.1.1. A rooted tree $T = (T^0, T)$ is a finite non-empty graph together with a distinguished vertex called the root $\iff$

(i) $T$ is connected.

(ii) If $k$ is an edge in $T$, then $T - \{k\} := (T^0, T - \{k\})$ is not connected.

The set of all vertices of $T$ is denoted by $T^0$ and the set of all edges by $T$.

For a rooted tree $T$ we denote the root by $\rho_T$. Connectedness means, that for any two vertices in $T$ there is a unique simple path connecting them. The unique path from $\lambda \in T^0$ to $\rho_T$ is denoted by $[\lambda, \rho_T]$.

Definition 1.1.2. Let $T$ be a rooted tree, then $\text{val}_T(\lambda) = \#\{k \in T : \lambda \in k\}$. $\text{val}_T(\lambda)$ is called the valence of $\lambda$ in $T$.

Definition 1.1.3. Let $T$ be a rooted tree and $\lambda$ a vertex in $T$. Then $\text{ar}_T(\lambda) = \begin{cases} \text{val}_T(\lambda) & : \lambda = \rho_T, \\ \text{val}_T(\lambda) - 1 & : \lambda \neq \rho_T, \end{cases}$ is called the arity of $\lambda$ in $T$.

Moreover $\text{ar}(T) = \text{ar}_T(\rho_T)$.

The vertices with the arity 0 are called the leaves and with $L(T)$ we denote the set of all leaves of $T$. The degree $|T|$ of a tree $T$ is the cardinality of the leave set of $T$. Furthermore the elements of $\text{In}(T) := T^0 - L(T)$ are called the inner vertices of $T$. If $\text{ar}(\lambda) = 2, \forall \lambda \in T^0$, then $T$ is called a binary tree.

For $a, b \in T$ we denote by $\text{dist}_T(a, b)$ the length of the shortest path between $a$ and $b$ in $T$, i.e. the number of edges between $a$ and $b$. $\text{dist}_T$ is a metric on $T^0$ and it determines $T$ completely, because $\text{dist}_T(a, b) = 1$ if and only if $\{a, b\}$ is an edge in $T$. 

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Theorem 1.1.4. For a connected graph $G$, it holds

(i) $\text{dist}_G$ is a distance function on $G^0$, i.e.
   \[
   \text{dist}_G(a, b) \geq 0 \text{ and } \text{dist}_G(a, b) = 0 \iff a = b;
   \]
   \[
   \text{dist}_G(a, b) = \text{dist}_G(b, a);
   \]
   \[
   \text{dist}_G(a, b) \leq \text{dist}_G(a, c) + \text{dist}_G(c, b), \forall a, b, c \in G^0.
   \]

(ii) If $\text{dist}_G(a, b) > 1$, there exists a $c \in G^0$ with $a \neq c \neq b$ such that:
   \[
   \text{dist}_G(a, b) = \text{dist}_G(a, c) + \text{dist}_G(c, b).
   \]

Theorem 1.1.5. Let $T$ be a rooted tree and $S$ a connected subtree of $T$, $S \neq \emptyset$, then it holds

(i) $S$ is a rooted tree.

(ii) $\exists$ a unique $\sigma \in S^0$ such that: $\text{dist}_T(\sigma, \rho_T)$ is minimal.

$\sigma$ is called the canonical root of $S$ with respect to $T$.

Definition 1.1.6. Let $T$ be a rooted tree and $S$ a subtree of $T$ considered as a rooted tree. Then it holds

(i) $S$ is called an open subtree of $T$, if $\rho_T \in S$ and $\text{ar}_S(a) = \text{ar}_T(a), \forall a \in \text{In}(S)$.

(ii) $S$ is called a closed subtree of $T$, if $\text{ar}_S(a) = \text{ar}_T(a), \forall a \in S^0$.

The root of $T$ and the whole $T$ are open subtrees of $T$. All other open subtrees are called proper. By $P(T)$ we denote the set of all subtrees of $T$ and by $O(T)$ the set of all open subtrees of $T$.

The whole $T$ is also a closed subtree of $T$. All other closed subtrees are called proper.

Example 1.1.7. Let $T$ be the following tree:

Then

Then
is the set of all proper open subtrees of \( T \) and

\[
\left\{ \begin{array}{c}
\bullet \rightarrow \rightarrow \rightarrow \rightarrow \downarrow \downarrow \downarrow \downarrow \bullet \\
\downarrow \downarrow \downarrow \downarrow \rightarrow \rightarrow \rightarrow \rightarrow \bullet \bullet \end{array} \right\}
\]

is the set of all proper closed subtrees of \( T \).

### 1.2 Planar reduced rooted trees and forests

**Definition 1.2.1.** Let \( T \) be a finite rooted tree and \( <_{L(T)} \) an order on the set of the leaves of \( T \). Then \( T \) together with this order is called planar, if it holds:

If \( S \) is a closed subtree of \( T \), then \( L(S) \) is an interval of \( L(T) \).

**Example 1.2.2.** The planar structure of a tree \( T \) corresponds to the chosen order on \( L(T) \). The following two trees represent the same tree \( T \), but different planar trees \( T_1 = x^3 \cdot x \) and \( T_2 = x \cdot x^3 \).

\[
\begin{array}{c}
\bullet \rightarrow \rightarrow \rightarrow \rightarrow \downarrow \downarrow \downarrow \downarrow \bullet \\
\downarrow \downarrow \downarrow \downarrow \rightarrow \rightarrow \rightarrow \rightarrow \bullet \bullet \\
\end{array}
\begin{array}{c}
\bullet \rightarrow \rightarrow \rightarrow \rightarrow \downarrow \downarrow \downarrow \downarrow \bullet \\
\downarrow \downarrow \downarrow \downarrow \rightarrow \rightarrow \rightarrow \rightarrow \bullet \bullet \\
\end{array}
\]

**Definition 1.2.3.** An ordered tuple \( F = (T_1, \ldots, T_k) \) of planar rooted trees is called a planar rooted forest (of length \( k \)).

The following definitions and theorems hold both for forests and for trees.

**Theorem 1.2.4.** Let \( T_1, \ldots, T_m \) be planar, finite rooted trees, \( m \geq 1 \). Then there is a unique planar rooted tree \( T \), such that

1. \( T - \{ k \in T : \rho_T \in k \} = \{ \rho_T \} \cup T_1 \cup \ldots \cup T_m \).
2. \( L(T) = L(T_1) \cup \ldots \cup L(T_m) \) and for \( a, b \in L(T_i) \) it holds: \( a <_{L(T_i)} b \iff a <_{L(T)} b \).
3. If \( a \in L(T_i) \) and \( b \in L(T_j) \) and \( i < j \), then \( a <_{L(T)} b \).

\( T \) is called the grafting of \( T_1, \ldots, T_m \) over \( x^m \); it will also be denoted by \( *_{m}(T_1, \ldots, T_m) = T_1 \cdot \ldots \cdot T_m \), if \( m \geq 2 \).

**Definition 1.2.5.** Let \( T \) be a rooted tree. \( T \) is called reduced \iff \( \ar_T(a) \neq 1, \forall a \in T^0 \).
Remark 1.2.6. It is well-known that the number of planar reduced rooted trees with \( n \) leaves is given by the \( n \)-th Super Catalan number (or little Schröder number) \( C_n \), see [Com]. The sequence starts with

\[
\begin{array}{cccccccccc}
  n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  C_n & 1 & 1 & 3 & 11 & 45 & 197 & 903 & 4279 & 20793 & 103049 \\
\end{array}
\]

The generating series \( f(t) = \sum_{n \geq 0} C_n t^n \) is given by

\[
\frac{1}{4}(1 + x - \sqrt{1 - 6x + x^2}).
\]

With \( \mathbb{P} \) we denote the set of all isomorphic classes of finite, planar, reduced, rooted trees. Moreover we define \( \mathbb{P}' := \mathbb{P} \cup \{1\} \), where 1 is the empty tree, which contains no vertices and no edges.

Theorem 1.2.7. Let \( S \in \mathbb{P} \) with \( |S| = m \) and \( V_1, \ldots, V_m \in \mathbb{P} \). Then there is a unique \( T \in \mathbb{P} \), such that

(i) \( S \) is an open subtree of \( T \).

(ii) \( T - \text{In}(S) \) is a planar rooted forest, which is isomorphic to \( V = (V_1, \ldots, V_m) \), where the order \( \prec_{L(V)} \) is defined by \( b \prec_{L(V)} b' \), if \( b \in L(V_i), b' \in L(V_{i+1}) \) for \( 1 \leq i < m \).

\( T \) is denoted by \( \cdot_S(V_1, \ldots, V_m) \) and is called the grafting of \( V_1, \ldots, V_m \) over \( S \).

Theorem 1.2.8. Let \( S, T \in \mathbb{P} \).

(i) If \( S \) is an open subtree of \( T \) and \( S \neq \rho_T \), then \( ar(S) = ar(T) \).

(ii) If \( ar(S) = ar(T), S = S_1 \cdots S_m, T = T_1 \cdots T_m \), then \( S \) is an open subtree of \( T \) if and only if \( S_i \) is an open subtree of \( T_i \) for all \( i \).

Definition 1.2.9. Let \( T \) be a finite, planar, rooted tree and \( S = (S_0, \ldots, S_r) \) a system of open subtrees \( S_i \in O(T) \) with \( S_0 = T, S_r = \rho_T \). Then \( S \) is called an open flag of \( T \) of length \( r \), if

\[
S_{i+1} \in O(S_i), S_{i+1} \neq S_i
\]

for all \( 1 \leq i \leq r - 1 \). The set of open flags of \( T \) of length \( r \) will be denoted by \( \Omega_r(T) \).

Proposition 1.2.10. Let \( T \in \mathbb{P}, r \geq 2 \). Then \( \Omega_r(T) \) corresponds bijectively to \( \Omega'_r(T) := \{(U, S) : U \in O(T), U \neq T, S \in \Omega_{r-1}(U)\} \).

Proof. If \( (U, S) \in \Omega'_r(T) \) and \( S = (S_0, \ldots, S_{r-1}) \), then \( (T, S_0, \ldots, S_{r-1}) \in \Omega_r(T) \) and \( (U, S) \to (U, S_0, \ldots, S_{r-1}) \) is a bijective map \( \Omega'_r(T) \to \Omega_r(T) \). \( \square \)
1.3 Reductions and contractions

Definition 1.3.1. Let $T \in \mathbb{P}$ and $I \subseteq L(T)$ be a non-empty subset of leaves of $T$. Then $I$ determines a rooted subtree

$$T_I := \bigcup_{\lambda \in I} [\lambda, \rho_T]_T$$

of $T$ with the root $\rho_T$. The tree $T_I$ is called the restriction of $T$ on $I \subseteq L(T)$.

The tree $T_I$ is the smallest subtree of $T$, which contains $I$ and $\rho_T$. Then $L(T_I) = I$, because all vertices $\lambda'$ of $[\lambda, \rho_T]_T$, $\lambda' \neq \lambda$ have the arity 1 in $[\lambda, \rho_T]_T$. Furthermore the arity of a vertex of $T_I$ is less or equal than the corresponding vertex in $T$.

Remark 1.3.2. The planar rooted tree $T_I$ is not necessarily reduced. There is a canonical map $Red$, which maps a planar rooted tree $T$ onto a planar reduced rooted tree $Red(T)$ by deleting all vertices of arity 1 together with the edges $K = \{ k \in T : a \in k, ar_T(a) = 1\}$, i.e. the set of vertices of $Red(T)$ is $T^0 - \{ a \in T^0 : ar_T(a) = 1\}$ and the set of edges is $T - \{ k \in T : a \in k, ar_T(a) = 1\}$, (see [Ger4]).

Definition 1.3.3. Let $T \in \mathbb{P}$ and $I \subseteq L(T)$ be a non-empty subset of leaves of $T$. The contraction $T|I$ of $T$ on $I$ is defined to be the reduction of the restriction of $T$ on $I$:

$$T|I = Red(T_I).$$

We set $T|\emptyset = 1$.

Example 1.3.4. Let $T \in \mathbb{P}$ with $|T| = 8$ as in the figure below. Then we get the following $T_I$ and $T|I$ for the subset $I = \{1, 3, 6, 8\} \subseteq L(T)$.

$$T =
\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & 5 & 8 & 1 & 4 \\
\bullet & 2 & 3 & 6 & 7 \\
\end{array}$$

$$T_I =
\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & 8 & 1 & 6 & 3 \\
\end{array}$$

$$T|I = Red(T_I) =
\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & 6 & 8 & 1 & 3 \\
\end{array}$$
Proposition 1.3.5. Let \( m \geq 2 \), \( T^1, ..., T^m \in \mathbb{P} \) and \( T = \cdot_m(T^1, ..., T^m) \). Moreover let \( I \subseteq L(T) = L(T^1) \cup ... \cup L(T^m) \) and \( I_j = I \cap L(T^j) \). Then
\[
T|I = \cdot_m(T^1|I_1, ..., T^m|I_m).
\]

Proof. We have \( T_I = \cdot_m(T^1_I, ..., T^m_I) \) and the result follows immediately. \( \square \)

1.4 Planar binomial coefficients

Definition 1.4.1. For any pair \( V, T \) of \( \mathbb{P} \) denote by
\[
\binom{V}{T}
\]
the number of \( I \subseteq L(V) \), such that the contraction \( V|I \) of \( V \) onto \( I \) is isomorphic with \( T \).
We call \( \binom{V}{T} \) the planar binomial coefficient of \( V \) over \( T \) of the first kind.

Proposition 1.4.2. The following elementary properties are easy to check:

(i) \( \binom{V}{T} = 0 \), if \( |T| > |V| \).

(ii) If \( |T| = |V| \), then \( \binom{V}{T} = 1 \), if \( T = V \) and \( \binom{V}{T} = 0 \), if \( T \neq V \).

(iii) \( \binom{V}{1} = 1 \).

(iv) If \( x \) denotes the tree with a single vertex, then \( \binom{V}{x} = |V| \).

(v) \( \binom{V}{x^2} = \binom{n}{2} \), if \( n = |V| \) and if \( x^2 \) denotes the unique tree of degree 2 in \( \mathbb{P} \).

(vi)
\[
\sum_{T \in \mathbb{P}, |T| = m} \binom{V}{T} = \binom{n}{m}, \quad \text{if } n = |V|.
\]

Let \( V_1, ..., V_m \in \mathbb{P} \) and denote by \( \cdot_m(V_1, ..., V_m) \) or \( V_1 \cdot ... \cdot V_m \) the grafting of \( V_1, ..., V_m \) over \( x^m \). There is a unique tree \( V \in \mathbb{P} \), for which \( V - \rho_V \) is the planar forest which is the disjoint union of the ordered system \( (V_1, ..., V_m) \) where \( V_i < V_j \) for \( i < j \).

Proposition 1.4.3. Let \( m, r \geq 2 \), \( V = V_1 \cdot ... \cdot V_m \) and \( T = T_1 \cdot ... \cdot T_r \in \mathbb{P} \) and \( V_i \neq 1, T_j \neq 1 \), for all \( i, j \). Then we have the following recursion formula
\[
\binom{V}{T} = \sum_{1 \leq i_1 < ... < i_r \leq m} \binom{V_{i_1}}{T_{i_1}} \cdot ... \cdot \binom{V_{i_r}}{T_{i_r}} + \sum_{i=1}^{m} \binom{V_i}{T}.
\]
1.5 Planar $\lambda$-deformed binomial coefficients

Proof. See [Ger2].

Example 1.4.4. Let $V = x^2 \cdot x \cdot (x \cdot x^2 \cdot x) \cdot x$ and $T = x \cdot x^2$. Then

$$\binom{V}{T} = \binom{x^2}{x} \cdot \binom{x \cdot x^2 \cdot x}{x^2} + \binom{x}{x} \cdot \binom{x \cdot x^2 \cdot x}{x^2} + \binom{x \cdot x^2 \cdot x}{x^2}
= 3 \cdot \left( \binom{x}{x} \cdot \binom{x^2}{x} + \binom{x}{x}^2 + \binom{x^2}{x} \cdot \binom{x}{x} + \binom{x^2}{x^2} \right) + \binom{x}{x} \cdot \binom{x^2}{x^2}
= 3 \cdot (2 + 1 + 2 + 1) + 1 = 19.$$  

Proposition 1.4.5. Let $V \in \mathbb{P}$. We denote the $V$-th power of $1 + x$ by $(1 + x)^V$. Then we obtain the planar binomial formula

$$(1 + x)^V = \sum_{T \in \mathbb{P}'} \binom{V}{T} x^T.$$  

Proof. Let $|V| = n$. We expand $\cdot_V(1 + x, ..., 1 + x)$ distributively to show that $(1 + x)^V$ is given by $2^n$ summands. These summands are $\cdot_V(R_1, ..., R_n)$ with $R_i = x$, if $i \in I$ and $R_i = 1$ else, where $I \subseteq L(V)$. We set $x^0 = 1$, because $\cdot_V(1, ..., 1) = 1$.

Proposition 1.4.6. Let $f \in K\{\{x\}\}$ be a planar power series (see Section 2.5) with the constant term 1 and $V \in \mathbb{P}$. We denote by $f^V$ the $V$-th power of $f$. Then we obtain

$$f^V = \sum_{T \in \mathbb{P}'} \left( \sum_{S \in O(T)} \binom{V}{S} \cdot \prod_{i=1}^m a(T_i) \right) \cdot x^T,$$  

where $a(T)$ is the coefficient of $T$ in $f$, $m = |S|$ and $T - S = (T_1, ..., T_m)$.

Proof. By Proposition 1.4.5 and Proposition 3.1.3 (see Section 3.1).

1.5 Planar $\lambda$-deformed binomial coefficients

Definition 1.5.1. For any $V, T \in \mathbb{P}$ with $|V| = n$ and $\lambda = (\lambda_1, ..., \lambda_n)$ we define

$$\binom{V}{T}_\lambda := \sum_{1 \leq k_1 < \ldots < k_m \leq n} \lambda_{k_1} \cdot \ldots \cdot \lambda_{k_m},$$  

where $|T| = m$, $I_T \subseteq L(V)$, such that the contraction $V|_{I_T}$ is isomorphic with $T$ and $(k_1, ..., k_m) \in I_T, k_i \in \mathbb{N}$.

We call $\binom{V}{T}_\lambda$ the planar $\lambda$-deformed binomial coefficient of the first kind.
Proposition 1.5.2. Let \( V \in P \) with \(|V| = n\) and \( \lambda = (\lambda_1, ..., \lambda_n) \). Then the following elementary properties are easy to check.

(i) \((V_T)_\lambda = 0\), if \(|T| > |V|\).

(ii) If \(|T| = |V|\), then \((V_T)_\lambda = \prod_{i=1}^{n} \lambda_i\), if \(T = V\) and \((V_T)_\lambda = 0\), if \(T \neq V\).

(iii) \((V_1)_\lambda = 1\).

(iv) \((V_x)_\lambda = \sum_{i=1}^{n} \lambda_i\).

(v) \((V_x^2)_\lambda = \sum_{1 \leq i < j \leq n} \lambda_i \cdot \lambda_j\).

Proposition 1.5.3. Let \( V, T \in P \) with \(|V| = n\), \( \lambda = (\lambda_1, ..., \lambda_n) \), \( V = V_1 \cdot ... \cdot V_m \) and \( T = T_1 \cdot ... \cdot T_r \) with \( r, m \geq 2\), \(|V_i| = n_i\) and \( V_i \neq 1, T_j \neq 1\), for all \( i, j \). Then we have the following recursion formula

\[
(V_T)_\lambda = \sum_{j=1}^{m} (V_j)_\lambda + \sum_{1 \leq j_1 < ... < j_r \leq m} (V_{j_1})_{\lambda_{j_1}} \cdot ... \cdot (V_{j_r})_{\lambda_{j_r}},
\]

where \( \lambda_J = (\lambda_{n_1+...+n_{j-1}+1}, ..., \lambda_{n_1+...+n_j}) \).

Proof. By Proposition 1.4.3 and substituting \( \lambda_i \) for every leaf \( k_i \in L(V)\), \( i = 1, ..., n \).

Example 1.5.4. Let \( V = x \cdot x^2 \cdot x^3 \) with \( \lambda = (2, 3, 1, 4, 2, 1) \) and \( T = x \cdot x^2 \). Then

\[
(V_T)_\lambda = 2 \cdot 3 \cdot 1 + 2 \cdot 4 \cdot 2 + 2 \cdot 4 \cdot 1 + 2 \cdot 2 \cdot 1 + 3 \cdot 4 \cdot 2 + 3 \cdot 4 \cdot 1 + 3 \cdot 2 \cdot 1 + 1 \cdot 4 \cdot 2 + 1 \cdot 4 \cdot 1 + 1 \cdot 2 \cdot 1
\]

\[
= 6 + 16 + 8 + 24 + 12 + 6 + 8 + 4 + 2 = 90.
\]

Proposition 1.5.5. Let \( V \in P\), \(|V| = n\), \( \lambda = (\lambda_1, ..., \lambda_n) \). Then we get

\[
\cdot V(1 + \lambda_1 x, ..., 1 + \lambda_n x) = \sum_{T \in P} (V_T)_\lambda x^T.
\]

Proof. Analogous to Proposition 1.4.5, by using Definition 1.5.1 and substituting \( \lambda_i \) for every leaf.
1.6 The composition of rooted trees

**Definition 1.5.6.** For any \( V, T \in \mathbb{P} \) with \( |V| = n \) and \( \lambda = (\lambda_1, ..., \lambda_n) \) we define

\[
\binom{V}{T}^\lambda := \sum_{S \in O(T)} \sum_{I_S \in \mathcal{L}(V)} \lambda_{k_1}^{r_1} \cdots \lambda_{k_m}^{r_m}
\]

where \( I_S \in \mathcal{L}(V) \), such that the contraction \( V | I_S \) is isomorphic with \( S \), \((k_1, ..., k_m) \in \mathbb{N}, |T| = r, |S| = m, |T_i| = r_i, \forall 1 \leq i \leq m, \) if \( T - S = (T_1, ..., T_m) \).

Further we define

\[
\binom{V}{T}^\lambda_s := \sum_{I_S} \lambda_{k_1}^{r_1} \cdots \lambda_{k_m}^{r_m}.
\]

Thus

\[
\binom{V}{T}^\lambda = \sum_{S \in O(T)} \binom{V}{T}^\lambda_s.
\]

We call \( \binom{V}{T}^\lambda \) the planar \( \lambda \)-deformed binomial coefficient of the second kind.

**Proposition 1.5.7.** Let \( f \in K\{\{x\}\} \) be a planar power series (see Section 2.5) with the constant term 1, \( V \in \mathbb{P} \) with \( n = |V| \) and \( \lambda = (\lambda_1, ..., \lambda_n) \). Then we obtain

\[
\mathcal{U}(f(\lambda_1 x), ..., f(\lambda_n x)) = \sum_{T \in \mathbb{P}'} \left( \sum_{S \in O(T)} \binom{V}{T}^\lambda_s \prod_{i=1}^{m} a(T_i) \right) \cdot x^T,
\]

where \( a(T) \) is the coefficient of \( T \) in \( f \), \( m = |S|, T - S = (T_1, ..., T_m) \).

**Proof.** By Proposition 1.5.5 and Proposition 3.1.3 (see Section 3.1). \qed

1.6 The composition of rooted trees

**Definition 1.6.1.** For any \( U, V \in \mathbb{P} \) we define the \(*\)-multiplication in the following way

\[ U * V = U^V, \]

which means that we replace every leaf \( k \in \mathcal{L}(V) \) with the tree \( U \).

The set of all planar reduced rooted trees \( \mathbb{P} \) together with the \(*\)-multiplication is a monoid. The unit is the tree \( x \) with only one vertex because it holds \( \forall U \in \mathbb{P} \)

\[ U * x = U^x = U \quad \text{and} \quad x * U = x^U = U. \]

The \(*\)-multiplication is associative, but non-commutative, because for \( U, V, W \in \mathbb{P} \) it holds

\[ (U * V) * W = U^V * W = (U^V)^W = U^{(V^W)} = U * (V * W) \]
but
\[ U \ast V = U^V \neq V^U = V \ast U, \text{ if } U \neq V \in \mathbb{P}. \]

Furthermore it is easy to check that \(|U \ast V| = |U| \cdot |V|\) and \((U \ast V) - V = (U, ..., U)\) is a planar rooted forest of length \(|V|\).

**Definition 1.6.2.** A rooted tree \(V \in \mathbb{P}\) is called irreducible if and only if \(V = U \ast W\), then \(|U| = 1\) or \(|W| = 1\).

**Proposition 1.6.3.** Let \(I\) be the set of all irreducible trees in \(\mathbb{P}\). Then \(\mathbb{P}\) is freely generated by \(I\) with respect to \(\ast\).

**Proof.** Let \(U \in \mathbb{P}, V_1, ..., V_r, W_1, ..., W_s \in I\) with \(U = V_1 \ast \ldots \ast V_r = W_1 \ast \ldots \ast W_s\), i.e. \(U\) is reducible. We have to show that \(r = s\) and \(V_i = W_i, \forall i\). Choose \(S \in O(U)\) with a maximal degree \(|S| = m\) such that \(U - S = (U_1, ..., U_m)\) is a forest containing only irreducible trees with \(|U_i| > 1\). Then \(U_i\) are closed subtrees in \(U\) and \(U_i = U_j, \forall 1 \leq i, j \leq m\). Thus \(V_1 = W_1 = U_1\). Now we get \(U = U_1 \ast V_2 \ast \ldots \ast V_r = U_1 \ast W_2 \ast \ldots W_s = U_1 \ast S\). If \(S \in I\) then \(r = s = 2\) and \(V_2 = W_2 = S\), otherwise choose a subtree \(R \in O(S)\) with maximal degree \(n\) such that \(S - R = (S_1, ..., S_n)\) is a forest containing only irreducible trees \(\neq x\). Then \(S_i = S_j, \forall 1 \leq i, j \leq n\). Go on inductively until finding all irreducible components of \(U\).

The free group \(F(I)\) is the set of all reduced words in one variable \(x\) over \(I\). It holds that \(I \subset \mathbb{P}\) and \(\mathbb{P}\) is a sub-monoid of \(F(I)\) freely generated by \(I\), see [LS].

**Notation 1.6.4.** For any \(U \in \mathbb{P}\) we call the inverse of \(U\) in \(F(I)\) the inverse tree and denote it by \(U^{-1}\).

We denote the set of all inverse trees with respect to \(\ast\) by \(\mathbb{P}^{-1}\).

**Proposition 1.6.5.** Let \(U, V, W \in \mathbb{P}, V^{-1} \in \mathbb{P}\). Then \(U \ast V^{-1} = W \iff U = W^V = W \ast V\) or analogously \(V^{-1} \ast U = W \iff U = V^W = V \ast W\).

**Proof.** Let \(U = W^V = W \ast V\). Then
\[ U \ast V^{-1} = (W \ast V) \ast V^{-1} = W \ast (V \ast V^{-1}) = W. \]
Chapter 2

Polynomials and power series

2.1 Classical algebra of polynomials \( K[x] \)

With \( K[\mathbb{N},+] = K[x] \) we denote the algebra of polynomials in one variable \( x \), where \( K \) is a field. This algebra has a \( K \)-vector space-basis \( E = \{ e_i : i \in \mathbb{N} \} \). We set \( x = e_1; x^i = e_i, \forall i \in \mathbb{N} \); and \( x^0 = e_0 \) is the unit of \( K[x] \).

Let \( c_i : K[x] \to K \) be the \( K \)-linear map, such that \( c_i(x^j) = \begin{cases} 1 : i = j, \\ 0 : i \neq j. \end{cases} \)

We call \( c_i \) the \( i \)-th coefficient.

Definition 2.1.1.
\[
\text{deg}(f) := \begin{cases} -\infty : f = 0, \\ \max \{ i \in \mathbb{N} : c_i(f) \neq 0 \} \cup \{ -\infty \} : \text{else}, \end{cases}
\]

is called the degree of \( f \) with respect to \( x \), for \( f \in K[x] \).

Definition 2.1.2. Let \( f \in K[x] \), \( \text{deg}(f) \geq 1 \). \( f \) is called irreducible in \( K[x] \iff \) if \( f = g \cdot h \) with \( g, h \in K[x] \), then \( \text{deg}(g) = 0 \) or \( \text{deg}(h) = 0 \).

Definition 2.1.3. \( f \) is called prime polynomial in \( K[x] \iff \) the ideal \( f \cdot K[x] \) in \( K[x] \) is a prime ideal in \( K[x] \).

Theorem 2.1.4.  
(i) If \( f, g \in K[x] \), then \( \text{deg}(f \cdot g) = \text{deg}(f) + \text{deg}(g) \).

(ii) The group \( (K[x])^* \) of the units of \( K[x] \) is \( \{ \lambda \cdot x^0 : \lambda \in K - \{ 0 \} \} \).

(iii) If \( f \in K[x], \text{deg}(f) \geq 1 \), then \( f \) is irreducible in \( K[x] \iff f \) is a prime polynomial in \( K[x] \).

Lemma 2.1.5. Let \( K \) be a field.

(i) For \( f \in K[x], g \neq 0 \) there exists a \( q \in K[x] \) such that \( \text{deg}(f - q \cdot g) < \text{deg}(g) \).

(ii) If \( I \) is an ideal in \( K[x] \), then there is a \( g \in K[x] \) such that: \( I = g \cdot K[x] \).
We say $K[x]$ is a principal ideal ring.

**Theorem 2.1.6.** $g \in K[x], \deg(g) = m \geq 1$. Let $A := K[x]/(g \cdot K[x])$ be a residue class algebra. Then

(i) $\dim_K(A) = m$.

(ii) If $g$ is irreducible, then $A$ is a field.

**Theorem 2.1.7.** Let $K$ be a field. We denote by $K^N$ the $K$-vector space of the $K$-significant sequence. Let $f, g \in K^N$. Then

(i) $f \cdot g \in K^N$.

(ii) $\cdot$ is a $K$-bilinear operation on $K^N$.

**Definition 2.1.8.** Let $f \in K^N$. $\ord(f) := \begin{cases} \infty & : f = 0, \\ \min\{i \in \mathbb{N} : f(i) \neq 0\} & : \text{else}, \end{cases}$ is called the order of $f$.

**Definition 2.1.9.** Let $f \in K^N$. $\supp(f) = \{k \in \mathbb{N} : f(k) \neq 0\}$ is called the support of $f$.

### 2.2 Classical algebra of power series $K[[x]]$

**Theorem 2.2.1.** (i) $(K^N, \cdot)$ is a commutative, associative $K$-algebra with a unit. It will be denoted by $K[[x]]$ if $x$ is a symbol for $e_1$, where $e_1(i) = \begin{cases} 1 & : i = 1, \\ 0 & : \text{else}. \end{cases}$ It is called the $K$-algebra of the formal power series in one variable $x$ over $K$.

(ii) $K^N := \{f \in K^N : \supp(f) \text{ is finite}\}$ is a $K$-subalgebra of $K[[x]]$. $K^N$ is isomorphic to $K[x]$.

(iii) $\ord(f \cdot g) = \ord(f) + \ord(g)$ for $f, g \in K[[x]]$.

**Notation 2.2.2.** Let $f \in K^N$. We write $f = \sum_{i=0}^{\infty} c_i(f)x^i$, where $c_i(f)$ is the $i$-th coefficient of $f$.

**Corollary 2.2.3.** Let $f \in K[[x]]$. Then $f$ is a unit in $K[[x]]$ $\iff$ $c_0(f) \neq 0$.

**Definition 2.2.4.** Let $f \in K[[x]]$, $K$ be a field. $|f|_K = |f| := \begin{cases} 0 & : f = 0, \\ (1/2)^{\ord(f)} & : \text{else}, \end{cases}$ is called the $x$-adic distance of $f$. It holds that $|f| \in \mathbb{Q}, 0 \leq |f| \leq 1$.

**Theorem 2.2.5.** Let $f, g \in K[[x]]$. Then
2.3 Derivations

(i) \(|f| \geq 0 \) and \(|f| = 0\), if \(f = 0\).

(ii) \(|f \cdot g| = |f| \cdot |g|\).

(iii) \(|f + g| \leq \max(|f|, |g|) \leq |f| + |g|\).

**Definition 2.2.6.** Let \((f_n)_{n \geq 0}\) be a sequence in \(K[[x]]\) and \(f \in K[[x]]\). \(\lim_{n \to \infty} f_n = f\), if \(\lim_{n \to \infty} |f_n - f| = 0\). We say \((f_n)_{n \geq 0}\) converges to \(f\).

**Theorem 2.2.7.** Let \((f_n)_{n \geq 0}\) be a sequence in \(K[[x]]\) such that \((f_n)_{n \geq 0}\) is a Cauchy sequence relative to the \(x\)-adic distance \(\implies\) there is a unique \(f \in K[[x]]\) such that \(\lim_{n \to \infty} f_n = f\). (We say \(K[[x]]\) is complete with regard to the \(x\)-adic distance.)

**Definition 2.2.8.** \((f_n)_{n \geq 0}\) is a Cauchy sequence relative to \(|\cdot|_x \iff\) there is a map \(k : \mathbb{N} \to \mathbb{N}\) such that: if \(i, j \geq k(n)\), then \(|f_i - f_j| \leq \frac{1}{n}\).

Let \(\mathbb{N}^n\) be the additive semigroup of the \(n\)-tuples of natural numbers. Then \(K[x_1, ..., x_n] := K[\mathbb{N}^n]\) is called the \(K\)-algebra of the polynomials in \(n\) independent variables \(x_1, ..., x_n\) over \(K\).

**Definition 2.2.9.** \(k \in \mathbb{N}^n, \quad k := (k_1, ..., k_n) \in \mathbb{N}^n; \quad |k| := \sum_{i=1}^{n} k_i \in \mathbb{N}. \quad \deg(f) := \max\{|k| : k \in \mathbb{N}^n, c_k(f) \neq 0\}, \quad \text{if } f \neq 0 \quad \text{and } \deg(0) := -\infty.\)

\(K[x_1, ..., x_n] := K[[\mathbb{N}^n]] := (K^{\mathbb{N}^n}, \cdot)\) is a commutative, associative \(K\)-algebra with unit; it is called the algebra of power series of \(\mathbb{N}^n\) over \(K\). The order and the \(x\)-adic distance are defined as in the 1-dimensional case.

**Definition 2.2.10.** \(f\) is called homogeneous of degree \(d \iff\) if \(k \in \mathbb{N}^n\) and \(c_k(f) \neq 0\), then \(|k| = d = k_1 + ... + k_n, k_i \in \mathbb{N}. \) It holds \(V_d := \{f \in K[x_1, ..., x_n] : f \text{ is homogeneous of degree } d\}\) is a \(K\)-sub vector space of \(K[x_1, ..., x_n]\) and \(\{x^k : k \in \mathbb{N}^n, |k| = d\}\) is a \(K\)-basis of \(V_d\).

**Lemma 2.2.11.** (i) If \(f \in V_d, f' \in V_{d'}, then f \cdot f' \in V_{d+d'}\).

(ii) If additionally \(f \neq 0 \neq f'\), then \(f \cdot f' \neq 0.\)

**Theorem 2.2.12.** Let \(X\) be a set, \(K\) a field. Let \(K[[x]]^*\) be the group of the units of \(K[[x]] \implies K[[x]]^* = \{f \in K[[x]] : \text{ord}(f) = 0\}\).

### 2.3 Derivations

**Definition 2.3.1.** Let \(A\) be a \(K\)-algebra, \(\theta : A \to A\) a \(K\)-linear map. \(\theta\) is called a derivation of \(A \iff \theta(f \cdot g) = f \cdot \theta(g) + \theta(f) \cdot g, \forall f, g \in A.\)
Theorem 2.3.2. Let $g_1, \ldots, g_n \in K[x_1, \ldots, x_n]$. Then there is a unique derivation $	heta$ on $K[x_1, \ldots, x_n]$ such that $\theta(x_i) = g_i$, $\forall i$. We write symbolically $\theta = \sum_{i=1}^{n} g_i \cdot \frac{\partial}{\partial x_i}$.

Particularly $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij} = \{ 1 : i = j, 0 : i \neq j \}$, it is called the partial derivation with respect to $x_i$.

Proof. Let $A = K[x_1, \ldots, x_n], g_1, \ldots, g_n \in A$.

1) $\exists$ a unique $K$-linear map $\theta : A \to A$ such that

$$\theta(w) = \sum_{i=1}^{n} w(1) \cdots w(i-1) \cdot g_{w(i)} \cdot w(i+1) \cdots w(n),$$

where $n = L(w), g_{w(i)} = g_k$, if $w(i) = x_k$. It holds $\theta(1) := 0$, because $W = W(\{x_1, \ldots, x_n\})$ is a $K$-vector space-basis of $A$.

2) $\theta$ in 1) is a derivation.

1. Prop.: Let $v, w \in W$. $\theta(v \cdot w) = v \cdot \theta(w) + \theta(v) \cdot w$. This holds, because

Induction over $L(v) = n$.

$n = 0$: (✓)

$n = 1$: because of the definition of $\theta$ (✓)

$n > 1$: Let $v = x_k \cdot v'$. Then

$$\theta(v \cdot w) = \theta(x_k \cdot v' \cdot w) = \theta(x_k) \cdot v' \cdot w + x_k \cdot \theta(v') \cdot w + x_k \cdot v' \cdot \theta(w)$$

$$= \theta(x_k v') \cdot w + x_k v' \cdot \theta(w) = \theta(v) \cdot w + v \cdot \theta(w).$$

2. Prop.: Let $f, g \in A$. $\theta(f \cdot g) = f \cdot \theta(g) + \theta(f) \cdot g$. This holds, because

$$\theta(f \cdot g) = \sum_{v \in supp(f), w \in supp(g)} c_v(f) c_w(g) \theta(v \cdot w)$$

$$= \sum_{v \in supp(f), w \in supp(g)} c_v(f) c_w(g) (\theta(v) \cdot w + v \cdot \theta(w))$$

$$= \sum_{v \in supp(f), w \in supp(g)} c_v(f) \theta(v) c_w(g) w + \sum_{v \in supp(f), w \in supp(g)} c_v(f) v c_w(g) \theta(w)$$

$$= \theta(f) \cdot g + f \cdot \theta(g).$$

$\square$

Proposition 2.3.3. (Chain rule) Let $P_n := K[[x_1, \ldots, x_n]]$ be $K$-algebras, for all $n \in \mathbb{N}, \varphi : P_n \to P_m$ a unitary $K$-algebra-homomorphism, $\theta \in \text{Der}(P_m)$ and $p_i = \theta(\varphi(x_i)) \implies$

$$\theta \circ \varphi = \sum_{i=1}^{n} p_i \cdot \left( \varphi \circ \frac{\partial}{\partial x_i} \right).$$
2.4 Planar algebra of polynomials $K\{x\}$

**Definition 2.4.1.** $A = (V, \mu)$ is called a $K$-algebra-system $\iff$

(i) $V$ is a $K$-vector space.

(ii) $\mu = (\mu_m)_{m \geq 2}$.

(iii) $\mu_m$ is a $K$-multi linear map $V^m \to V$.

**Corollary 2.4.2.** $K\{x\} := (K[\mathbb{P}], (\cdot)_m)_{m \geq 2}$ is a $K$-algebra-system. It is called the $K$-algebra of the planar (= non-associative, non-commutative) polynomials in $x$ over $K$.

**Remark 2.4.3.** We consider $K\{x\}$ not as an algebra in the usual sense, but as an algebra over a free operad $\mathcal{P}$ generated by the system $(\mu_m)_{m \geq 2}$. For more detailed description of the operad theory and an algebra over an operad see [MSS], [Hol1] and [Hol2].

**Definition 2.4.4.** Let $A$ be a $K$-algebra-system, $1_A \in A$. $1_A$ is called the unit of $A$ $\iff$ $\mu_2(a_1, a_2) = \begin{cases} a_1 : a_2 = 1_A, \\ a_2 : a_1 = 1_A \end{cases}$ and for $m \geq 3$:

$\mu_m(a_1, \ldots, a_m) = \mu_{m-1}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m)$, if $a_i = 1_A$.

**Theorem 2.4.5.** (Degree formula) Let $f_1, \ldots, f_m \in K\{x\}$, $f = f_1 \cdot \ldots \cdot f_m$. Then

$$\deg(f) = \sum_{i=1}^m \deg(f_i).$$

**Definition 2.4.6.** Let $A$ be a $K$-algebra-system, $\theta : A \to A$ a $K$-linear map. $\theta$ is called a derivation $\iff$ $\forall f_i \in A, 1 \leq i \leq m$ it holds that

$\theta(f_1 \cdot \ldots \cdot f_m) = \sum_{i=1}^m f_1 \cdot \ldots \cdot f_{i-1} \cdot \theta(f_i) \cdot f_{i+1} \cdot \ldots \cdot f_m$.

**Proposition 2.4.7.** Let $g \in K\{x\}$ $\implies$ $\exists$ a unique derivation $\theta_g$ of $K\{x\}$ such that $\theta_g(x) = g$. We write symbolically $\theta_g = g \cdot \frac{d}{dx}$.

2.5 Planar algebra of power series $K\{\{x\}\}$

**Theorem 2.5.1.** Let $K$ be a field, $K^\mathbb{P} := K$-vector space of all maps $f : \mathbb{P} \to K$. We denote the value of $f$ at the point $T \in \mathbb{P}$ by $c_T(f)$ and call it the coefficient of $f$ with respect to $T$. Let $f_1, \ldots, f_m \in K^\mathbb{P}$. We set $f := f_1 \cdot \ldots \cdot f_m = \cdot_m(f_1, \ldots, f_m)$, where

$$c_T(f) := \sum_{\langle S_1, \ldots, S_m \rangle \in \mathbb{P}^m}^{T = S_1 \ldots S_m} c_{S_1}(f_1) \cdot \ldots \cdot c_{S_m}(f_m).$$

Then
(i) $\cdot_m : (K^P)^m \to K^P$ is $K$-multi linear, $\forall m$.

(ii) $K\{\{x\}\} := (K^P, (\cdot_m)_{m \geq 2})$ is a $K$-algebra-system with unit. It is called the $K$-algebra-system of the planar power series in $x$ over $K$.

Remark 2.5.2. Here we also regard $K\{\{x\}\}$ as an algebra over a free operad $P$.

Proposition 2.5.3. Let $f \in K\{x\}$. The order of $f$ in $K\{x\}$ is defined by

$$\text{ord}(f) := \begin{cases} \min \{|T| : T \in P, c_T(f) \neq 0\} & : f \neq 0 \\ \infty & : f = 0. \end{cases}$$

Define a distance function on $K\{x\}$ by

$$|f, g| = \begin{cases} (\frac{1}{2})^{\text{ord}(f-g)} & : f \neq g \\ 0 & : f = g. \end{cases}$$

for $f, g \in K\{x\}$.

Remark 2.5.4. $||$ is an ultrametric distance. $A = K\{x\}$ has a natural topology called the $x$-adic topology. It corresponds to the metric $||$ induced by the order function. Clearly, the operations of $A$ are continuous with respect to this topology. This topology has the following property with respect to infinite sums: if $(f_i)_{i \in I}$ is a family of series such that for each neighborhood of 0, all but a finite number of these series are in this neighborhood, then the family $(f_i)_{i \in I}$ is summable, and its sum $f$ is defined for any tree $T$ by

$$\text{coeff}_T(f) = \sum_{i \in I} \text{coeff}_T(f_i).$$

I.e. given a sequence $(f_n)_{n \geq 0}$ in $A$ and $f \in A$. Then convergence

$$\lim_{n \to \infty} f_n = f$$

means that

$$\lim_{n \to \infty} \text{ord}(f_n - f) = \infty$$

and by standard arguments this holds if and only if $(f_n)_{n \geq 0}$ is a Cauchy-sequence.

Partial sums $p_n = \sum_{i=0}^{n} f_i$ are convergent in $A$ if and only if $\lim_{n \to \infty} \text{ord}(f_n) = \infty$. We denote by

$$\hat{A} = K\{\{x\}\}$$

the $x$-adic completion of $A = K\{x\}$. Equivalently, this is the smallest topology on $K\{\{x\}\}$ such that each mapping $f \mapsto \text{coeff}_T(f)$, $K\{\{x\}\} \to K$ is continuous ($K$ with the discrete topology), see [Sch].
Theorem 2.5.5. Let \( f_1, \ldots, f_m \in K\{\{x\}\} \). Then

(i) \( \text{ord}(f_1 \cdots f_m) = \sum_{i=1}^{m} \text{ord}(f_i) \).

(ii) \( |f_1 \cdots f_m| = \prod_{i=1}^{m} |f_i| \).

It holds: Let \( \lambda \in K, \lambda \neq 0 \). The map \( \varphi_\lambda : K\{\{x\}\} \rightarrow K\{\{x\}\} \) defined by \( f \mapsto \sum_{T \in P} c_T(f) \cdot \lambda^{|T|} \cdot T =: f(\lambda \cdot x) \) is a \( K \)-algebra-automorphism and \( \varphi_\lambda \circ \varphi_\mu = \varphi_{\lambda \mu} \).

Definition 2.5.6. Let now \( K \) be a field with \( \text{char}(K) \neq 2 \) and \( P = K\{\{x\}\} \). \( f \in P \) is called even (resp. odd), if \( f(-x) = f(x) \) (resp. \( f(-x) = -f(x) \)).

By \( P_0 \) we denote the set of all even power series in \( P \), and it is a subalgebra of \( P \). By \( P_1 \) we denote the set of all odd power series and it is a right- and left-module over \( P_0 \). It holds for \( f, g \in P_1 : f \cdot g \in P_0 \), because \( (f \cdot g)(-x) = f(-x) \cdot g(-x) \). Further any \( f \in P \) has a unique decomposition \( f = f_0 + f_1 \), where \( f_0 \in P_0, f_1 \in P_1 \). Actually it holds \( f_0(x) = \frac{1}{2}(f(x) + f(-x)) \) and \( f_1 = \frac{1}{2}(f(x) - f(-x)) \). \( f_0 \) (resp. \( f_1 \)) is called the even (resp. odd) part of \( f \). For example it is true that the even part of \( f^2 \) is \( f_0^2 + f_1^2 \), while the odd part is \( f_1 f_0 + f_0 f_1 \); and the even part of \( f(x)f(-x) \) is \( f_0^2 - f_1^2 \), while the odd part is \( f_1 f_0 - f_0 f_1 \), see [Ger7].

Proposition 2.5.7. There is a canonical, continuous \( K \)-algebra-homomorphism

\[
\eta : K\{\{x\}\} \rightarrow K[[x]],
\]

which maps a tree \( T \in P \) onto \( x^n \), where \( n = |T| \).

For any \( f \in K\{\{x\}\} \) we call \( \eta(f) \) the classical series of \( f \).

Proposition 2.5.8. There is a unique, continuous \( K \)-linear map \( \frac{d}{dx} : K\{\{x\}\} \rightarrow K\{\{x\}\} \), such that

\[
\frac{d}{dx}(x) = 1, \quad \frac{d}{dx}(f \cdot g) = \frac{d}{dx}(f) \cdot g + f \cdot \frac{d}{dx}(g)
\]

for any \( f, g \in K\{\{x\}\} \). This map is called the derivative of \( f \) with respect to \( x \).

There is also such a map \( \frac{d}{dx} : K[[x]] \rightarrow K[[x]] \) with the same properties.

This map commutes with the map \( \eta \) in the Proposition 2.5.7.
Chapter 3

Substitution homomorphisms

Most propositions and proofs in this Chapter, except section 3.4, can also be found in [Ger3] and [Ger5].

3.1 Substitution endomorphisms

Notation 3.1.1. Let $K\{x\} := A$. By $\text{Hom}(A, A')$ we denote the monoid of all $K$-algebra homomorphisms. $\text{End}(A) := \text{Hom}(A, A)$ together with $\circ = "\text{composition of maps}"$ as the operation is a $K$-algebra, and it is called the monoid of the unitary $K$-algebra endomorphisms of $A$. The composition $(f \circ g)(x)$ of two maps $f(x), g(x) \in \text{End}(A)$ means that we substitute the map $g(x)$ for $x$ in $f$.

Proposition 3.1.2. Let $g \in A$. Then there is a unique continuous unitary $K$-algebra homomorphism $\varphi_g : A \to A$ with $\varphi_g(x) = g$. It is called the substitution endomorphism induced by $g$. With $f \in A$ we use $f(g(x))$ or $f(g)$ for the power series $\varphi_g(f)$. This is obtained by substituting $g$ for $x$ in $f$.

Proof. 1) Let $(h_n)_{n \geq 0}$ be a sequence in $A$ and $h \in A$. Then $\lim_{n \to \infty} h_n = h$ is defined to mean that

$$\lim_{n \to \infty} \text{ord}(h_n - h) = \infty.$$

Now it is easy to check that a sequence $(h_n)_{n \geq 0}$ in $A$ has a unique limit $h \in A$ if and only if

$$\lim_{n \to \infty} \text{ord}(h_n - h_m) \to \infty, \text{ for } n, m \to \infty.$$

It follows that the partial sums

$$p_n = \sum_{i=0}^{n} h_i$$

of a sequence $(h_n)_{n \geq 0}$ in $A$ are convergent in $A$ if and only if

$$\lim_{n \to \infty} \text{ord}(h_n) = \infty.$$
2) Let $V \in \mathbb{P}'$. 
Define $\varphi_g(V) = 1$, if $V = 1$ and $\varphi_g(V) = g$, if $V = x$.
If $ar(V) = m \geq 2, V = V_1 \cdot \ldots \cdot V_m$, then $\varphi_g(V) := \varphi_g(V_1) \cdot \ldots \cdot \varphi_g(V_m)$.
By induction on $|V|$ this defines $\varphi_g(V)$ for all $V \in \mathbb{P}'$.

3) If $f \in A$, then
$$h_n := \sum_{|V|=n} c_V(f) \cdot \varphi_g(V)$$
is a power series of order $\geq n$, as $ord(\varphi_g(V)) \geq n$, if $|V| \geq n$.
Thus
$$\sum_{n=0}^{\infty} h_n$$
is a series in $A$, which is defined to be $\varphi_g(f)$.

4) It is easy to check that $\varphi_g$ is a $K$-algebra homomorphism. \(\square\)

It also holds that $\text{End}(A) = \{\varphi_g : g \in A, ord(g) \geq 1\}$ by the above proposition.
Also $\varphi_x = id$ and for $g, h \in K[x]$ is $\varphi_g \circ \varphi_h = \varphi_{h(g)}$, because
$$(\varphi_g \circ \varphi_h)(x) = \varphi_g(h) = h(g) = \varphi_{h(g)}.$$
3.1 Substitution endomorphisms

To $x$; it consists only of the root of $T$. Further $T - V = T$, and we have to show that $c_T(\varphi_g(x)) = c_T(g)$ is equal to $b(T)$ which is trivial by definition.

3) Let now $|V| > 1$, $ar(V) = m \geq 2$ and $V = V_1 \cdot \cdots \cdot V_m$ be the unique factorization of $V$, where $V_i \in \mathbb{P}$. If $V$ is isomorphic to an open subtree of $T \in \mathbb{P}$, then $ar(T) = m, T = T_1 \cdot \cdots \cdot T_m, T_i \in \mathbb{P}$, and $V_i$ is isomorphic to an open subtree of $T_i$. Moreover $T - V = F_1 \cup \cdots \cup F_m$, if $F_i = T_i - In(V_i)$. Thus

$$b(T - V) = b(F_1) \cdot \cdots \cdot b(F_m) = \prod_{i=1}^{m} b(T_i - V_i).$$

By induction on the degree we can assume that

$$c_{T_i}(\varphi_g(V_i)) = b(T_i - V_i).$$

Thus it is here that the multiplication comes into play

$$c_T(\varphi_g(V)) = \prod_{i=1}^{m} b(T_i - V_i) = b(T - V).$$

\[\square\]

**Theorem 3.1.4.** It holds that

(i) Is $\varphi \in \text{End}(A)$, then $\varphi(x) \in x \cdot A = \{g \in A : \text{ord}(g) \geq 1\}$.

(ii) The map $\text{End}(A) \to x \cdot A$ with $\varphi \to \varphi(x)$ is bijective.

**Proof.** 1) Let $\varphi \in \text{End}A; g = \varphi(x)$.

Assumption: $\text{ord}(g) = 0 \implies \lambda = c_0(g) \neq 0, \lambda \in K \implies f := x - \lambda = x - \lambda \cdot x^0$

is a unit in $A$. $\implies \varphi(f \cdot f^{-1}) = \varphi(f) \cdot \varphi(f^{-1}) = 1 \implies \varphi(f)$ is a unit in $A$ with $\text{ord}(\varphi(f)) = 0$.

But $\varphi(f) = g - \lambda$ and $c_0(g - \lambda) = c_0(g) - \lambda = 0$.

This is a contradiction so $\text{ord}(g) \geq 1$.

Thus (i) holds.

2) Let $g \in x \cdot A$ and $\varphi : A \to A$ be a map given by

$$\varphi(f) = \sum_{i=0}^{\infty} c_i(f) \cdot g^i.$$ 

It is easy to compute that $\varphi \in \text{End}(A)$.

3) Let $\varphi, \psi \in \text{End}(A)$ with $\varphi(x) = \psi(x) = g$.

Assumption: $\varphi \neq \psi$. Then there is a $f \in A$ with $\varphi(f) \neq \psi(f)$.

Let $n = \text{ord}(\varphi(f) - \psi(f)) \implies n \in \mathbb{N}$ (i.e. $n \neq \infty$).

Let $f = f' + f''$ with $f' = \sum_{i=0}^{n} c_i(f) \cdot x^i$ and $f'' = f - f'$; $\text{ord}(f'') > n$.

Then $\varphi(f') = \psi(f')$, because $f'$ is a polynomial $\implies$
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\[ \varphi(f) - \psi(f) = \varphi(f'') - \psi(f''); \]
But it also holds that \( \varphi(x^{n+1} \cdot A) \subseteq x^{n+1} \cdot A \)
and also \( \varphi(x^{n+1} \cdot h) = g^{n+1} \cdot \varphi(h) \in x^{n+1} \cdot A \), because
\( g = x \cdot g_1, g_1 \in A; g^{n+1} = x^{n+1} \cdot g_1^{n+1}. \)
Further \( \psi(x^{n+1} \cdot A) \subseteq x^{n+1} \cdot A. \)
This is a contradiction because \( \varphi(x^{n+1}) \in x^{n+1} \cdot A. \)
From 2) and 3) \( (ii) \) holds.

3.2 Substitution automorphisms

Notation 3.2.1. By \( \text{Aut}(A) \) we denote the group of all \( K \)-algebra automorphisms of \( A \). It is also called the group of planar diffeomorphisms in one variable.

Proposition 3.2.2. It is true that \( \text{Aut}(A) = \{ \varphi_g : g \in A; \text{ord}(g) = 1 \} \).

Proof. 1) Let \( \text{ord}(g) = 1. \) We will show that there is a \( f \in A \) with \( \text{ord}(f) > 0 \) such that
\[ \varphi_g(f) = x. \]
We consider the coefficients \( a(T) = c_T(f) \) as unknown which have to satisfy the system of linear equations of Proposition 3.1.3. This means
\[ a(x) = (b(x))^{-1}, \text{ if } b(T) = c_T(g) \]
and
\[ a(T) \cdot (b(x))^{|T|} = - \sum_{S \subseteq O(T) \setminus \{T\}} a(S) \cdot b(T - S), \]
for any \( T \in \mathbb{P} \) of degree \( n > 1 \). This system of equations has a unique solution as \( b(x) \neq 0 \) and \( |S| < |T| \), for any open subtree \( S \) of \( T \), which is different from \( T \).
2) \( (\varphi_g \circ \varphi_f)(x) = \varphi_g(f)(x) = x \) and thus \( \varphi_g \circ \varphi_f = \varphi_x \), where \( \varphi_x \) is the identity in \( \text{Aut}(A) \).
As \( \text{ord}(f) = 1 \) there is a power series \( h \) in \( A \) of order 1 such that
\[ \varphi_f \circ \varphi_h = \varphi_x. \]
Therefore \( \varphi_h = \varphi_g \) and \( \varphi_f \) is the inverse of \( \varphi_g \) in \( \text{Aut}(A) \). \( \square \)

Let \( g \in A, \text{ord}(g) = 1, b(T) = c_T(g) \) be the coefficient of \( g \) relative to \( T \in \mathbb{P} \) and assume that \( b(x) = 1. \) Let \( f \in A \), such that \( \varphi_g(f) = x \) and \( a(T) = c_T(f) \) be the coefficient of \( f \) relative to \( T \in \mathbb{P} \). For any \( S \in \Omega_r(T), S = (S_0, ..., S_r) \) let
\[ \hat{b}_S := \prod_{i=0}^{r-1} b(S_i - S_{i+1}), \]
where
3.2 Substitution automorphisms

\[ b(S_i - S_{i+1}) = b(F_1) \cdot \ldots \cdot b(F_n); \]
\[ S_i - S_{i+1} = (F_1, \ldots, F_n) \]

and all \( F_i \) are connected graphs.

**Proposition 3.2.3.** Let

\[ \beta_r(T) = \sum_{S \in \Omega_r(T)} \hat{b}_S. \]

Then

\[ a(T) = \sum_{r=1}^{\infty} (-1)^r \beta_r(T). \]

**Proof.** From Proposition 3.1.3 it follows that

\[ c_T(f(g(x))) = \sum_{S \in \Omega(T)} a(S) \cdot b(T - S) = \begin{cases} 1 & : T = x \\ 0 & : \text{else.} \end{cases} \]

If \(|T| > 1\) and \( S = T \), then we get \( T - S \) is a disjoint system of isolated vertices. Thus \( b(T - S) = 1 \), because \( b(x) = 1 \).

If \( S = \rho_T \), then \( T - \rho_T = T, a(\rho_T) = 1 \) and thus

\[ a(T) = -b(T) - \sum_{S \in \Omega(T) \atop x \neq S \neq T} a(S) \cdot b(T - S). \]

2) In order to prove the formula we proceed by induction on \( n = |T| \):

\( n = 1 \): This case is trivial.

\( n > 1 \): We start with the formula in 1), and we may assume by induction hypothesis that for any proper open subtree \( S \) of \( T \) we have

\[ a(S) = \sum_{r=1}^{\infty} (-1)^r \beta_r(S). \]

Substituting this expression into the formula in 1) gives

\[ a(T) = -\sum_{r=1}^{\infty} (-1)^r \sum_{S \in \Omega(T) \atop S \neq T} \beta_r(S) \cdot b(T - S). \]

After applying Proposition 1.2.10 we get

\[ a(T) = \sum_{r=0}^{\infty} (-1)^{r+1} \beta_{r+1}(T). \]
3.3 Special planar chain rule

Let $h \in A$ and $\theta = \frac{d}{dx} : A \to A$ be a derivation. Then there is a unique $K$-algebra homomorphism $\varphi_h = \varphi : A \to A$, such that $\varphi(x) = h$. Then

$$\theta_h := \varphi \circ \theta : A \to A$$

is a derivation on $A$, i.e. $\theta_h$ is $K$-linear, continuous and satisfies the general product rule for all $f_1, \ldots, f_m \in A$:

$$\theta_h(f_1 \cdot \ldots \cdot f_m) = \sum_{i=1}^{m} f_1 \cdot \ldots \cdot f_{i-1} \cdot \theta_h(f_i) \cdot f_{i+1} \cdot \ldots \cdot f_m.$$  

This derivation will be formally denoted by $h \frac{d}{dx}$.

Be aware that $(h \frac{d}{dx})(f) \neq h \cdot \frac{d}{dx}(f)$ in general.

We denote $\frac{d}{dx}(f)$ also by $f'$ and call it the derivative of $f$ with respect to $x$.

**Proposition 3.3.1.** Let $g \in A$, such that $g' = 1 + g$. Then

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) + \left( x \frac{d}{dx} (f)(g(x)) = \left( (1 + x) \frac{d}{dx} \right) (f)(g(x)).$$

We call this rule the special planar chain rule.

**Proof.** (1) First we will prove the formula in case $f$ is a tree $T$ with $|T| = n$.

If $n = 1$, then $T = x$ and $\frac{d}{dx}x = 1$, while $\left( (1 + x) \frac{d}{dx} \right) (T) = 1 + x$. Thus $f(g(x)) = g(x)$ and $\frac{d}{dx}(g) = 1 + g$ from which the formula follows. Let now $n > 1$. Then

$$\frac{d}{dx}(T) = \sum_{i=1}^{n} T^{(i)},$$

where $T^{(i)} = \cdot_T(Z_1, \ldots, Z_n)$ with

$$Z_j = \begin{cases} x : j \neq i \\ 1 : j = i \end{cases}$$

and

$$\left( x \frac{d}{dx} \right) (T) = n \cdot T.$$  

Also

$$\frac{d}{dx}(T(g(x))) = \sum_{i=1}^{n} S^{(i)}$$

with

$$S^{(i)} = \cdot_T(W_1, \ldots, W_n).$$
and
\[ W_j = \begin{cases} g(x) : j \neq i \\ g'(x) : j = i. \end{cases} \]

As \( g' = 1 + g \), we obtain from the multilinearity of \( \cdot \) that
\[ S^{(i)} = \cdot_T(g, \ldots, g) + \cdot_T(i, g, \ldots, g). \]

From these computations it follows that the proposition holds, if \( f \) is a tree.

(2) From (1) one can extend the result to polynomials by multilinearity. As both sides are continuous in \( f \) the result follows for power series.

\[ \square \]

### 3.4 The \( k \)-th derivative

**Proposition 3.4.1.** Let \( \frac{d}{dx} \) be the derivation with respect to the variable \( x \) and \( T \in \mathbb{P} \) with \( n = |T| \). Then
\[
\frac{d}{dx}(T) = \frac{d}{dx}(x^T) = \sum_{S \in \mathbb{P}'} \binom{T}{S} x^S.
\]

**Proof.** See [Ger2]. \[ \square \]

**Corollary 3.4.2.** Let \( V \in \mathbb{P} \). Then it holds
\[
\frac{d}{dx}((1 + x)^V) = \sum_{T \in \mathbb{P}'} \binom{V}{T} \cdot \left( \sum_{S \in \mathbb{P}'} \binom{T}{S} \cdot x^S \right)
\]
\[
= \sum_{T \in \mathbb{P}'} \binom{V}{T} \cdot \left( \sum_{S \in \mathbb{P}'} \binom{T}{S} \cdot x^S \right).
\]

**Proof.** On the one hand we get by using Proposition 3.4.1
\[
\frac{d}{dx}((1 + x)^V) = \frac{d}{dx} \left( \sum_{T \in \mathbb{P}'} \binom{V}{T} \cdot x^T \right)
\]
\[
= \sum_{T \in \mathbb{P}'} \binom{V}{T} \cdot \left( \frac{d}{dx}(x^T) \right) = \sum_{T \in \mathbb{P}'} \binom{V}{T} \cdot \left( \sum_{S \in \mathbb{P}'} \binom{T}{S} \cdot x^S \right).
\]

On the other hand we obtain by Proposition 3.4.1 and Proposition 3.1.3
\[
\frac{d}{dx}((1 + x)^V) = \sum_{T \in \mathbb{P}'} \binom{V}{T} \cdot (1 + x)^T.
\]
Substitution homomorphisms

\[ = \sum_{T \in \mathbb{P}', |T|=|V|-1} \binom{V}{T} \cdot \left( \sum_{S \in \mathbb{P}'} \binom{T}{S} x^S \right). \]

\[ \square \]

Proposition 3.4.3. Let \( T, S \in \mathbb{P}', \ n = |T| \). Then

\[
\left( \frac{d}{dx} \right)^k (x^T) = T^{(k)}(x, ..., x)
\]

\[ = k! \cdot \sum_{I \subseteq \mathbb{n} \atop |I|=n-k} T(I) = k! \cdot \sum_{|S|=|T|-k} \binom{T}{S} \cdot x^S,
\]

where \( \mathbb{n} = \{1, ..., n\} \), \( x^T = T(x, ..., x) \), \( T^{(k)}(x, ..., x) \) is the \( k \)-th derivative of \( T \)
and

\[ T(I) = T(i_1, ..., i_n) = \begin{cases} x & : j \in I \\ 1 & : j \notin I. \end{cases} \]

Proof. Induction over \( k \):

- Induction begin: \( k = 0 \): \( T^0(x, ..., x) = 1 \cdot \sum_{I=\mathbb{n}} T(I) = T(x, ..., x) \)

\[ = 1 \cdot \sum_{|S|=n} \binom{T}{S} \cdot S. \]

- Induction hypothesis:

\[ T^{(k)}(x, ..., x) = k! \cdot \sum_{I \subseteq \mathbb{n} \atop |I|=n-k} T(I) \]

is true for one \( k \in \mathbb{N} \).

- Induction step: \( k \rightarrow k + 1 \): it is to show that

\[ T^{(k+1)}(x, ..., x) = (k + 1)! \cdot \sum_{I \subseteq \mathbb{n} \atop |I|=n-(k+1)} T(I) = (k + 1)! \cdot \sum_{|S|=n-(k+1)} \binom{T}{S} \cdot S. \]

It holds

\[
T^{(k+1)}(x, ..., x) = \left( T^{(k)}(x, ..., x) \right)' \overset{I.H.}{=} \left( k! \cdot \sum_{I \subseteq \mathbb{n} \atop |I|=n-k} T(I) \right)' = k! \cdot \sum_{I \subseteq \mathbb{n} \atop |I|=n-k} \sum_{J \subseteq I \atop |J|=k} T(I - J) = k! \cdot (k + 1) \cdot \sum_{K \subseteq \mathbb{n} \atop |K|=n-(k+1)} T(K) = (k + 1)! \cdot \sum_{|S|=n-(k+1)} \binom{T}{S} \cdot S.
\]
3.4 The $k$-th derivative

$\star$ is true as for a fixed $K$ with $|K| = n - (k + 1)$, $K^c$ with $|K^c| = k + 1$ and $i \in K^c$ there is exactly one $I := K \cup \{i\}$ with $J = \{i\}$. 

\[ \sum_{|S_1| = |T| - 1} \frac{T}{S_1} \cdots \frac{S_{k-1}}{S_k} = k! \cdot \sum_{|S| = |T| - k} \frac{T}{S}. \]

**Proof.** The result follows immediately from the above Proposition and

\[
\left( \frac{d}{dx} \right)^k (x^T) = \left( \frac{d}{dx} \right)^{k-1} \left( \sum_{|S_1| = |T| - 1} \frac{T}{S_1} \cdot x^{S_1} \right)
= \sum_{|S_1| = |T| - 1} \cdots \sum_{|S_k| = |S_{k-1}| - 1} \frac{T}{S_1} \cdot \frac{S_1}{S_2} \cdots \frac{S_{k-1}}{S_k} \cdot x^S.
\]

**Corollary 3.4.5.** Let $V \in \mathbb{P}$. Then it follows

\[
\left( \frac{d}{dx} \right)^k ((1 + x)^V) = k! \cdot \sum_{T \in \mathbb{P}'} \frac{V}{T} \cdot \left( \sum_{|S| = |T| - k} \frac{T}{S} \cdot S \right)
= k! \cdot \sum_{|T| = |V| - k} \frac{V}{T} \cdot \left( \sum_{S \in \mathbb{P}'} \frac{T}{S} \cdot S \right).
\]

**Proof.** Analogous to Corollary 3.4.2 by using Proposition 3.4.3. 

\[ \square \]
Chapter 4

Exponential and logarithm series

In this and all the following chapters $K$ will be a field with $\text{char}(K) = 0$.

4.1 Planar exponential series

**Proposition 4.1.1.** Let $V \in \mathbb{P}$ and $k = |V|$ with $k \in \mathbb{N}_{\geq 2}$. Then there is a unique power series $\exp_V(x) \in K\{x\}$, such that

(i) $\text{ord}(\exp_V(x) - (1 + x)) \geq 2$,

(ii) $(\exp_V(x))^V = \exp_V(kx)$.

Moreover $\frac{d}{dx}(\exp_V(x)) = \exp_V(x)$. $\exp_V(x)$ is called the planar exponential series with respect to $V$.

We denote by $a_V(T)$ the coefficient of $\exp_V(x)$ at $T \in \mathbb{P}$.

**Theorem 4.1.2.** Let $V \in \mathbb{P}$ and $k = |V|$. Then for the coefficient $a_V(T)$ of $\exp_V(x)$ it holds that

$$a_V(x) = 1$$

and

$$a_V(T) = \sum_S \binom{V}{S} \cdot \frac{(k^n - k)^{\frac{V}{S}}}{k^n} \cdot a_V(T - S),$$

where $S \in O(T), S \neq x, 2 \leq m = |S|, n = |T|$ and $a_V(T - S) = \prod_{i=1}^{m} a_V(T_i)$, if $T - S = (T_1, ..., T_m)$.

**Proof.** (1) Let $F \in K\{x\}$, $\text{ord}(F) > 0$ and $\exp_V = E = 1 + F$. For any $V \in \mathbb{P}$ with $k = |V|$ we get

$$E^V = 1 + \sum_S \binom{V}{S} \cdot F^S.$$
where \( F^S := \cdot_S(F,\ldots,F) \).

For \( T \in \mathbb{P} \) with \( 1 \leq ar(T) \leq k \) we get

\[
\text{coeff}_T(E^V) = \sum_S \binom{V}{S} \cdot \prod_{i=1}^{m} \text{coeff}_{T_i}(F) + k \cdot \text{coeff}_T(F),
\]

where \( S \in O(T), S \neq x, |S| = m \geq 2 \) and \( T - S = (T_1,\ldots,T_m) \). If \( ar(T) > k \) or \( ar(T_i) > k \) for any \( T_i \in P(T) \), then \( \text{coeff}_T(E^V) = 0 \).

(2) Define a map \( a_V : \mathbb{P} \to K \) as in Theorem 4.1.2.

Let \( F := \sum_{T \in \mathbb{P}} a_V(T) \cdot T \) and \( E = 1 + F \).

Then \( \text{ord}(F) \geq 1 \) and for \( T \in \mathbb{P} \) with \( |T| = n \) we obtain

\[
\text{coeff}_T(E^V) = \text{coeff}_T \left( \sum_S \binom{V}{S} \cdot F^S \right) + \text{coeff}_T(k \cdot F) = \sum_S \binom{V}{S} \cdot \prod_{i=1}^{m} \text{coeff}_{T_i}(F) + k \cdot \text{coeff}_T(F) = \sum_S \binom{V}{S} \cdot \prod_{i=1}^{m} a_V(T_i) + k \cdot a_V(T) = (k^n - k) \cdot a_V(T) + k \cdot a_V(T) = k^n \cdot a_V(T).
\]

As \( \text{coeff}_T(\exp_V(kx)) = k^n \cdot \text{coeff}_T(\exp_V(x)) = k^n \cdot a_V(T) \) we get the equation

\[
(\exp_V(x))^V = \exp_V(kx).
\]

(3) Assume that \( (\exp_V(x))^V = \exp_V(kx) \) and \( \text{ord}(\exp_V(x) - (1 + x)) \geq 2 \), then one gets from the computation in (2) that

\[
\text{coeff}_T(E^V) = \sum_S \binom{V}{S} \cdot \prod_{i=1}^{m} \text{coeff}_{T_i}(E),
\]

from which follows that

\[
a_V(T) = \text{coeff}_T(E) = \text{coeff}_T(\exp_V(x)),
\]

for all \( T \in \mathbb{P} \).

The uniqueness follows immediately from (1).

(4) Let \( \exp_V(x) = E \) and \( F = E - 1 \). Denote by \( F_n \) the homogeneous part of \( F \) of degree \( n \). Thus

\[
F_n = \sum_{T \in \mathbb{P} \atop |T| = n} \text{coeff}_T(F) \cdot T(x).
\]
Then
\[ F_n(kx) = k^n \cdot F_n(x) \]
and
\[ F_n(kx) = \sum_{m=1}^{k} \sum_{S \in \mathcal{P} \mid |S|=m} \left( \sum_{V \subseteq S} \cdot (F^S)_n(x) \right), \]
where \((F^S)_n\) is the homogeneous part of degree \(n\) of \(F^S\). Now
\[ (F^S)_n = \sum_{\nu \in M_S(m,n)} F_{\nu}, \]
where \(m = |S|\), \(M_S(m,n) = \{\nu = (\nu_1, ..., \nu_m) : \nu_i \in \mathbb{N}^{\geq 1}, |\nu| = \nu_1 + ... + \nu_m = n\}\)
and \(F_\nu := \cdot S(F_{\nu_1}, ..., F_{\nu_m})\) for \(\nu = (\nu_1, ..., \nu_m) \in M_S(m,n)\).
We prove by induction on \(n\) that
\[ F'_n := \frac{d}{dx}(F_n) = F_{n-1}. \]

**Induction begin: \(n = 1\):** This is obvious as \(F_1 = x\) and \(F'_1 = 1\).

**Induction step:** Let \(n > 1\). As
\[ (k^n - k)F_n(x) = \sum_{m=1}^{k} \sum_{S \in \mathcal{P} \mid |S|=m} \left( \sum_{V \subseteq S} \cdot (F^S)_n(x) \right) \]
we get
\[ (k^n - k)F'_n(x) = \sum_{m=1}^{k} \sum_{S \in \mathcal{P} \mid |S|=m} \left( \sum_{V \subseteq S} \cdot \frac{d}{dx}((F^S)_n(x)) \right). \]

We will prove in part (5) below that
\[ \frac{d}{dx}((F^S)_n) = m \cdot (F^S)_{n-1} + \sum_{T \in \mathcal{P} \mid |T|=m-1} \left( \sum_{T \in \mathcal{P} \mid |T|=m-1} \left( \sum_{S \in \mathcal{P} \mid |S|=m-1} \cdot (F^T)_{n-1}, \right) \right). \]
∀ 2 ≤ m ≤ k, where m = |S|. Thus

\[
(k^n - k)F_n'(x) = \sum_{m=2}^{k} \sum_{S \in \mathcal{P}} \binom{V}{S} \cdot \frac{d}{dx}((F^S)_n)
\]

\[
= \sum_{m=2}^{k} \sum_{S \in \mathcal{P}, |S| = m} \binom{V}{S} \cdot \left( m \cdot (F^S)_{n-1} + \sum_{T \in \mathcal{P}, |T| = m-1} \binom{S}{T} \cdot (F^T)_{n-1} \right)
\]

\[
= k \cdot (F^V)_{n-1} + \sum_{m=2}^{k-1} \sum_{S \in \mathcal{P}, |S| = m} \binom{V}{S} \cdot m \cdot (F^S)_{n-1}
\]

\[
+ \sum_{m=3}^{k} \sum_{S \in \mathcal{P}, |S| = m+1} \binom{V}{S} \cdot \left( \sum_{T \in \mathcal{P}, |T| = m} \binom{S}{T} \cdot (F^T)_{n-1} \right) + 2 \cdot \binom{k}{2} \cdot F_{n-1}
\]

\[
= k \cdot (F^V)_{n-1} + \sum_{m=2}^{k-1} \sum_{S \in \mathcal{P}, |S| = m} \binom{V}{S} \cdot m \cdot (F^S)_{n-1}
\]

\[
+ \sum_{S \in \mathcal{P}, |S| = m+1} \binom{V}{S} \cdot \left( \sum_{T \in \mathcal{P}, |T| = m} \binom{S}{T} \cdot (F^T)_{n-1} \right) + 2 \cdot \binom{k}{2} \cdot F_{n-1}
\]

After renaming we obtain

\[
= k \cdot (F^V)_{n-1} + \sum_{m=2}^{k-1} \sum_{S \in \mathcal{P}, |S| = m} \binom{V}{S} \cdot m \cdot (F^S)_{n-1}
\]

\[
+ \sum_{W \in \mathcal{P}, \text{deg}(W) = m+1} \binom{V}{W} \cdot \left( \sum_{S \in \mathcal{P}, |S| = m} \binom{W}{S} \cdot (F^S)_{n-1} \right) + 2 \cdot \binom{k}{2} \cdot F_{n-1}.
\]

Then

\[
\sum_{S \in \mathcal{P}, |S| = m} \binom{V}{S} \cdot m + \sum_{W \in \mathcal{P}, \text{deg}(W) = m+1} \binom{V}{W} \cdot \left( \sum_{S \in \mathcal{P}, |S| = m} \binom{W}{S} \right)
\]
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\[
(\binom{k}{m} \cdot m + \binom{k}{m+1} \cdot \frac{m+1}{m}) = k \cdot \frac{k-1}{m-1} + k \cdot \frac{k-1}{m} = k \cdot \left(\binom{k}{m}\right)
\]

for all \(2 \leq m \leq k-1\) and thus

\[
(k^n - k)F'_n(x) = k \cdot \sum_{m=2}^{k} \sum_{S \in P, |S|=m} \left(\binom{V}{S}\right) \cdot (F^S)_{n-1} + (k^2 - k) \cdot F_{n-1}
\]

\[
= k \cdot (k^{n-1} - k) \cdot F_{n-1}(x) + (k^2 - k) \cdot F_{n-1}(x)
\]

\[
= (k^n - k) \cdot F_{n-1}(x),
\]

which proves that

\[
F'_n(x) = F_{n-1}(x).
\]

(5) Now we prove \(\forall 2 \leq m \leq k\) that

\[
\frac{d}{dx} ((F^S)_n) = m \cdot (F^S)_{n-1} + \sum_{T \in P, |T|=m-1} \left(\binom{S}{T}\right) \cdot (F^T)_{n-1}.
\]

By definition

\[
(F^S)_n = \sum_{\nu \in M_S(m,n)} F_\nu,
\]

where \(m = |S|, M_S(m,n) = \{\nu = (\nu_1, ..., \nu_m) : \nu_i \in \mathbb{N}^\geq, |\nu| = \nu_1 + ... + \nu_m = n\}\)

and \(F_\nu := \cdot S(F_{\nu_1}, ..., F_{\nu_m})\) for \(\nu = (\nu_1, ..., \nu_m) \in M_S(m,n)\). We have

\[
\frac{d}{dx}(F_\nu) = \sum_{i=1}^{m} F_{\nu-e_i},
\]

where \(e_i = (\delta_{i1}, ..., \delta_{im}) = (0, ..., 0, 1, 0, ..., 0)\), \(1\) is at the \(i\)-th place, as by induction hypothesis \(\forall j < n:\)

\[
\frac{d}{dx}(F_j) = F_{j-1}
\]

and

\[
\frac{d}{dx}(F_\nu) = \sum_{i=1}^{m} \cdot S(F_{\nu_1}, ..., F_{\nu_{i-1}}, F'_{\nu_i}, F_{\nu_{i+1}}, ..., F_{\nu_m}).
\]

Let \(N_S(m,n) := M_S(m,n) \times \{1, ..., m\}, N_S(m,n,i) = \{(\nu,i) \in N_S(m,n) : \text{the } i\text{-th component } \nu_i \text{ of } \nu \text{ is equal to } 1 \}\) and \(N^*_S(m,n) = N_S(m,n) - \bigcup_{i=1}^{m} N_S(m,n,i)\).

Now

\[
\frac{d}{dx} ((F^S)_n) = \sum_{i=1}^{m} \sum_{\nu \in N^*_S(m,n)} F_{\nu-e_i} + \sum_{i=1}^{m} \sum_{\nu \in N_S(m,n,i)} F_{\nu-e_i}.
\]
Let $\mu \in M_S(m, n-1)$. For any $1 \leq i \leq m$ there is exactly one $\nu = \nu(\mu, i) = \mu + e_i$, such that $F_{\nu-i} = F_\mu$. Then $\nu \notin N_S(m, n, i)$ and it follows that

$$
\sum_{i=1}^{m} \sum_{\nu \in N_S(m, n)} F_{\nu-i} = m \cdot (F^n S)_{n-1}.
$$

For any $\nu \in N_S(m, n, i)$ let $\bigwedge_i(\nu) = (\nu_1, \ldots, \nu_{i-1}, \nu_{i+1}, \ldots, \nu_m) \in N_T(m-1, n-1)$, where $T \in \mathbb{P}$ with $|T| = m-1$. Then $F_{\nu-i} = F_{\bigwedge_i(\nu)}$, for all $\nu \in N_S(m, n, i)$. Be aware that $\bigwedge_i(\nu)$ is not unique if $\sharp(T \in \mathbb{P} : |T| = m-1) > 1$, but there are as much possibilities as its cardinality. Thus we get

$$
\sum_{i=1}^{m} \sum_{\nu \in N_S(m, n, i)} F_{\nu-i} = \sum_{T \in \{ \bigwedge_i(\nu) \}} \binom{S}{T} \cdot (F^n T)_{n-1}.
$$

Altogether we obtain

$$
\frac{d}{dx} ((F^n S)_n) = m \cdot (F^n S)_{n-1} + \sum_{T \in \{ \bigwedge_i(\nu) \}} \binom{S}{T} \cdot (F^n T)_{n-1}.
$$

□

Example 4.1.3. (i) Let $V = x \cdot x^2$. Then

$$
\exp_{x \cdot x^2}(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} \left( \frac{5}{8} x \cdot x^2 + \frac{3}{8} x^3 \cdot x \right) + \frac{1}{4!} \left( \frac{1}{4} x \cdot (x \cdot x^2) + \frac{5}{52} (x \cdot x^2) \cdot x + \frac{11}{52} x \cdot (x^2 \cdot x) + \frac{3}{52} (x^2 \cdot x) \cdot x + \frac{5}{13} x^2 \cdot x^2 \right)
+ \frac{1}{5!} \left( \frac{13}{192} x \cdot (x \cdot (x \cdot x^2)) + \frac{1}{64} (x \cdot (x \cdot x^2)) \cdot x + \frac{145}{2496} x \cdot (x \cdot x^2) \cdot x \right)
+ \frac{5}{832} (x \cdot (x^2 \cdot x)) \cdot x + \frac{37}{832} x \cdot (x \cdot (x^2 \cdot x)) + \frac{11}{832} (x \cdot (x^2 \cdot x)) \cdot x
+ \frac{29}{832} x \cdot ((x^2 \cdot x) \cdot x) + \frac{3}{32} ((x^2 \cdot x) \cdot x) \cdot x + \frac{31}{208} x \cdot (x^2 \cdot x^2)
+ \frac{5}{208} (x^2 \cdot x^2) \cdot x + \frac{13}{64} x^2 \cdot (x \cdot x^2) + \frac{25}{192} (x \cdot x^2) \cdot x^2
+ \frac{11}{64} x^2 \cdot (x^2 \cdot x) + \frac{5}{64} (x^2 \cdot x) \cdot x \right) + \text{higher terms}.
$$
(ii) Let \( V = (x \cdot x^3) \cdot x \). Then
\[
\exp_{(x \cdot x^3) \cdot x}(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} \left( \frac{2}{5} x \cdot x^2 + \frac{11}{20} x^2 \cdot x + \frac{1}{20} x^3 \right) \\
+ \frac{1}{4!} \left( \frac{13}{155} x \cdot (x \cdot x^2) + \frac{8}{31} (x \cdot x^2) \cdot x + \frac{29}{310} x \cdot (x^2 \cdot x) \\
+ \frac{47}{310} (x^2 \cdot x) \cdot x + \frac{13}{310} x \cdot x^3 + \frac{13}{310} x^3 \cdot x + \frac{42}{155} x^2 \cdot x^2 \\
+ \frac{3}{155} x \cdot x \cdot x^2 + \frac{3}{155} x \cdot x^2 \cdot x + \frac{3}{155} x^3 \cdot x \cdot x \right) + \text{higher terms.}
\]

\textbf{Proposition 4.1.4.} Let \( V \in \mathbb{P} \) and \( V_n \in \mathbb{P} \) with \( V_1 = V \) and \( V_{n+1} = V \ast V_n \) for \( n \in \mathbb{N}^\geq 1 \). Then
\[
\exp_{V_n}(x) = \exp_V(x), \quad \forall n \in \mathbb{N}^> 1.
\]

\textbf{Proof.} Let \(|V| = r\), then \(|V_n| = r^n\).
Induction on \( n \):
Induction begin: \( n = 2 \): then
\[
(\exp_{V_2}(x))^{V_2} = \exp_{V_2}(r^2 \cdot x)
\]
and
\[
(\exp_{V}(x))^{V_2} = \underbrace{\exp_{V}(r x) \cdot \ldots \cdot \exp_{V}(r x)}_{r \text{-times}} = \exp_{V}(r^2 \cdot x).
\]

As these two series satisfy the same functional equation
\( \implies \exp_{V}(x) = \exp_{V_2}(x) \).

Induction hypothesis: Let the assumption be true for one \( n \in \mathbb{N} \).

Induction step: \( n \rightarrow n + 1 \): then
\[
(\exp_{V_{n+1}}(x))^{V_{n+1}} = \exp_{V_{n+1}}(r^{n+1} \cdot x)
\]
and
\[
(\exp_{V}(x))^{V_{n+1}} = \underbrace{(V_n \cdot \ldots \cdot V_n)(\exp_{V}(x))}_{r \text{-times}} = \underbrace{V_n(\exp_{V}(x)) \cdot \ldots \cdot V_n(\exp_{V}(x))}_{r \text{-times}}
\]
\( \overset{I.H.}{=} \)
\[
\underbrace{\exp_{V}(r^n \cdot x) \cdot \ldots \cdot \exp_{V}(r^n \cdot x)}_{r \text{-times}} = \exp_{V}(r^{n+1} \cdot x).
\]

As these two series also satisfy the same functional equation
\( \implies \exp_{V}(x) = \exp_{V_{n+1}}(x) \). \( \square \)
The following proposition is a special case of Proposition 4.1.4 where $V$ is a corona. Then we call $V_n$ a full-symmetric tree for all $n \in \mathbb{N} > 1$. It follows that all full-symmetric trees of arity $r$ have the same planar exponential series as the corona $x^r$.

**Proposition 4.1.5.** Let $T^r_n \in \mathbb{P}$ with $T^r_{n+1} = T^r_n \cdot \ldots \cdot T^r_n$, $r \in \mathbb{N} > 1$ and $T^r_1 = x^r$.

Then

\[ exp_{T^r_n}(x) = exp_{x^r}(x), \forall n \in \mathbb{N} > 1. \]

**Proof.** It holds $\deg(T^r_n) = r^n$.

Induction on $n$:

Induction begin: $n = 2$: then

\[
\left( exp_{T^r_2}(x) \right)^{T^r_2} = exp_{T^r_2}(r^2 \cdot x)
\]

and

\[
\left( exp_{x^r}(x) \right)^{T^r_2} = \underbrace{exp_{x^r}(rx) \cdot \ldots \cdot exp_{x^r}(rx)}_{r \text{-times}} = exp_{x^r}(r^2 \cdot x).
\]

As these two series satisfy the same functional equation

\[ \Rightarrow exp_{x^r}(x) = exp_{T^r_2}(x). \]

Induction hypothesis: Let the assumption be true for one $n \in \mathbb{N}$.

Induction step: $n \rightarrow n + 1$: then

\[
\left( exp_{T^r_{n+1}}(x) \right)^{T^r_{n+1}} = exp_{T^r_{n+1}}(r^{n+1} \cdot x)
\]

and

\[
\left( exp_{x^r}(x) \right)^{T^r_{n+1}} = \underbrace{(T^r_n \cdot \ldots \cdot T^r_n)}_{r \text{-times}} \left( exp_{x^r}(x) \right)
\]

\[ = \underbrace{T^r_n \left( exp_{x^r}(x) \right) \cdot \ldots \cdot T^r_n \left( exp_{x^r}(x) \right)}_{r \text{-times}}
\]

$\text{I.H.}$

\[ = \underbrace{exp_{x^r}(r^n \cdot x) \cdot \ldots \cdot exp_{x^r}(r^n \cdot x)}_{r \text{-times}}
\]

\[ = exp_{x^r}(r^{n+1} \cdot x). \]

As these two series also satisfy the same functional equation

\[ \Rightarrow exp_{x^r}(x) = exp_{T^r_{n+1}}(x). \]

$\square$
Proposition 4.1.6. Let $a_V(T)$ be the coefficient of $T \in \mathbb{P}$ in the planar exponential series $\exp_V(x)$. Then

$$\sum_{T \in \mathbb{P}} a_V(T) = \frac{1}{n!}.$$ 

Proof. By Proposition 2.5.7 it follows that $\eta(\exp_V(x))$ is the classical Euler exponential series and thus the above proposition follows immediately. $\square$

Remark 4.1.7. It is possible to consider the planar exponential series over the field $\mathbb{C}$ of complex numbers. We can assert that this is a planar analytic function which radius of convergence is equal to $\infty$. Furthermore we get the equation

$$\exp_V(\alpha + x) = e^\alpha \cdot \exp_V(x), \quad \forall \alpha \in \mathbb{C}.$$ 

For more details see [Ger9].

4.2 Planar logarithm series

Now we can find the compositional inverse of the planar exponential series.

Proposition 4.2.1. Let $V \in \mathbb{P}$. Then there is a unique power series $\log_V(1 + x) \in K\{\{x\}\}$, such that

(i) $\log_V(\exp_V(x)) = x$ and $\exp_V(\log_V(1 + x)) = 1 + x$.

(ii) $\log_V((1 + x)^V) = k \cdot \log_V(1 + x)$, for $k = |V|$.

Moreover

$$\left((1 + x) \frac{d}{dx}\right) (\log_V(1 + x)) = 1, \quad \forall \quad V \in \mathbb{P}.$$ 

We call $\log_V(1 + x)$ the planar logarithm series with respect to $V$.

Proof. (1) (i) follows immediately by applying the substitution automorphism. (2) (ii) follows from the properties of the exponential series. It holds

$$\exp_V(\log_V((1 + x)^V)) = (1 + x)^V$$

and

$$\exp_V(k \cdot \log_V(1 + x)) = (\exp_V(\log_V(1 + x)))^V = (1 + x)^V.$$ 

Thus

$$\log_V((1 + x)^V) = k \cdot \log_V(1 + x).$$

(3) Let

$$g = \exp_V(x) - 1.$$
Then

$$\log_V(\exp_V(x)) = x.$$  

From the special planar chain rule in Proposition 3.3.1 one can conclude that

$$\left( (1 + x) \frac{d}{dx} \right) (\log_V)(\exp_V(x)) = 1.$$  

Let \( f = \left( (1 + x) \frac{d}{dx} \right) (\log_V) \). Thus as \( f(1 + g(x)) = 1 \) if and only if \( f(1 + x) = 1, \forall x \), it follows that the derivation property of \( \log_V(1 + x) \) is true. \( \square \)

**Corollary 4.2.2.** Let \( h_n \) be the homogeneous component of \( \log_V(1 + x) \) of degree \( n \). Then

\[
h_0 = 0
\]

and for \( n \geq 1 \)

\[
h'_{n+1} = -nh_n.
\]

**Proof.** It holds that

\[
\log_V(1 + x) = \sum_{n=0}^{\infty} h_n =: h
\]

and

\[
\left( (1 + x) \frac{d}{dx} \right) (h_n) = h'_n + nh_n,
\]

because

\[
\left( x \frac{d}{dx} \right) (h_n) = n \cdot h_n.
\]

Moreover \( h'_{n+1} \) is homogeneous of degree \( n \), because it holds for all \( n \geq 1 \) that

\[
h'_{n+1} + nh_n = 0,
\]

because

\[
\left( (1 + x) \frac{d}{dx} \right) (h) = 1.
\]

\( \square \)

Denote by \( c_V(T) \) the coefficient of \( \log_V(1 + x) \) at \( T \in \mathbb{P} \). By considering the functional equation of the logarithm series we get the following

**Theorem 4.2.3.** Let \( V \in \mathbb{P} \) and \( k = |V| \). Then for the coefficient \( c_V(T) \) of \( \log_V(1 + x) \) it is true that

\[
c_V(x) = 1
\]
4.2 Planar logarithm series

and

\[ c_V(T) = \frac{\sum_{S \in \mathcal{O}(T)} \left( c_V(S) \cdot \prod_{i=1}^{m} \left( \frac{V}{T_i} \right) \right)}{k - k^n}, \]

where \( n = |T|, m = |S| \) and \( T - S = (T_1, \ldots, T_m) \).

**Proof.** This follows immediately from Proposition 4.2.1 (ii), because

\[ \text{coef}_{f_T}(k \cdot \log_V(1 + x)) = k \cdot \text{coef}_{f_T}(\log_V(1 + x)) = k \cdot c_V(T) \]

and

\[ \text{coef}_{f_T}(\log_V((1 + x)^V)) = \text{coef}_{f_T}\left( \log_V\left(1 + \sum_{S \in \mathcal{P}(V)} \left( \frac{V}{S} \right) \cdot x^S \right) \right) \]

\[ = \sum_{S \in \mathcal{O}(T)} c_V(S) \cdot \prod_{i=1}^{m} \left( \frac{V}{T_i} \right) \]

\[ = \sum_{S \in \mathcal{O}(T)} c_V(S) \cdot \prod_{i=1}^{m} \left( \frac{V}{T_i} \right) + k^n \cdot c_V(T). \]

By Proposition 4.2.1 (ii) the equality follows, thus

\[ k \cdot c_V(T) = \sum_{S \in \mathcal{O}(T)} c_V(S) \cdot \prod_{i=1}^{m} \left( \frac{V}{T_i} \right) + k^n \cdot c_V(T). \]

\[ \square \]

**Remark 4.2.4.** If one tree in the forest \( T - S \) in the formula in Theorem 4.2.3 is not a subtree of \( V \), then the summand is equal to 0. It follows that if

\[ T_i \in P(V), \quad \forall i, \]

then the summand does not vanish.

**Remark 4.2.5.** As \( \log_V(1 + x) \) is the compositional inverse of \( \exp_V(x) \), we can also compute the logarithm coefficients by applying the exponential coefficients, see also [Ger3], thus

1.) Degree 1: \( c_V(x) = 1; \)

2.) Degree 2: \( c_V(x^2) = -\frac{1}{2}; \)
3.) Degree 3:
\[c_V(x \cdot x^2) = -(a_V(x \cdot x^2) - \frac{1}{2} \cdot a_V(x) \cdot a_V(x^2));\]
\[c_V(x^2 \cdot x) = -(a_V(x^2 \cdot x) - \frac{1}{2} \cdot a_V(x^3) \cdot a_V(x));\]
\[c_V(x^3) = -a_V(x^3);\]

4.) Degree 4:
\[c_V(x \cdot (x \cdot x^2)) = - \left( a_V(x \cdot (x \cdot x^2)) - \frac{1}{2} \cdot a_V(x) \cdot a_V(x \cdot x^2) + \frac{1}{2} \cdot c_V(x \cdot x^2) \right);\]
\[c_V((x \cdot x^2) \cdot x) = - \left( a_V((x \cdot x^2) \cdot x) - \frac{1}{2} \cdot a_V(x \cdot x^2) \cdot a_V(x) + \frac{1}{2} \cdot c_V(x^2 \cdot x) \right);\]
\[c_V(x \cdot (x^2 \cdot x)) = - \left( a_V(x \cdot (x^2 \cdot x)) - \frac{1}{2} \cdot a_V(x) \cdot a_V(x^2) \cdot a_V(x) + \frac{1}{2} \cdot c_V(x \cdot x^2) \right);\]
\[c_V((x^2 \cdot x) \cdot x) = - \left( a_V((x^2 \cdot x) \cdot x) - \frac{1}{2} \cdot a_V(x^2 \cdot x) \cdot a_V(x) + \frac{1}{2} \cdot c_V(x \cdot x^2) \right);\]
\[c_V(x^2 \cdot x^2) = - \left( a_V(x^2 \cdot x^2) - \frac{1}{2} \cdot a_V(x^2) \cdot a_V(x^2) + \frac{1}{2} \cdot (c_V(x \cdot x^2) + c_V(x^2 \cdot x)) \right);\]
\[c_V(x \cdot x^3) = - \left( a_V(x \cdot x^3) - \frac{1}{2} \cdot a_V(x) \cdot a_V(x^3) \right);\]
\[c_V(x^3 \cdot x) = - \left( a_V(x^3 \cdot x) - \frac{1}{2} \cdot a_V(x^3) \cdot a_V(x) \right);\]
\[c_V(x \cdot x \cdot x^2) = - \left( a_V(x \cdot x \cdot x^2) + \frac{1}{2} \cdot c_V(x^3) \right);\]
\[c_V(x \cdot x^2 \cdot x) = - \left( a_V(x \cdot x^2 \cdot x) + \frac{1}{2} \cdot c_V(x^3) \right);\]
\[c_V(x^2 \cdot x \cdot x) = - \left( a_V(x^2 \cdot x \cdot x) + \frac{1}{2} \cdot c_V(x^3) \right);\]
\[c_V(x^4) = -a_V(x^4).\]

**Example 4.2.6.**  (i) Let \( V = x \cdot x^2 \). Then
\[
\log_V(1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} \cdot \left( \frac{7}{16} x \cdot x^2 + \frac{9}{16} x^2 \cdot x \right)
- \frac{1}{4} \cdot \left( \frac{1}{8} x \cdot (x \cdot x^2) + \frac{19}{104} (x \cdot x^2) \cdot x + \frac{21}{104} x \cdot (x^2 \cdot x)ight.
+ \frac{27}{104} (x^2 \cdot x) \cdot x + \frac{3}{13} x^2 \cdot x^2)
+ \text{ higher terms.}
\]
(ii) Let $V = x \cdot x \cdot x^2$. Then
\[
\log_V(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3} \left( \frac{1}{2}x \cdot x^2 + \frac{3}{5}x^2 \cdot x - \frac{1}{10}x^3 \right)
- \frac{1}{4} \left( \frac{4}{21}x \cdot (x \cdot x^2) + \frac{76}{315}(x \cdot x^2) \cdot x + \frac{16}{63}x \cdot (x^2 \cdot x) 
+ \frac{32}{105}(x^2 \cdot x) \cdot x + \frac{86}{315}x^2 \cdot x^2 - \frac{4}{63}x \cdot x^3 - \frac{4}{63}x^3 \cdot x 
- \frac{11}{315}x \cdot x \cdot x^2 - \frac{16}{315}x \cdot x^2 \cdot x - \frac{16}{315}x^2 \cdot x \cdot x \right)
+ \text{higher terms.}
\]

**Proposition 4.2.7.** Let $V \in \mathbb{P}$ and $V_n \in \mathbb{P}$ with $V_1 = V$ and $V_{n+1} = V \ast V_n$ for $n \in \mathbb{N}_{\geq 1}$. Then
\[
\log_{V_n}(1 + x) = \log_V(1 + x), \quad \forall n \in \mathbb{N}_{> 1}.
\]

**Proof.** Let $|V| = r$, then $|V_n| = r^n$.

Induction on $n$:

**Induction begin**: $n = 2$: then
\[
\log_{V_2}((1 + x)^V_2) = r^2 \cdot \log_{V_2}(1 + x)
\]

and
\[
\log_V((1 + x)^V_2) = \log_V(((1 + x)^V)^V)
= r \cdot \log_V((1 + x)^V) = r^2 \cdot \log_V(1 + x).
\]

As these two series satisfy the same functional equation
\[
\Rightarrow \log_V(1 + x) = \log_{V_2}(1 + x).
\]

**Induction hypothesis**: Let the assumption be true for one $n \in \mathbb{N}$.

**Induction step**: $n \rightarrow n + 1$: then
\[
\log_{V_{n+1}}((1 + x)^{V_{n+1}}) = r^{n+1} \cdot \log_{V_{n+1}}(1 + x)
\]

and
\[
\log_V((1 + x)^{V_{n+1}}) = \log_V(((1 + x)^{V_n})^V)
= r \cdot \log_V((1 + x)^{V_n})
\overset{IH}{=} r \cdot r^n \cdot \log_V(1 + x)
= r^{n+1}\log_V(1 + x).
\]

As these two series also satisfy the same functional equation
\[
\Rightarrow \log_V(1 + x) = \log_{V_{n+1}}(1 + x).
\]
Proposition 4.2.8. Let $c_V(T)$ be the coefficient of $T \in \mathbb{P}$ in the planar logarithm series $\log_V(1 + x)$. Then

$$\sum_{T \in \mathbb{P}, |T| = n} c_V(T) = (-1)^{n-1} \cdot \frac{1}{n}.$$ 

Proof. By Proposition 2.5.7 the proof follows immediately as $\eta(\log_V(1 + x))$ is the classical logarithm. \qed

4.3 Planar $\lambda$-deformed exponential series

Let $\lambda : L(V) \to K$ be a map on the set of leaves of $V$.

Let $\lambda_i = \lambda(b_i)$, where $b_i$ is the $i$-th leave of $T$. Then one can write $\lambda = (\lambda_1, ..., \lambda_n)$.

Moreover we define

$$\delta_r = \delta_r(\lambda) := (\lambda_1 + ... + \lambda_n)^r - (\lambda_1^r + ... + \lambda_n^r),$$

for all $r \geq 2$.

Proposition 4.3.1. Let $V \in \mathbb{P}$ with $n = |V|$ and $\lambda = (\lambda_1, ..., \lambda_n)$ with $\delta_r(\lambda) \neq 0$, $\forall r \geq 2$. Then there is a unique power series $\exp_{(V, \lambda)}(x) \in K \{\{x\}\}$, such that

(i) $\text{ord}(\exp_{(V, \lambda)}(x) - (1 + x)) \geq 2$,

(ii) $\cdot_V \left( \exp_{(V, \lambda)}(\lambda_1 \cdot x), ..., \exp_{(V, \lambda)}(\lambda_n \cdot x) \right) = \exp_{(V, \lambda)}\left( (\lambda_1 + ... + \lambda_n) \cdot x \right)$.

Moreover $\frac{d}{dx}(\exp_{(V, \lambda)}(x)) = \exp_{(V, \lambda)}(x)$. $\exp_{(V, \lambda)}(x)$ is called the planar $\lambda$-deformed exponential series with respect to $V$.

Theorem 4.3.2. Let $V \in \mathbb{P}$ with $|V| = n$. Then for the coefficient $a_{(V, \lambda)}(T)$ at $T$ of the exponential series $\exp_{(V, \lambda)}(x)$ with $\lambda = (\lambda_1, ..., \lambda_n)$, $\delta_r(\lambda) \neq 0$ it is true that

$$a_{(V, \lambda)}(x) = 1$$

and

$$\delta_r(\lambda) \cdot a_{(V, \lambda)}(T) = \sum_{S \in O(T)} \binom{V}{T}^\lambda_s \cdot a_{(V, \lambda)}(T - S),$$

where $a_{(V, \lambda)}(T - S) = \prod_{i=1}^{m} a_{(V, \lambda)}(T_i)$, if $T - S = (T_1, ..., T_m)$ and $r_i = |T_i|$. 

4.3 Planar $\lambda$-deformed exponential series

Proof. (1) Let $\exp_{(V,\lambda)}(x) = E \in K\{(x)\}$. Then

$$\cdot_V\left(E(\lambda_1 x), ..., E(\lambda_n x)\right)$$

$$= \sum_{U \in \mathcal{P}'} \sum_{S \in O(U)} \left(V\right)_U^S \cdot a_{(V,\lambda)}(U - S) \cdot x_U,$$

by Proposition 1.5.7, where $a_{(V,\lambda)}(T) = \text{coeff}_T(E(x))$.

Further for $T \in \mathcal{P}$ with $1 \leq ar(T) \leq n$ and $r = |T|$ it holds that

$$\text{coeff}_T\left(\cdot_V\left(E(\lambda_1 x), ..., E(\lambda_n x)\right)\right) = \delta_r \cdot a_{(V,\lambda)}(T) + (\lambda_1^r + ... + \lambda_n^r) \cdot a_{(V,\lambda)}(T),$$

where $T - S = (T_1, ..., T_m)$ is a planar rooted forest.

If $ar(T) > n$ or $ar(T_i) > n$ for any $T_i \in P(T)$, then

$$\text{coeff}_T\left(\cdot_V\left(E(\lambda_1 x), ..., E(\lambda_n x)\right)\right) = 0.$$

(2) Define a map $a_{(V,\lambda)} : \mathcal{P} \to K$ as in Theorem 4.3.2.

Let $E = \sum_{r=0}^{\infty} E_r := \sum_{T \in \mathcal{P}} a_{(V,\lambda)}(T) \cdot T(x)$, i.e. $\text{coeff}_T(E(x)) = a_{(V,\lambda)}(T)$.

Then for $T \in \mathcal{P}$ it holds that

$$\text{coeff}_T\left(\cdot_V\left(E(\lambda_1 x), ..., E(\lambda_n x)\right)\right)$$

$$= \text{coeff}_T\left(\sum_{U \in \mathcal{P}'} \sum_{S \in O(U)} \left(V\right)_U^S \cdot a_{(V,\lambda)}(U - S) \cdot x_U\right)$$

$$= \sum_{\substack{S \in O(T) \\ S \neq x}} \left(V\right)_T^S \cdot \prod_{i=1}^{m} a_{(V,\lambda)}(T_i) + (\lambda_1^r + ... + \lambda_n^r) \cdot a_{(V,\lambda)}(T),$$

$$= \delta_r \cdot a_{(V,\lambda)}(T) + (\lambda_1^r + ... + \lambda_n^r) \cdot a_{(V,\lambda)}(T)$$

As $\text{coeff}_T\left(\exp_{(V,\lambda)}((\lambda_1 + ... + \lambda_n) \cdot x)\right) = (\lambda_1 + ... + \lambda_n)^r \cdot a_{(V,\lambda)}(T)$ one obtains the equation

$$\cdot_V\left(\exp_{(V,\lambda)}(\lambda_1 x), ..., \exp_{(V,\lambda)}(\lambda_n x)\right) = \exp_{(V,\lambda)}((\lambda_1 + ... + \lambda_n) \cdot x).$$
(3) The uniqueness follows immediately from (1).
From the computations in (2) it follows that \( a_{(V,\lambda)}(T) = \text{coeff}_T(E) \), for all \( T \in \mathbb{P} \) and thus the claim in the Theorem is true.

(4) Let \( exp_{(V,\lambda)}(x) = E \in K\{\{x\}\} \) and \( E = \sum_{r=0}^{\infty} E_r \), where \( E_r \) is the homogeneous component of degree \( r \), \( \forall \ r \geq 0 \). Then

\[
E_r(x) = \delta_r^{-1} \cdot \left[ \sum_{T \in \mathbb{P}} \sum_{S \in O(T)} \left( \frac{V}{T} \right)^{\lambda} \cdot a_{(V,\lambda)}(T - S) \cdot x^T \right]
\]

\[
= \delta_r^{-1} \cdot \left[ \sum_{m=2}^{n} \sum_{S \in \mathbb{P}} \sum_{S \neq x} 1 \leq i_1 < ... < i_m \leq n \lambda_{i_1}^{r_1} \cdot ... \cdot \lambda_{i_m}^{r_m} \cdot \left( E_{r_1}(x), ..., E_{r_m}(x) \right) \right],
\]

where \( m = |S| \) and \((i_1, ..., i_m) \in I_S\). Moreover

\[
\frac{d}{dx}(E_r(x)) = \delta_r^{-1} \cdot \left[ \sum_{m=2}^{n} \sum_{S \in \mathbb{P}} \sum_{r_1 + ... + r_m = r \text{ and } r_j \neq 1} 1 \leq i_1 < ... < i_m \leq n \lambda_{i_1}^{r_1} \cdot ... \cdot \lambda_{i_m}^{r_m} \cdot \left( \sum_{j=1}^{m} \cdot S \left( E_{r_1}(x), ..., \hat{E}_{r_j}(x), ..., E_{r_m}(x) \right) \right) \right]
\]

\[
= \delta_r^{-1} \cdot \left[ \sum_{m=2}^{n} \sum_{S \neq x} \sum_{r_1 + ... + r_m = r \text{ and } r_j = 1} 1 \leq i_1 < ... < i_m \leq n \lambda_{i_1}^{r_1} \cdot ... \cdot \lambda_{i_m}^{r_m} \cdot \left( \sum_{j=1}^{m} \cdot T \left( E_{r_1}(x), ..., \hat{E}_{r_j}(x), ..., E_{r_m}(x) \right) \right) \right]
+ \sum_{m=2}^{n} \sum_{S \neq x} \sum_{r_1 + ... + r_m = r \text{ and } r_j = 1} 1 \leq i_1 < ... < i_m \leq n \lambda_{i_1}^{r_1} \cdot ... \cdot \lambda_{i_m}^{r_m} \cdot \left( \sum_{j=1}^{m} \cdot T \left( E_{r_1}(x), ..., \hat{E}_{r_j}(x), ..., E_{r_m}(x) \right) \right)
\]

where \( T = \text{Red}(S - (j - \text{th leave})), |T| = m - 1 \) and \((E_1, ..., \hat{E}_j, ..., E_m) = (E_1, ..., E_{j-1}, E_{j+1}, ..., E_m)\).
\[\begin{align*}
\delta_r^{-1} & \cdot \left[ \sum_{r_1 + \ldots + r_n = r} \lambda_1^{r_1} \cdot \ldots \lambda_n^{r_n} \cdot \left( \sum_{j=1}^{m} \cdot T \left( E_{r_1}, \ldots, E_{r_j-1}, \ldots, E_{r_m} \right) \right) \right] \\
& + \sum_{m=2}^{n-1} \sum_{S,S \neq x,S \neq V} \sum_{r_1 + \ldots + r_m = r} \lambda_1^{r_1} \cdot \ldots \lambda_m^{r_m} \cdot \left( \sum_{j=1}^{m} \cdot S \left( E_{r_1}, \ldots, E_{r_j-1}, \ldots, E_{r_m} \right) \right) \\
& + \sum_{m=3}^{n-1} \sum_{S,S \neq x,S \neq x^2} \sum_{r_1 + \ldots + r_m = r} \lambda_1^{r_1} \cdot \ldots \lambda_j^{r_j} \cdot \ldots \lambda_m^{r_m} \cdot \left( \sum_{j=1}^{m} \cdot T \left( E_{r_1}, \ldots, E_{r_j-1}, \ldots, E_{r_m} \right) \right) \\
& \quad \cdot E_{r-1} \right] \\
& = \delta_r^{-1} \cdot \left[ \left( \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1} \cdot \lambda_{i_2}^{r-1} + \lambda_{i_1}^{r-1} \cdot \lambda_{i_2} \right) \cdot E_{r-1} \right] \\
& + \left( \sum_{j=1}^{n} \lambda_j \right) \cdot \left( \sum_{r_1 + \ldots + r_n = r-1} \lambda_1^{r_1} \cdot \ldots \lambda_n^{r_n} \cdot S \left( E_{r_1}, \ldots, E_{r_n} \right) \right) \\
& + \sum_{m=2}^{n-1} \sum_{S,S \neq x,S \neq V} \sum_{j=1}^{m} \lambda_j \cdot \left( \sum_{r_1 + \ldots + r_m = r-1} \lambda_1^{r_1} \cdot \ldots \lambda_m^{r_m} \cdot S \left( E_{r_1}, \ldots, E_{r_m} \right) \right) \\
& + \sum_{m=2}^{n-1} \sum_{S,S \neq x,S \neq x^2} \sum_{j=1}^{m} \lambda_j \cdot \left( \sum_{r_1 + \ldots + r_m = r-1} \lambda_1^{r_1} \cdot \ldots \lambda_m^{r_m} \cdot S \left( E_{r_1}, \ldots, E_{r_m} \right) \right),
\end{align*}\]

where in the last row an index shift and a renaming from \( T \) to \( S \) was made.
= \delta_r^{-1} \cdot \left[ \left( \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1} \cdot \lambda_{i_2}^{r-1} + \lambda_{i_1}^{r-1} \cdot \lambda_{i_2} \right) \cdot E_{r-1} \right.
+ \left( \sum_{j=1}^{n} \lambda_j \right) \cdot \left( \sum_{m=2}^{r} \sum_{S,S \neq x_{r_1}+...+x_{r_m} \neq r-1}^{n} \lambda_{i_1}^{r_1} \cdot ... \cdot \lambda_{i_m}^{r_m} \cdot S(x_{r_1}, ..., x_{r_m}) \right)
\left. \right] \cdot E_{r-1}
= \delta_r^{-1} \cdot \left[ \left( \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1} \cdot \lambda_{i_2}^{r-1} + \lambda_{i_1}^{r-1} \cdot \lambda_{i_2} \right) \cdot E_{r-1} \right.
+ \left( \sum_{j=1}^{n} \lambda_j \right) \cdot \delta_{r-1} \cdot E_{r-1}
\left. \right] \cdot E_{r-1}
= E_{r-1},

as

\left( \sum_{j=1}^{n} \lambda_j \right) \cdot \delta_{r-1} \quad = \quad (\lambda_1 + ... + \lambda_n) \cdot ((\lambda_1 + ... + \lambda_n)^{r-1} - (\lambda_1^{r-1} + ... + \lambda_n^{r-1}))

= \quad (\lambda_1 + ... + \lambda_n)^{r} - (\lambda_1 + ... + \lambda_n) \cdot (\lambda_1^{r-1} + ... + \lambda_n^{r-1})

= \quad (\lambda_1 + ... + \lambda_n)^{r} - (\lambda_1^{r} + ... + \lambda_n^{r})

= \quad \left( \sum_{1 \leq i_1 < i_2 \leq n} (\lambda_{i_1} \cdot \lambda_{i_2}^{r-1} + \lambda_{i_1}^{r-1} \cdot \lambda_{i_2}^{r}) \right)
= \quad \delta_r - \left( \sum_{1 \leq i_1 < i_2 \leq n} (\lambda_{i_1} \cdot \lambda_{i_2}^{r-1} + \lambda_{i_1}^{r-1} \cdot \lambda_{i_2}^{r}) \right).

\text{Example 4.3.3. (i) Let } V = x \cdot x^2 \text{ and } \lambda = (2, 1, 3). \text{ Then}

\exp_{(x, x^2, \lambda)}(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} \cdot \left( \frac{41}{60} x \cdot x^2 + \frac{19}{60} x^2 \cdot x \right)
+ \frac{1}{4!} \cdot \left( \frac{6643}{17970} x \cdot (x \cdot x^2) + \frac{287}{3594} (x \cdot x^2) \cdot x + \frac{2657}{17970} x \cdot (x^2 \cdot x) \right.
\left. + \frac{133}{3594} (x^2 \cdot x) \cdot x + \frac{219}{599} x^2 \cdot x^2 \right) + \text{higher terms.}
4.3 Planar $\lambda$-deformed exponential series

(ii) Let now $V = (x \cdot x^3) \cdot x$ and $\lambda = (3, 1, 4, 2, 1)$. Then

\[
\exp_{(x \cdot x^3) \cdot x}(\lambda)(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} \left( \frac{193}{410} x \cdot x^2 + \frac{201}{410} x^2 \cdot x + \frac{8}{205} x^3 \right)
\]

\[
+ \frac{1}{4!} \left( \frac{65703}{488105} x \cdot (x^2 \cdot x) + \frac{102243}{488105} (x \cdot x^2) \cdot x + \frac{69051}{488105} x \cdot (x^2 \cdot x) \right)
\]

\[
+ \frac{69751}{488105} (x^2 \cdot x) \cdot x + \frac{21456}{488105} x \cdot x^3 + \frac{8816}{488105} x^3 \cdot x + \frac{625}{2381} x^2 \cdot x^2
\]

\[
+ \frac{32}{2381} x \cdot x^2 + \frac{64}{2381} x^2 \cdot x + \frac{16}{2381} x^2 \cdot x \right) + \text{higher terms.}
\]

Remark 4.3.4. Let $V \in \mathbb{P}$ with $|V| = n$, $\lambda = (\lambda_1, ..., \lambda_n)$ with $\delta_r(\lambda) \neq 0$, $\forall r$ and $\rho \in \mathbb{Q}$. Then

\[
\exp_{(V, \lambda)}(x) = \exp_{(V, \rho \cdot \lambda)}(x),
\]

where $\rho \cdot \lambda = (\rho \cdot \lambda_1, ..., \rho \cdot \lambda_n)$. This is true as

\[
\delta_r(\rho \cdot \lambda) = (\rho \cdot \lambda_1 + ... + \rho \cdot \lambda_n)^r - ((\rho \cdot \lambda_1)^r + ... + (\rho \cdot \lambda_n)^r)
\]

\[
= \rho^r \cdot ((\lambda_1 + ... + \lambda_n)^r - (\lambda_1^r + ... + \lambda_n^r)) = \rho^r \cdot \delta_r(\lambda)
\]

and

\[
\delta_r(\rho \cdot \lambda) \cdot a_{(V, \rho \cdot \lambda)}(T) = \rho^r \cdot \delta_r(\lambda) \cdot a_{(V, \rho \cdot \lambda)}(T)
\]

\[
= \sum_{S} \sum_{r_1 + ... + r_m = r \atop 1 \leq t_1 < ... < t_m \leq n} (\rho \cdot \lambda_{t_1})^{r_1} \cdot ... \cdot (\rho \cdot \lambda_{t_m})^{r_m} \cdot a_{(V, \rho \cdot \lambda)}(T - S)
\]

\[
= \sum_{S} \sum_{r_1 + ... + r_m = r \atop 1 \leq t_1 < ... < t_m \leq n} \rho^{r_1} \cdot ... \cdot \rho^{r_m} \cdot \lambda_{t_1}^{r_1} \cdot ... \cdot \lambda_{t_m}^{r_m} \cdot a_{(V, \rho \cdot \lambda)}(T - S)
\]

\[
= \rho^r \cdot \left( \sum_{S} \sum_{r_1 + ... + r_m = r \atop 1 \leq t_1 < ... < t_m \leq n} \lambda_{t_1}^{r_1} \cdot ... \cdot \lambda_{t_m}^{r_m} \cdot a_{(V, \rho \cdot \lambda)}(T - S) \right).
\]

Thus we get

\[
a_{(V, \rho \cdot \lambda)}(T) = a_{(V, \lambda)}(T),
\]

because

\[
a_{(V, \lambda)}(x) = a_{(V, \rho \cdot \lambda)}(x) = 1
\]

and the other coefficients are generated by this initial value and do not depend on $\rho$.

Proposition 4.3.5. Let $V, V_1, ..., V_k \in \mathbb{P}$ with $k = |V|$. Then there is a unique power series $\exp_{(V, V_1, ..., V_k)}(x) \in K\{x\}$, such that

(i) $\text{ord}(\exp_{(V, V_1, ..., V_k)}(x) - (1 + x)) \geq 2$,
Moreover \( \frac{d}{dx}(\exp(V_1, ..., V_k)(x)) = \exp(V_1, ..., V_k)(x) \).

\( \exp(V_1, ..., V_k)(x) \) is called the product of planar exponential series with respect to \( V_1, ..., V_k \) over \( V \).

**Theorem 4.3.6.** Let \( T, V, V_1, ..., V_k \in \mathbb{P} \) with \( k = |V| \). Then for the coefficient \( a_{(V,V_1,\ldots,V_k)}(T) \) of \( \exp(V_1, ..., V_k)(x) \) it is true that

\[
a_{(V,V_1,\ldots,V_k)}(x) = 1
\]

and

\[
a_{(V,V_1,\ldots,V_k)}(T) = \frac{\sum_{i=1}^{k} a_{V_i}(T) + \sum_{S} a_{V,S}(T - S)}{k^{n}}
\]

where \( n = |T|, S \in O(T), S \neq x, |S| = m \geq 2 \) and \( a_{V,S}(T - S) = \sum_{t_{i} \in I_{S}} a_{V_{t_{i}}}(T_{i}) \),

where \( V|S = \{I_{S} \subseteq L(V) : V|I_{S} = S\} \) and \( I_{S} = \{(r_1, ..., r_m) \in \mathbb{N}^m : r_i \in L(V)\} \).

The set of leaves \( L(V) \) of \( V \) is ordered naturally from left (1) to right (= k).

**Proof.** (1) Let \( V, V_1, ..., V_k \in \mathbb{P} \) with \( k = |V| \) and \( \exp_{V_i}(x) = \sum_{T \in \mathbb{P}} a_{V_i}(T) \cdot T(x) = E_i(x) \in K\{x\}, \forall 1 \leq i \leq k \). Then

\[
\cdot_{V}(E_1(x), ..., E_k(x)) = \sum_{S} \sum_{i \in V|S} \cdot_{S}(E_{r_1}(x), ..., E_{r_m}(x)),
\]

where \( S \in P(V), 2 \leq m = |S|, V|S = \{I \subseteq L(V) : V|I = S\} \) and \( I = \{(r_1, ..., r_m) \in \mathbb{N}^m : r_i \in L(V)\} \).

For \( T \in \mathbb{P} \) with \( 1 \leq ar(T) \leq k \) and \( n = |T| \) it holds that

\[
coef_{T}\left(\cdot_{V}(E_1(x), ..., E_k(x))\right)
\]

\[
= \sum_{S} \sum_{I \in V|S} coef_{(T-S)}\left(\cdot_{S}(E_{r_1}(x), ..., E_{r_m}(x))\right) + coef_{T}\left(\sum_{i=1}^{k} E_i(x)\right),
\]

where \( T - S = (T_1, ..., T_m) \) is a planar rooted forest.

If \( ar(T) > k \) or \( ar(T_i) > k \) one \( T_i \in P(V) \), then

\[
coef_{T}\left(\cdot_{V}(E_1(x), ..., E_k(x))\right) = 0.
\]

(2) Define a map \( a_{(V,V_1,\ldots,V_k)} : \mathbb{P} \to K \) as in Theorem 4.3.6.

Let \( E_i(x) = \sum_{T \in \mathbb{P}} a_{V_i}(T) \cdot T(x), \forall i \)
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and $E_{(V, v_1, \ldots, v_k)}(x) = \sum_{T \in \mathbb{P}} a_{(V, v_1, \ldots, v_k)}(T) \cdot T(x)$.

Then it holds for $T \in \mathbb{P}$:

$$
\text{coef}_{f_T} \left( \cdot_V (E_1(x), \ldots, E_k(x)) \right)
= \text{coef}_{f_T} \left( \sum_S \sum_{I \in V \setminus S} \left( \cdot_S (E_{r_1}(x), \ldots, E_{r_m}(x)) \right) \right) + \text{coef}_{f_T} \left( \sum_{i=1}^k E_i(x) \right)
= \sum_S \sum_{I \in V \setminus S} \text{coef}_{f_{T-S}} \left( \cdot_S (E_{r_1}(x), \ldots, E_{r_m}(x)) \right) + \text{coef}_{f_T} \left( \sum_{i=1}^k E_i(x) \right)
= \sum_S \prod_{i=1}^m \text{coef}_{f_{T_i}}(E_{r_i}(x)) + \sum_{i=1}^k \text{coef}_{f_T}(E_i(x))
= \sum_S \prod_{i=1}^m a_{V_i}(T_i) + \sum_{i=1}^k a_{V_i}(T)
= k^n \cdot a_{(V, v_1, \ldots, v_k)}(T).
$$

As $\text{coef}_{f_T}(E_{(V, v_1, \ldots, v_k)}(kx)) = k^n \cdot a_{(V, v_1, \ldots, v_k)}(T)$ one obtains the equation

$$
\cdot_V (E_1(x), \ldots, E_k(x)) = E_{(V, v_1, \ldots, v_k)}(kx).
$$

(3) From the computations in (2) it follows that the claim in the Theorem 4.3.6 is true.

The uniqueness follows immediately from (1).

(4)

$$
E'(kx) = \sum_{i=1}^k \cdot_V (E_1(x), \ldots, E_i'(x), \ldots, E_k(x))
= \sum_{i=1}^k \cdot_V (E_1(x), \ldots, E_i(x), \ldots, E_k(x))
= k \cdot \cdot_V (E_1(x), \ldots, E_k(x)) = k \cdot E(kx).
$$

Thus

$$
E'(x) = E(x).
$$
Example 4.3.7. Let \( V = x \cdot x^2 \cdot x, V_1 = x \cdot x^2, V_2 = x^2, V_3 = x^3, V_4 = x \cdot x^3 \). Then

\[
\exp(V, V_1, V_2, V_3, V_4)(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} \left( \frac{261}{640} x \cdot x^2 + \frac{511}{1280} x^2 \cdot x + \frac{247}{1280} x^3 \right) \\
+ \frac{1}{4!} \left( \frac{11307}{116480} x \cdot (x \cdot x^2) + \frac{103}{1040} (x \cdot x^2) \cdot x + \frac{19277}{232960} x \cdot (x^2 \cdot x) \\
+ \frac{20281}{232960} (x^2 \cdot x) \cdot x + \frac{9127}{698880} x \cdot x^3 + \frac{2861}{698880} x^3 \cdot x \\
+ \frac{5581}{232960} x^2 \cdot x + \frac{6607}{698880} x \cdot x^2 + \frac{13159}{698880} x \cdot x^2 \cdot x \\
+ \frac{6607}{698880} x^2 \cdot x \cdot x \right) + \text{higher terms.}
\]

4.4 Planar \( \lambda \)-deformed logarithm series

As in case of the planar exponential series, we can define the planar \( \lambda \)-deformed logarithm series with respect to a rooted tree \( V \). But here we have two possibilities. The first one is the planar logarithm series which is inverse to the corresponding exponential series with respect to \( \circ \). The second one is the planar logarithm series which satisfies a functional equation like in the classical case. The notations are the same as in the above section.

Proposition 4.4.1. Let \( V \in \mathbb{P} \) with \( |V| = n \), \( \lambda = (\lambda_1, \ldots, \lambda_n) \) with \( \delta_r(\lambda) \neq 0 \), \( \forall r \geq 2 \) and \( \exp_{(V, \lambda)}(x) \) the planar \( \lambda \)-deformed exponential series. Then there is a unique power series \( \log_{(V, \lambda)}(1 + x) \in K\{\{x\}\} \), such that

\[
\exp_{(V, \lambda)}(\log_{(V, \lambda)}(1 + x)) = 1 + x
\]

and

\[
\log_{(V, \lambda)}(\exp_{(V, \lambda)}(x)) = x.
\]

Moreover \((1 + x) \frac{d}{dx} \log_{(V, \lambda)}(1 + x) = 1\).

\(\log_{(V, \lambda)}(1 + x)\) is called the planar \( \lambda \)-deformed logarithm series with respect to \( V \) of the first kind.

Proof. The derivative property follows immediately by using Proposition 3.3.1 and the fact that \( \log_{(V, \lambda)}(1 + x) \) is the inverse of \( \exp_{(V, \lambda)}(x) \) and that \( \frac{d}{dx}(\exp_{(V, \lambda)}(x)) = \exp_{(V, \lambda)}(x) \). \(\square\)

Theorem 4.4.2. Let \( V \in \mathbb{P} \) with \( |V| = n \). Then for the coefficients \( c_{(V, \lambda)}(T) \) of \( T \) of the planar \( \lambda \)-deformed logarithm series of the first kind \( \log_{(V, \lambda)}(1 + x) \) it is true that

\[
c_{(V, \lambda)}(x) = 1
\]
and
\[ c_{(V,\lambda)}(T) = \sum_{r \geq 1} (-1)^r \sum_{S \in \Omega_r(T)} \hat{a}_S, \]
where
\[ \hat{a}_S = \prod_{i=1}^{r-1} a_{(V,\lambda)}(S_i - S_{i+1}), \]
and
\[ a_{(V,\lambda)}(S_i - S_{i+1}) = \prod_{i=1}^{n} a_{(V,\lambda)}(F_i), \]
if \( S \in \Omega_r(T), S = (S_0, ..., S_r) \) and \( S_i - S_{i+1} = (F_1, ..., F_n) \).

**Proof.** This follows immediately by Proposition 3.2.3. \( \square \)

**Proposition 4.4.3.** Let \( V \in \mathbb{P} \) with \( |V| = n \) and \( \lambda = (\lambda_1, ..., \lambda_n) \) with \( \delta_r(\lambda) \neq 0, \forall r \geq 2 \). Then there is a unique power series \( \text{Log}_{(V,\lambda)}(1 + x) \in K\{\{x\}\} \), such that
\[ \sum_{i=1}^{n} \text{Log}_{(V,\lambda)}(1 + \lambda_i x) = \text{Log}_{(V,\lambda)} \left( \cdot_V ((1 + \lambda_1 x), ..., (1 + \lambda_n x)) \right). \]

\( \text{Log}_{(V,\lambda)}(1 + x) \) is called the planar \( \lambda \)-deformed logarithm series with respect to \( V \) of the second kind.

**Theorem 4.4.4.** Let \( V \in \mathbb{P} \) with \( |V| = n \). Then for the coefficients \( C_{(V,\lambda)}(T) \) of \( T \) of the planar \( \lambda \)-deformed logarithm series of the second kind \( \text{Log}_{(V,\lambda)}(1 + x) \) with \( \lambda = (\lambda_1, ..., \lambda_n), \delta_r(\lambda) \neq 0, \forall r \geq 2 \) it is true that
\[ C_{(V,\lambda)}(x) = 1 \]
and
\[ \delta_r(\lambda) \cdot C_{(V,\lambda)}(T) = - \left( \sum_{S \in \Omega(T)} C_{(V,\lambda)}(S) \cdot \prod_{i=1}^{m} \left( \frac{V}{T_i} \right)_\lambda \right), \]
where \( r = |T|, m = |S|, T - S = (T_1, ..., T_m) \) and \( \left( \frac{V}{T} \right)_\lambda \) is the planar \( \lambda \)-deformed binomial coefficient of the first kind.
The summand is equal to 0 if there is a \( T_i \) with \( T_i \notin P(V) \).

**Proof.** On the left side it holds
\[ \text{coef}_{f_T} \left( \sum_{i=1}^{n} \text{Log}_{(V,\lambda)}(1 + \lambda_i x) \right) = (\lambda^*_1 + \cdots + \lambda^*_n) \cdot C_{(V,\lambda)}(T). \]
On the right side it holds
\[ \text{Log}_{(V,\lambda)} \left( \cdot_V ((1 + \lambda_1 x), ..., (1 + \lambda_n x)) \right) \]
\[ \log_{(V,\lambda)}(1 + x) = Log_{(V,\lambda)} \left( \sum_{U \in P'} \frac{V}{U} \cdot x^U \right), \]

see Proposition 1.5.5. Thus by Proposition 3.1.3

\[ coef_T \left( \log_{(V,\lambda)} \left( (1 + \lambda_1 x), \ldots, (1 + \lambda_n x) \right) \right) = coef_T \left( \log_{(V,\lambda)} \left( \sum_{U \in P'} \frac{V}{U} \cdot x^U \right) \right) \]

\[ = \sum_{S \in O(T)} C_{(V,\lambda)}(S) \cdot \left( \frac{V}{T - S} \right) \lambda \]

\[ = \sum_{S \in O(T)} C_{(V,\lambda)}(S) \cdot \prod_{i=1}^{m} \left( \frac{V}{T_i} \right) \lambda \]

\[ = -\delta_\epsilon(\lambda) \cdot C_{(V,\lambda)}(T) + (\lambda_1 + \ldots + \lambda_n)^r \cdot C_{(V,\lambda)}(T) \]

\[ = -\left( (\lambda_1 + \ldots + \lambda_n)^r - (\lambda_1^r + \ldots + \lambda_n^r) \right) \cdot C_{(V,\lambda)}(T) + (\lambda_1 + \ldots + \lambda_n)^r \cdot C_{(V,\lambda)}(T) \]

\[ = (\lambda_1^r + \ldots + \lambda_n^r) \cdot C_{(V,\lambda)}(T). \]

\[ \square \]

**Remark 4.4.5.** (1) The planar \( \lambda \)-deformed logarithm series \( \log_{(V,\lambda)}(1 + x) \) of the first kind does not satisfy any visible and known functional equation.

(2) The logarithm series \( \log_{(V,\lambda)}(1 + x) \) of the second kind does not satisfy any visible and known derivative property. Moreover it is not inverse to \( \exp_{(V,\lambda)}(x) \) with respect to \( \circ \). But it holds

\[ \eta \left( \log_{(V,\lambda)}(\exp_{(V,\lambda)}(x)) \right) = x \quad \text{and} \quad \eta \left( \exp_{(V,\lambda)}(\log_{(V,\lambda)}(1 + x)) \right) = 1 + x, \]

where \( \eta \) is the projection on the classical algebra of power series \( K[[x]] \).

(3) It is possible to compute the inverse series of \( \log_{(V,\lambda)}(1 + x) \) with respect to \( \circ \), but the inverse series does not have any visible and interesting properties neither a functional equation nor derivative property.

**Example 4.4.6.** (i) Let \( V = x \cdot x^2 \) and \( \lambda = (2, 1, 3) \). Then

\[ \log_{(V,\lambda)}(1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} \cdot \left( \frac{49}{120} x \cdot x^2 + \frac{71}{120} x^2 \cdot x \right) \]

\[ - \frac{1}{4} \cdot \left( \frac{2287}{21564} x \cdot (x \cdot x^2) + \frac{3881}{21564} (x \cdot x^2) \cdot x \right) \]

\[ + \frac{20627}{107820} x \cdot (x^2 \cdot x) + \frac{31813}{107820} (x^2 \cdot x) \cdot x + \frac{409}{1797} x^2 \cdot x^2 \]

\[ + \text{higher terms}. \]
and

\[
\log_{(V,\lambda)}(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3} \left( \frac{9}{20}x \cdot x^2 + \frac{11}{20}x^2 \cdot x \right) \\
- \frac{1}{4} \cdot \left( \frac{414}{2995}x \cdot (x \cdot x^2) + \frac{546}{2995}(x \cdot x^2) \cdot x \\
+ \frac{594}{2995}x \cdot (x^2 \cdot x) + \frac{726}{2995}(x^2 \cdot x) \cdot x + \frac{143}{599}x^2 \cdot x^2 \right) \\
+ \text{ higher terms.}
\]

(ii) Let now \( V = x \cdot x^2 \) and \( \lambda = \left( \frac{1}{2}, -2, \frac{1}{3} \right) \). Then

\[
\log_{(V,\lambda)}(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3} \left( \frac{29}{50}x \cdot x^2 + \frac{21}{50}x^2 \cdot x \right) \\
- \frac{1}{4} \cdot \left( \frac{2121}{12800}x \cdot (x \cdot x^2) + \frac{1729}{12800}(x \cdot x^2) \cdot x \\
+ \frac{1421}{12800}x \cdot (x^2 \cdot x) + \frac{1029}{12800}(x^2 \cdot x) \cdot x + \frac{65}{128}x^2 \cdot x^2 \right) \\
+ \text{ higher terms.}
\]

**Remark 4.4.7.** Let \( V \in \mathbb{P} \) with \( |V| = n \), \( \lambda = (\lambda_1, ..., \lambda_n) \) with \( \delta_r(\lambda) \neq 0 \), \( \forall r \) and \( \rho \in \mathbb{Q} \). Then

\[
\log_{(V,\lambda)}(1 + x) = \log_{(V,\rho \cdot \lambda)}(1 + x),
\]

where \( \rho \cdot \lambda = (\rho \cdot \lambda_1, ..., \rho \cdot \lambda_n) \). This is true as

\[
\delta_r(\rho \cdot \lambda) = (\rho \cdot \lambda_1 + ... + \rho \cdot \lambda_n)^r - ((\rho \cdot \lambda_1)^r + ... + (\rho \cdot \lambda_n)^r) \\
= \rho^r \cdot \left( (\lambda_1 + ... + \lambda_n)^r - (\lambda_1^r + ... + \lambda_n^r) \right) = \rho^r \cdot \delta_r(\lambda)
\]
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\[ \delta_r(\rho \cdot \lambda) \cdot C_{(V,\rho \cdot \lambda)}(T) = \rho^r \cdot \delta_r(\lambda) \cdot C_{(V,\lambda)}(T) \]

\[ = - \left( \sum_{S \in O(T)} C_{(V,\rho \cdot \lambda)}(S) \cdot \prod_{i=1}^m \left( \sum_{1 \leq k_1 < \ldots < k_{r_i} \leq n} (\rho \cdot \lambda_{k_1} \cdot \ldots \cdot (\rho \cdot \lambda_{k_{r_i}})) \right) \right) \]

\[ = - \left( \sum_{S \in O(T)} C_{(V,\rho \cdot \lambda)}(S) \cdot \prod_{i=1}^m \left( \rho^{r_i} \cdot \left( \sum_{1 \leq k_1 < \ldots < k_{r_i} \leq n} \lambda_{k_1} \cdot \ldots \cdot \lambda_{k_{r_i}} \right) \right) \right) \]

\[ = - \rho^r \cdot \left( \sum_{S \in O(T)} C_{(V,\rho \cdot \lambda)}(S) \cdot \prod_{i=1}^m \left( \sum_{1 \leq k_1 < \ldots < k_{r_i} \leq n} \lambda_{k_1} \cdot \ldots \cdot \lambda_{k_{r_i}} \right) \right) \]

Thus we get

\[ C_{(V,\rho \cdot \lambda)}(T) = C_{(V,\lambda)}(T), \]

because

\[ C_{(V,\lambda)}(x) = C_{(V,\rho \cdot \lambda)}(x) = 1 \]

and the other coefficients are generated by this initial value and do not depend on \( \rho \).

**Proposition 4.4.8.** Let \( V_1, \ldots, V_k \in \mathbb{P} \). Then there is a unique power series \( \log(V_1, \ldots, V_k)(1 + x) \in K\{\{x\}\} \), such that

(i) \( \sum_{i=1}^k \log(V_i)(1 + x) = k \cdot \log(V_1, \ldots, V_k)(1 + x) \).

(ii) \( (1 + x) \frac{d}{dx} (\log(V_1, \ldots, V_k)(1 + x)) = 1 \).

\( \log(V_1, \ldots, V_k)(1 + x) \) is called the planar logarithm series of sums with respect to \( V_1, \ldots, V_k \).

**Theorem 4.4.9.** Let \( T, V_1, \ldots, V_k \in \mathbb{P} \). Then for coefficients \( c_{(V_1, \ldots, V_k)}(T) \) of \( \log(V_1, \ldots, V_k)(1 + x) \) it holds

\[ c_{(V_1, \ldots, V_k)}(x) = 1 \]

and

\[ c_{(V_1, \ldots, V_k)}(T) = \frac{\sum_{i=1}^k c_{V_i}(T)}{k}. \]
Proof. (1) Let $V_1, \ldots, V_k \in \mathbb{P}$ and $\log V_i(1 + x) = \sum_{T \in \mathbb{P}} c_{V_i}(T) \cdot T(x) = L_i(1 + x) \in K\{\{x\}\}$, $\forall 1 \leq i \leq k$.

Define a map $c_{(V_1, \ldots, V_k)} : \mathbb{P} \to K$ as in Theorem 4.4.9.

Then for $T \in \mathbb{P}$

$$
\text{coeff}_T \left( \sum_{i=1}^{k} L_i(1 + x) \right) = \sum_{i=1}^{k} \text{coeff}_T (L_i(1 + x)) = \sum_{i=1}^{k} c_{V_i}(T).
$$

On the other side

$$
\text{coeff}_T \left( k \cdot \log_{(V_1, \ldots, V_k)}(1 + x) \right) = k \cdot \text{coeff}_T (\log_{(V_1, \ldots, V_k)}(1 + x))
$$

$$
= k \cdot c_{(V_1, \ldots, V_k)}(T) = \sum_{i=1}^{k} c_{V_i}(T).
$$

(2) Let

$$
\log_{(V_1, \ldots, V_k)}(1 + x) = \sum_{r=0}^{\infty} h_r =: h
$$

and for all $1 \leq i \leq k$

$$
\log V_i(1 + x) = \sum_{r=0}^{\infty} (h_i)_r =: h_i.
$$

Thus

$$
h_r = \sum_{|T|=r} c_{(V_1, \ldots, V_k)} \cdot T(x)
$$

and

$$(h_i)_r = \sum_{|T|=r} c_{V_i} \cdot T(x).
$$

It holds because of (1) that

$$
k \cdot h = \sum_{i=1}^{k} h_i
$$

and thus $\forall r$

$$
k \cdot h_r = \sum_{i=1}^{k} (h_i)_r.
$$

It follows

$$
k \cdot h'_r + 1 = \sum_{i=1}^{k} \frac{d}{dx} ((h_i)_r + 1) \overset{(a)}{=} \sum_{i=1}^{k} (-r(h_i)_r)
$$

$$
= -r \sum_{i=1}^{k} (h_i)_r = -r \cdot k \cdot h_r.
$$
(\ast) holds because of Corollary 4.2.2.

\[ h'_{r+1} = -rh_r. \]

Example 4.4.10. Let \( V_1 = x \cdot x^2, V_2 = x^2, V_3 = x \cdot (x \cdot x^2), V_4 = x \cdot x^3 \). Then

\[
\log_{(V_1, \ldots, V_4)}(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3} \cdot \left( \frac{143}{320}x \cdot x^2 + \frac{181}{320}x^2 \cdot x - \frac{1}{80}x^3 \right)
\]

\[ - \frac{1}{4} \cdot \left( \frac{1363}{10080}x \cdot (x \cdot x^2) + \frac{3559}{18720}(x \cdot x^2) \cdot x + \frac{3857}{18720}x \cdot (x^2 \cdot x) \right)
\]

\[ + \frac{551}{2080}(x^2 \cdot x) \cdot x + \frac{961}{4095}x^2 \cdot x^2 - \frac{1}{252}x \cdot x^3 - \frac{1}{126}x^3 \cdot x
\]

\[ - \frac{2}{315}x \cdot x \cdot x^2 - \frac{2}{315}x \cdot x^2 \cdot x - \frac{2}{315}x^2 \cdot x \cdot x
\]

+ higher terms.

Remark 4.4.11. This sum of planar logarithm series is not the inverse of the product of planar exponential series, but it holds

\[ \eta \left( \log_{(V_1, \ldots, V_k)}(\exp_{(V, V_1, \ldots, V_k)}(x)) \right) = x \]

and

\[ \eta \left( \exp_{(V, V_1, \ldots, V_k)}(\log_{(V_1, \ldots, V_k)}(1 + x)) \right) = 1 + x, \]

for \( \eta \) see Theorem 2.5.7.

The reason for the fact that these series are not inverse is that the product of exponential series depend on one more rooted tree \( V \) over which the product is taken but this rooted tree does not play any role in the sum of logarithm series, because the addition is associative and commutative.

So if the rooted trees \( V_1, \ldots, V_k \) are given we have only one sum of logarithm series, but we have \( C_k \) possible exponential series, where \( C_k \) is the \( k \)-th Super Catalan number or little Schröder number.

4.5 Planar hyperbolic sine and hyperbolic cosine

Let \( \Phi \subseteq \mathbb{P}' \) be the set of all planar rooted trees with an even degree. Then \( \Phi \) is a submagma of \( \mathbb{P} \).

Let \( \Psi := \mathbb{P} - \Phi \) be the set of all planar rooted trees with an odd degree. Then it holds that \( \Gamma := \Psi \cdot \Psi := \{ T_1 \cdot T_2 : T_1, T_2 \in \Psi \} \subseteq \Phi \).

Proposition 4.5.1. \( \Gamma \) is the unique minimal set of generators of the magma \( \Phi \).

Moreover \( \Phi \) regarded as a magma with unit is freely generated by \( \Gamma \).
4.5 Planar hyperbolic sine and hyperbolic cosine

Proof. Let \( \Phi^+ := \Phi - \{1\} \). The grafting multiplication is an injective map

\[
\Phi^+ \times \Phi^+ \rightarrow \Phi^+, 
\]

which image is the complement of \( \Gamma \) in \( \Phi^+ \). By induction over the degree it follows that \( \Gamma \) generates the magma \( \Phi^+ \) freely. \( \Box \)

**Definition 4.5.2.** The planar hyperbolic sine ”sinh” and hyperbolic cosine ”cosh” are defined in the following way

\[
\sinh_V(x) := \frac{1}{2}(\exp_V(x) - \exp_V(-x))
\]

and

\[
\cosh_V(x) := \frac{1}{2}(\exp_V(x) + \exp_V(-x)).
\]

**Remark 4.5.3.** As described in Chapter 2 there are odd and even power series. \( \sinh_V(x) \) is the odd part and \( \cosh_V(x) \) the even part of the exponential series \( \exp_V(x) \). Thus

\[
\sinh_V(x) = \sum_{T \in \Psi} a_V(T) \cdot T
\]

and

\[
\cosh_V(x) = \sum_{T \in \Phi} a_V(T) \cdot T,
\]

where \( a_V(T) \) is the coefficient of \( T \) in the planar exponential series with regard to \( V \).

**Theorem 4.5.4.** Let \( V,T \in \mathbb{P} \) and \( a_V(T_i) \) be the coefficient of \( T_i \) of the planar exponential series \( \exp_V(x) \), \( \forall i = 1,2 \). Then

\[
c_T(\sinh^2_V(x)) = \begin{cases} 
 a_V(T_1) \cdot a_V(T_2) & : T = T_1 \cdot T_2 \in \Gamma \\
 0 & : \text{else}
\end{cases}
\]

Proof. \( \sinh^2_V(x) \in \mathbb{P}_0 \), i.e. \( \sinh^2_V(x) \) is an even power series, and thus \( c_T(\sinh^2_V(x)) = 0, \forall T \in \Psi \). It is also obvious that \( c_2(\sinh^2_V(x)) = 0 \). Let now \( \Phi^+ = \Phi - \{1\}, T \in \Phi^+, T = T_1 \cdot T_2 \) with \( T_i \in \mathbb{P} \). Then

\[
c_T(\sinh^2_V(x)) = c_{T_1}(\sinh^2_V(x)) \cdot c_{T_2}(\sinh^2_V(x))
\]

\[
= \begin{cases} 
 a_V(T_1) \cdot a_V(T_2) & : T_1, T_2 \in \Psi \\
 0 & : \text{else}
\end{cases}
\]

Thus

\[
c_T(\sinh^2_V(x)) = a_V(T_1) \cdot a_V(T_2)
\]

if \( T \in \Gamma \). It follows that in \( \sinh^2_V(x) \) there are only trees of \( \Gamma \), because in \( \sinh_V(x) \) there are only trees of \( \Psi \). \( \Box \)
Theorem 4.5.5. Let $V, T \in \mathbb{P}$, $ar(T) = m$ and $a_V(T_i)$ be the coefficient of $T_i$ of the planar exponential series $exp_V(x)$, $\forall i = 1, 2$. Then

$$c_T(cosh_V^2(x)) = \begin{cases} 
1 & : T = \frac{1}{2} 
2 \cdot a_V(T) & : T \in \Gamma \text{ and } T \in \Phi^+ \text{ with } m > 2 
2 \cdot a_V(T) + a_V(T_1) \cdot a_V(T_2) & : T = T_1 \cdot T_2 \in \Phi^- - \Gamma \text{ with } m = 2 
0 & : T \in \Psi.
\end{cases}$$

Proof. Let $\Phi^+ = \Phi - \{1\}$ and $g(x) = \sum_{T \in \Phi^+} a_V(T) \cdot T$. Then

$$cosh_V(x) = 1 + g(x) \text{ and } cosh_V^2(x) = 1 + 2 \cdot g(x) + g^2(x).$$

As $cosh_V^2(x)$ is an even series it follows that $c_T(cosh_V^2(x)) = 0$, $\forall T \in \Psi$. It also holds that $c_T(cos^2_V(x)) = 1$.

Let now $T \in \Phi^+$. Then $c_T(g^2) = c_{T_1}(g) \cdot c_{T_2}(g)$, if $T = T_1 \cdot T_2$, $T_i \in \mathbb{P}$ and thus

$$c_T(g^2) = 0, \forall T \in \Gamma.$$

It also holds that $c_T(g^2) = 0$ if $ar(T) > 2$, because the coefficient of $g^2$ is only defined for trees of arity 2. Thus

$$c_T(cosh_V^2(x)) = 2 \cdot a_V(T)$$

for $T \in \Gamma$ and for $T \in \Phi^+$ with $ar(T) > 2$.

But if $T \in \Phi^+ - \Gamma, ar(T) = 2$, then $T = T_1 \cdot T_2$ with $T_i \in \Phi^+$ and

$$c_T(g^2) = c_{T_1}(g) \cdot c_{T_2}(g) = a_V(T_2) \cdot a_V(T_2).$$

Thus

$$c_T(cosh_V^2(x)) = 2 \cdot c_T(g) + c_T(g^2)$$

$$= 2 \cdot a_V(T) + a_V(T_1) \cdot a_V(T_2).$$

Remark 4.5.6. Like in the case of the planar exponential series it is possible to define the planar $\lambda$-deformed hyperbolic sine and cosine. The coefficients $c_T(sinh_{(V,\lambda)}(x))$, resp. $c_T(cosh_{(V,\lambda)}(x))$ and $c_T(sinh^2_{(V,\lambda)}(x))$, resp. $c_T(cosh^2_{(V,\lambda)}(x))$ are defined in the same way as the coefficients of the planar hyperbolic sine and cosine, but instead of the coefficients $a_V(T)$ of the planar exponential series $exp_V(x)$ we use the coefficients $a_{(V,\lambda)}(T)$ of the planar $\lambda$-deformed exponential series $exp_{(V,\lambda)}(x)$.\qed
4.6 Planar sine and cosine series

Definition 4.6.1. The planar sine and cosine are defined in the following way

\[ \sin_V(x) := \frac{1}{2i}(\exp_V(ix) - \exp_V(-ix)) \]

and

\[ \cos_V(x) := \frac{1}{2}(\exp_V(ix) + \exp_V(-ix)), \]

where \( i \) is the imaginary number with \( i = \sqrt{-1} \).

Remark 4.6.2. (i) By these definitions it holds that

\[ \sin_V(x) = \frac{1}{i} \sinh_V(ix) \quad \text{and} \quad \cos_V(x) = \cosh_V(ix), \]

and it follows that \( \sin_V(x) \) is an odd and \( \cos_V(x) \) an even power series.

(ii) By Remark 4.5.3 it follows

\[ \sin_V(x) = \sum_{T \in \Phi} (i)^{n-1} \cdot a_V(T) \cdot T \]

and

\[ \cos_V(x) = \sum_{T \in \Phi} (i)^n \cdot a_V(T) \cdot T, \]

where \( n = |T| \) and \( a_V(T) \) is the coefficient of \( T \) in the planar exponential series \( \exp_V(x) \).

Theorem 4.6.3. Let \( V, T \in \mathbb{P}, n = |T| \). Then

\[ c_T(\sin_V^2(x)) = \begin{cases} (i)^{n-2} \cdot a_V(T_1) \cdot a_V(T_2) & : T = T_1 \cdot T_2 \in \Gamma \\ 0 & : \text{else} \end{cases} \]

Proof. By Remark 4.6.2 and Theorem 4.5.4 it holds

\[ c_T(\sin_V^2(x)) = c_T\left(\left(\frac{1}{i}\sinh_V(ix)\right)^2\right) = c_T(-\sinh_V^2(ix)) \]

\[ = \begin{cases} -a_V(iT_1) \cdot a_V(iT_2) & : T = T_1 \cdot T_2 \in \Gamma \\ 0 & : \text{else} \end{cases} \]

\[ = \begin{cases} -(i)^{n_1+n_2} \cdot a_V(T_1) \cdot a_V(T_2) & : T = T_1 \cdot T_2 \in \Gamma \\ 0 & : \text{else} \end{cases} \]

\[ = \begin{cases} (i)^{n_1+n_2} \cdot a_V(T_1) \cdot a_V(T_2) & : T = T_1 \cdot T_2 \in \Gamma \\ 0 & : \text{else} \end{cases} \]

where \( n_i = |T_i| \) and thus \( n_1 + n_2 = n \) as \( T = T_1 \cdot T_2 \).
Theorem 4.6.4. Let \( V, T \in \mathbb{P}, n = |T|, ar(T) = m \). Then

\[
c_T(\cos_V^2(x)) = \begin{cases} 
1 & : \quad T = \frac{1}{2} \\
2 \cdot (i)^n \cdot a_V(T) & : \quad T \in \Gamma \text{ and } T \in \Phi^+ \text{ with } m > 2 \\
2 \cdot (i)^n \cdot a_V(T) + (i)^n \cdot a_V(T_1) \cdot a_V(T_2) & : \quad T = T_1 \cdot T_2 \in \Phi^+ - \Gamma \text{ with } m = 2 \\
0 & : \quad T \in \Psi
\end{cases}
\]

Proof. By Remark 4.6.2 and Theorem 4.5.5 it holds

\[
c_T(\cosh_V^2(x)) = c_T(\cosh_V^2(ix))
\]

\[
= \begin{cases} 
1 & : \quad T = \frac{1}{2} \\
2 \cdot a_V(iT) & : \quad T \in \Gamma \text{ and } T \in \Phi^+ \text{ with } m > 2 \\
2 \cdot a_V(iT) + a_V(iT_1) \cdot a_V(iT_2) & : \quad T = T_1 \cdot T_2 \in \Phi^+ - \Gamma \text{ with } m = 2 \\
0 & : \quad T \in \Psi
\end{cases}
\]

\[
= \begin{cases} 
1 & : \quad T = \frac{1}{2} \\
2 \cdot (i)^n \cdot a_V(T) & : \quad T \in \Gamma \text{ and } T \in \Phi^+ \text{ with } m > 2 \\
2 \cdot (i)^n \cdot a_V(T) + (i)^{n_1+n_2} \cdot a_V(T_1) \cdot a_V(T_2) & : \quad T = T_1 \cdot T_2 \in \Phi^+ - \Gamma \text{ with } m = 2 \\
0 & : \quad T \in \Psi
\end{cases}
\]

where \( n_i = |T_i| \) and thus \( n_1 + n_2 = n \) as \( T = T_1 \cdot T_2 \). \( \square \)

Theorem 4.6.5. Let \( V, T \in \mathbb{P}, n = |T|, ar(T) = m \). Then

\[
c_T(\sin_V^2(x) + \cos_V^2(x)) = \begin{cases} 
1 & : \quad T = \frac{1}{2} \\
2 \cdot (i)^n \cdot a_V(T) & : \quad T \in \Phi^+ \text{ with } m > 2 \\
(i)^{n-2} \cdot a_V(T_1) \cdot a_V(T_2) + (i)^n \cdot a_V(T) & : \quad T = T_1 \cdot T_2 \in \Gamma \\
2 \cdot (i)^n \cdot a_V(T) + (i)^n \cdot a_V(T_1) \cdot a_V(T_2) & : \quad T = T_1 \cdot T_2 \in \Phi^+ - \Gamma \text{ with } m = 2 \\
0 & : \quad T \in \Psi
\end{cases}
\]

Proof. By Theorems 4.6.3 and 4.6.4. \( \square \)

Remark 4.6.6. (i) For planar sine and cosine it is not true that

\[
\sin_V^2(x) + \cos_V^2(x) = 1.
\]
4.6 Planar sine and cosine series

(ii) By the same computations as in Theorem 4.6.5 it follows that

\[ \cosh^2_V(x) - \sinh^2_V(x) \neq 1 \]

and it holds

\[ c_T(\cosh^2_V(x) - \sinh^2_V(x)) = (-1)^{\frac{n}{2}} \cdot c_T(\sin^2_V(x) + \cos^2_V(x)). \]

**Theorem 4.6.7.** Let \( V \in \mathbb{P} \) and the coefficients of \( \sin^2_V(x) \) and \( \cos^2_V(x) \) be defined as above. Then \( \forall \ n \geq 1 \)

\[ \sum_{T \in \mathbb{P} \atop |T|=n} c_T(\sin^2_V(x)) = - \sum_{T \in \mathbb{P} \atop |T|=n} c_T(\cos^2_V(x)). \]

**Proof.** By Proposition 2.5.7 it follows that \( \eta(\sin_V(x)) \) (resp. \( \eta(\cos_V(x)) \)) is the classical sine (resp. cosine). Thus it holds that \( \eta(\sin^2_V(x) + \cos^2_V(x)) = 1 \) and it follows

\[ \sum_{T \in \mathbb{P} \atop |T|=n} c_T(\sin^2_V(x) + \cos^2_V(x)) = 0, \]

for \( n \geq 1. \)

**Remark 4.6.8.** By Proposition 4.1.6 it follows that

\[ \sum_{T \in \Psi \atop |T|=n} c_T(\sin_V(x)) = c_{x^n}(\sin(x)) = (i)^{n-1} \cdot \frac{1}{n!} \]

and

\[ \sum_{T \in \Phi \atop |T|=n} c_T(\cos_V(x)) = c_{x^n}(\cos(x)) = (i)^n \cdot \frac{1}{n!}. \]

**Proposition 4.6.9.** Let \( V \in \mathbb{P} \) and \( \sin_V(x) \) and \( \cos_V(x) \) be the planar sine and cosine. Then

(i) \[ \frac{d}{dx}(\sin_V(x)) = \cos_V(x). \]

(ii) \[ \frac{d}{dx}(\cos_V(x)) = -\sin_V(x). \]

**Proof.** (i) \[ \frac{d}{dx}(\sin_V(x)) = \frac{d}{dx}(\frac{1}{2i} \cdot (\exp_V(ix) - \exp_V(-ix))) \]

\[ = \frac{1}{2i} \cdot (i \cdot \exp_V(ix) + i \cdot \exp_V(-ix)) = \frac{1}{2} \cdot (\exp_V(ix) + \exp_V(-ix)) = \cos_V(x). \]

(ii) \[ \frac{d}{dx}(\cos_V(x)) = \frac{d}{dx}(\frac{1}{2} \cdot (\exp_V(ix) + \exp_V(-ix))) \]

\[ = \frac{1}{2} \cdot (i \cdot \exp_V(ix) + i \cdot \exp_V(-ix)) = \frac{1}{2} \cdot (\exp_V(ix) - \exp_V(-ix)) = -\sin_V(x). \]
Remark 4.6.10.  (i) Like in the classical case it holds that

\[ \sin_V(x) = -\sin_V(-x) \quad \text{and} \quad \cos_V(x) = \cos_V(-x). \]

(ii) Also like in the classical case it is possible to present the planar exponential series of \( ix \) using the planar sine and cosine as follows

\[ \exp_V(ix) = \cos_V(x) + i \cdot \sin_V(x). \]
Chapter 5

Planar root series

5.1 Planar root functions

Proposition 5.1.1. Let \( V \in \mathbb{P} \). Then there is a unique power series \( f \in K\{\{x\}\} \), such that \( f^V = 1 + x \) and the constant term is \( f(0) = 1 \). We denote \( f \) by \( \sqrt[1]{1 + x} \) or \( (1 + x)^{V^{-1}} \) and call it the \( V \)-th root series of \( 1 + x \).

Theorem 5.1.2. Let \( V \in \mathbb{P} \) and \( k = |V| \). Then for the coefficients \( b_V(T) \) of \( \sqrt[1]{1 + x} \) it is true that

\[
b_V(x) = \frac{1}{k}
\]

and

\[
b_V(T) = \frac{-\sum_S \binom{V}{S} \cdot \prod_{i=1}^{m} b_V(T_i)}{k},
\]

where \( 2 \leq m = |S|, T - S = (T_1, ..., T_m) \) is a planar rooted forest and \( \binom{V}{S} \) the planar binomial coefficient.

If \( S \notin P(V) \cap O(T) \), then the summand is equal to 0.

Proof. (1) Let \( f \in K\{\{x\}\} \) and \( f = \sqrt[1]{1 + x} \). For all \( V \in \mathbb{P} \) with \( k = |V| \) it holds

\[
f^V = 1 + \sum_S \binom{V}{S} \cdot f^S,
\]

where \( f^S : = \cdot_S (f, ..., f) \).

For \( T \in \mathbb{P} \) with \( 1 \leq ar(T) \leq k \) it holds

\[
coef_{f_T}(f^V) = \sum_S \binom{V}{S} \cdot \prod_{i=1}^{m} coef_{f_{T_i}}(f) + k \cdot coef_{f_T}(f),
\]

where \( S \in P(V) \cap O(T), S \neq x, |S| = m \geq 2 \) and \( T - S = (T_1, ..., T_m) \).

(2) Define a map \( b_V : \mathbb{P} \to K \) as in Theorem 5.1.2.
Let $f := \sum_{T \in \mathcal{P}} b_V(T) \cdot T$, then

$$
\text{coef}_T(f^V) = \text{coef}_T\left( \sum_{S,S \neq x} \binom{V}{S} \cdot f^S \right) + k \cdot \text{coef}_T(f)
$$

$$
= \sum_{S,S \neq x} \binom{V}{S} \cdot \prod_{i=1}^m \text{coef}_{T_i}(f) + k \cdot \text{coef}_T(f)
$$

$$
= \sum_{S,S \neq x} \binom{V}{S} \cdot \prod_{i=1}^m b_V(T_i) + k \cdot b_V(T)
$$

$$
= -k \cdot b_V(T) + k \cdot b_V(T) = 0.
$$

Moreover

$$
\text{coef}_{f_1}(f^V) = 1
$$

and

$$
\text{coef}_{f_x}(f^V) = k \cdot \text{coef}_{f_x}(f) = k \cdot \frac{1}{k} = 1.
$$

Thus $f^V = 1 + x$. \hfill \Box

**Proposition 5.1.3.** Let $V \in \mathbb{P}$ with $|V| = k$ and $\sqrt[1+x]{}$ be the $V$-th root of $1 + x$. Then

(i)

$$
\log_V\left( \sqrt[1+x]{1+x} \right) = \frac{1}{k} \cdot \log_V(1 + x).
$$

(ii)

$$
\exp_V\left( \frac{1}{k} \cdot x \right) = \sqrt[k]{\exp_V(x)}.
$$

(iii)

$$
\sqrt[1+x]{(1+x)^V} = 1 + x.
$$

**Proof.** (1) Because of Proposition 5.1.1 it is true that

$$
\left( \sqrt[1+x]{1+x} \right)^V = 1 + x.
$$

Thus

$$
\exp_V\left( k \cdot \log_V\left( \sqrt[1+x]{1+x} \right) \right) = 1 + x
$$

$$
\iff \log_V\left( \exp_V\left( k \cdot \log_V\left( \sqrt[1+x]{1+x} \right) \right) \right) = \log_V(1 + x)
$$

$$
\iff k \cdot \log_V\left( \sqrt[1+x]{1+x} \right) = \log_V(1 + x)
$$

$$
\iff \log_V\left( \sqrt[1+x]{1+x} \right) = \frac{1}{k} \cdot \log_V(1 + x).
$$
(2) From \((i)\) it follows
\[
\log_V \left( \exp_V \left( \frac{1}{k} \cdot x \right) \right) = \frac{1}{k} \cdot x
\]
and
\[
\log_V \left( \sqrt[\sqrt{\exp_V(x)}] \right) = \frac{1}{k} \cdot \log_V \left( \exp_V(x) \right) = \frac{1}{k} \cdot x.
\]
Thus
\[
\exp_V \left( \frac{1}{k} \cdot x \right) = \sqrt[\sqrt{\exp_V(x)}].
\]

(3) From \((i)\) it follows
\[
\sqrt[\sqrt{(1+x)^V}] = \exp_V \left( \log_V \left( \sqrt[\sqrt{(1+x)^V}] \right) \right) = \exp_V \left( \frac{1}{k} \cdot \log_V \left( (1+x)^V \right) \right)
\]
\[
= \exp_V \left( \frac{1}{k} \cdot k \cdot \log_V (1+x) \right) = 1 + x.
\]

Proposition 5.1.4. Let \(V \in \mathbb{P}\) with \(|V| = k\). Then it holds for the planar root with respect to \(V\)
\[
\left( (1 + x) \frac{d}{dx} \right) \left( \sqrt[\sqrt{1+x}] \right) = \frac{1}{k} \sqrt[\sqrt{1+x}] = 1 + x.
\]

Proof. Recall the special planar chain rule from Proposition 3.3.1. Here
\[
g = \exp_V(x) - 1,
\]
then
\[
\sqrt[\sqrt{\exp_V(x)}] = \exp_V \left( \frac{1}{k} \cdot x \right).
\]

Thus
\[
\left( (1 + x) \frac{d}{dx} \right) \left( \sqrt[\sqrt{\exp_V(x)}] \right) = \frac{d}{dx} \left( \sqrt[\sqrt{\exp_V(x)}] \right) = \frac{d}{dx} \left( \exp_V \left( \frac{1}{k} \cdot x \right) \right)
\]
\[
= \frac{1}{k} \cdot \exp_V \left( \frac{1}{k} \cdot x \right) = \frac{1}{k} \sqrt[\sqrt{\exp_V(x)}].
\]

Let \(f = ((1 + x) \frac{d}{dx}) \left( \sqrt[\sqrt{\exp_V(x)}] \right)\). It holds that \(f(1 + g(x)) = \frac{1}{k} \sqrt[\sqrt{1+g(x)}] = \sqrt[\sqrt{1+g(x)}]\) and with the substitution endomorphism \(\varphi_g\) we get \(\varphi_g(f(1 + x)) = \varphi_g(1 + \sqrt[\sqrt{1+x}])\) and finally the demanded identity. \(\square\)
Example 5.1.5. (i) Let $V = x^2$. Then
\[ \sqrt[2]{1 + x} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} \cdot \frac{1}{2} x \cdot x^2 + \frac{1}{2} x \cdot x \\
- \frac{5}{128} \cdot \left( \frac{1}{5} x \cdot (x \cdot x) + \frac{1}{5} x \cdot x^2 \cdot x \right) \\
+ \frac{1}{5} (x^2 \cdot x) \cdot x + \frac{1}{5} x^2 \cdot x^2 \]
+ higher terms.

(ii) Let now $V = x^2 \cdot x$. Then
\[ x^2 \sqrt[2]{1 + x} = 1 + \frac{1}{3} x - \frac{1}{9} x^2 + \frac{5}{81} \cdot \frac{3}{5} x \cdot x + \frac{2}{5} x \cdot x^2 \cdot x \\
- \frac{10}{243} \cdot \left( \frac{3}{10} x \cdot (x \cdot x) + \frac{1}{5} x \cdot x^2 \cdot x + \frac{1}{5} x \cdot x \cdot x^2 \cdot x \right) \\
+ \frac{1}{10} (x^2 \cdot x) \cdot x + \frac{1}{5} x^2 \cdot x^2 \]
+ higher terms.

5.2 Planar $\lambda$-deformed root functions

The notations are as in Chapter 4.

Proposition 5.2.1. Let $V \in \mathbb{P}, |V| = n$, $\lambda = (\lambda_1, ..., \lambda_n)$ and $\delta_r(\lambda) \neq 0, \forall r \geq 2$. Then there is a unique power series $f \in K\{\{x\}\}$, such that
\[ \sqrt[\lambda]{V}(f(\lambda_1x), ..., f(\lambda_nx)) = f(\sqrt[\lambda]{V}(1 + \lambda_1x, ..., 1 + \lambda_nx)) \]
and the constant term is $f(0) = 1$. We denote $f$ by $\sqrt[\lambda]{V}1 + x$ and call it the planar $\lambda$-deformed $V$-th root series of $1 + x$.

Theorem 5.2.2. Let $V \in \mathbb{P}, |V| = n$. Then for the coefficients $b_{(V, \lambda)}(T)$ of $T$ of $\sqrt[\lambda]{V}1 + x$ with $\lambda = (\lambda_1, ..., \lambda_n), \delta_r(\lambda), \forall r \geq 2$ it is true that
\[ b_{(V, \lambda)}(x) = \frac{1}{\lambda_1 + ... + \lambda_n} \]
and
\[ \delta_r(\lambda) \cdot b_{(V, \lambda)}(T) = \sum_{S \in \Omega(T)} \left( \begin{array}{c} V \\ T \end{array} \right)_s^\lambda \cdot \prod_{i=1}^m b_{(V, \lambda)}(T_i) - \sum_{S \in \Omega(T)} b_{(V, \lambda)}(S) \cdot \prod_{i=1}^m \left( \begin{array}{c} V \\ T_i \end{array} \right)^\lambda, \]
where $r = |T|, m = |S|$ and $T - S = (T_1, ..., T_m)$. 
5.2 Planar $\lambda$-deformed root functions

Proof. For the right side it holds because of the substitution automorphism that

\[
\text{coef}_{T}(\sqrt[1+\lambda x]\text{V}(1+\lambda_1 x, ..., 1+\lambda_n x)) = \sum_{S \in O(T)} b_{(V,\lambda)}(S) \cdot \prod_{i=1}^{m} \left(V_{T_i}\right)_{\lambda} + b_{(V,\lambda)}(T) \cdot (\lambda_1 + ... + \lambda_n)\tau.
\]

On the other side it holds

\[
\text{coef}_{T}(\sqrt[1+\lambda x]\text{V}(1+\lambda_1 x, ..., 1+\lambda_n x)) = \sum_{S \in O(T)} \left(V_{T}\right)_{S}^{\lambda} \cdot \prod_{i=1}^{m} b_{(V,\lambda)}(T_{i}) + b_{(V,\lambda)}(T) \cdot (\lambda_1^\tau + ... + \lambda_n^\tau) \cdot b_{(V,\lambda)}(T).
\]

As the left and the right side of the equation should be equal, it follows

\[
((\lambda_1 + ... + \lambda_n)^\tau - (\lambda_1^\tau + ... + \lambda_n^\tau)) \cdot b_{(V,\lambda)}(T)
\]

\[
= \sum_{S \in O(T)} \left(V_{T}\right)_{S}^{\lambda} \cdot \prod_{i=1}^{m} b_{(V,\lambda)}(T_{i}) - \sum_{S \in O(T)} b_{(V,\lambda)}(S) \cdot \prod_{i=1}^{m} \left(V_{T_i}\right)_{\lambda}.
\]

\[\square\]

Example 5.2.3. Let $V = x^2 \cdot x$ and $\lambda = (2, 1, 3)$. Then

\[
\sqrt[1+x]{x^2} = 1 + \frac{1}{6} x - \frac{5}{72} x^2 + \frac{55}{1296} \left(\frac{367}{660} x \cdot x^2 + \frac{293}{660} x^2 \cdot x\right)
\]

\[- \frac{935}{31104} \cdot \left(\frac{841531}{3360390} x \cdot (x \cdot x^2) + \frac{684403}{3360390} (x \cdot x^2) \cdot x\right)
\]

\[+ \frac{614393}{3360390} x \cdot (x^2 \cdot x) + \frac{454793}{3360390} (x^2 \cdot x) \cdot x + \frac{2319}{10183} x^2 \cdot x^2\]

+ higher terms.

Remark 5.2.4. Let $V \in \mathbb{P}$ with $|V| = n$, $\lambda = (\lambda_1, ..., \lambda_n)$ with $\delta_r(\lambda) \neq 0$, $\forall r$ and $\rho \in \mathbb{Q}$. Then

\[
\sqrt[1+x]{x^\rho} = \sqrt[1+x]{\rho \cdot x},
\]

where $\rho \cdot \lambda = (\rho \cdot \lambda_1, ..., \rho \cdot \lambda_n)$.

The proof is analogous to these for $\exp_{(V,\lambda)}(x)$ and $\log_{(V,\lambda)}(1 + x)$.
5.3 Planar algebraic power series

Definition 5.3.1. Let $F(x, y) \in K\{x, y\}$ be a polynomial in two variables $x, y$ over $K$ and $f(x) \in K\{x\}$ be a planar power series. Then $f(x)$ is called a planar algebraic power series if and only if

$$F(x, f(x)) = 0.$$ 

In this work, we consider some special cases of planar algebraic power series. The first case is, if $F(x, y) = y^T - (1 + x)$, $T \in \mathbb{P}$. Then it is obvious that the planar root functions with respect to $T \in \mathbb{P}$ are algebraic, as

$$F(x, \sqrt[1]{1 + x}) = (\sqrt[1]{1 + x})^T - (1 + x) = 1 + x - (1 + x) = 0.$$ 

The other case is, if we choose $F(x, y) = \sum_{i=1}^{n} y^V_i - (n + x)$, $V_i \in \mathbb{P}$. Then we find the following series, which are algebraic.

Proposition 5.3.2. Let $V_1, ..., V_n \in \mathbb{P}$ with $|V_i| = k_i, \forall i = 1, ..., k$. Then there is a unique power series $f \in K\{x\}$ with the constant term 1 and the property

$$f^{V_1} + ... + f^{V_n} = n + x.$$ 

Theorem 5.3.3. Let $V_1, ..., V_n \in \mathbb{P}$ with $|V_i| = k_i, \forall i = 1, ..., k$. Then for the coefficients of $f \in K\{x\}$ with $f^{V_1} + ... + f^{V_n} = n + x$ it is true that

$$\text{coeff}_{f^{V_1}}(f) = 1,$$

$$\text{coeff}_{f_x}(f) = \frac{1}{k_1 + ... + k_n}$$

and

$$\text{coeff}_T(f) = -\frac{\sum_{S \in O(T) \atop S \neq S} \left(\binom{V_1}{S} + ... + \binom{V_n}{S}\right)}{k_1 + ... + k_n} \cdot \prod_{i=1}^{m} \text{coeff}_{T_i}(f),$$

where $m = |S|$ and $T - S = (T_1, ..., T_m)$.

Proof. It holds

$$\text{coeff}_T(f^{V_1} + ... + f^{V_n}) = \sum_{S \in O(T)} \left(\binom{V_1}{S} + ... + \binom{V_n}{S}\right) \cdot \prod_{i=1}^{m} \text{coeff}_{T_i}(f),$$
where \( m = |S| \) and \( T - S = (T_1, ..., T_m) \), because

\[
\text{coeff}_T(f^{V_i}) = \sum_{S \in O(T)} \binom{V_i}{S} \cdot \prod_{i=1}^m \text{coeff}_{T_i}(f), \quad \forall \ i = 1, ..., n.
\]

\( \text{coeff}_{f_{x}}(f) = 1 \) by assumption, as \( f \) should have the constant term 1. As \( \text{coeff}_x(f^{V_1} + ... + f^{V_n}) = \left( \binom{V_1}{x} + ... + \binom{V_n}{x} \right) \cdot \text{coeff}_x(f) = 1 \) and \( \binom{V_i}{x} = k_i, \ \forall \ i = 1, ..., n \),

\[ \Rightarrow \text{coeff}_x(f) = \frac{1}{k_1 + ... + k_n}. \]

Moreover

\[
\text{coeff}_T(f^{V_1} + ... + f^{V_n}) = \sum_{S \in O(T)} \binom{V_1}{S} + ... + \binom{V_n}{S} \cdot \prod_{i=1}^m \text{coeff}_{T_i}(f) + (k_1 + ... + k_n) \cdot c_T(f) = 0
\]

\[ \Rightarrow c_T(f) = - \frac{\sum_{S \in O(T)} \binom{V_1}{S} + ... + \binom{V_n}{S}}{k_1 + ... + k_n} \cdot \prod_{i=1}^m \text{coeff}_{T_i}(f), \ \forall \ T \in \mathbb{P}, |T| \geq 2. \]

Example 5.3.4. Let \( V_1 = x \cdot x^2 \) and \( V_2 = x^2 \cdot x \). Then the planar power series \( f \) with \( f^{x \cdot x^2} + f^{x^2 \cdot x} = 2 + x \) is

\[
f = 1 + \frac{1}{6} x - \frac{1}{36} x^2 + \frac{5}{1296} \cdot (x \cdot x^2 + x^2 \cdot x)
- \frac{1}{1944} \cdot (x \cdot (x \cdot x^2) + (x \cdot x^2) \cdot x + x \cdot (x^2 \cdot x) + (x^2 \cdot x) \cdot x + x^2 \cdot x^2)
+ \text{higher terms}.
\]

Proposition 5.3.5. Let \( a, b, c \in K\{x\} \) with \( a_1 = b_1 = 1 \) and \( c_1 = 3 \). Then there is a unique power series \( f \in K\{x\} \) such that the constant term of \( f \) is \( f(0) = 1 \) and \( f^2 + a \cdot f + f \cdot b = c \).

Theorem 5.3.6. Denote by \( a_T, b_T, c_T \) the coefficients of \( T \in \mathbb{P} \) in the power series \( a, b, c \in K\{x\} \) and by \( f_T \) the coefficient of \( T \in \mathbb{P} \) in the power series \( f \). Then we get the recursion formula

\[
f_T = \frac{c_T - a_T - b_T}{4}, \quad \text{if} \quad T \in \mathbb{P}, \ ar(T) \neq 2
\]

and

\[
f_T = \frac{c_T - a_T - b_T - f_{T_1} \cdot f_{T_2} - a_{T_1} \cdot f_{T_2} - f_{T_1} \cdot b_{T_2}}{4}, \quad \text{if} \quad T \in \mathbb{P}, \ T = T_1 \cdot T_2.
\]
Planar root series

Proof. It holds
\[
\text{coeff}_{f_T}(c) = \text{coeff}_{f_T}(f^2) + \text{coeff}_{f_T}(a \cdot f) + \text{coeff}_{f_T}(f \cdot b)
\]
\[
= \text{coeff}_{f_{T_1}}(f) \cdot \text{coeff}_{f_{T_2}}(f) + 2 \cdot \text{coeff}_{f_T}(f)
\]
\[
+ \text{coeff}_{f_{T_1}}(a) \cdot \text{coeff}_{f_{T_2}}(f) + \text{coeff}_{f_T}(a) + \text{coeff}_{f_T}(f)
\]
\[
+ \text{coeff}_{f_{T_1}}(f) \cdot \text{coeff}_{f_{T_2}}(b) + \text{coeff}_{f_T}(f) + \text{coeff}_{f_T}(b),
\]
since
\[
f^2 = (1 + f')^2 = 1 + 2 \cdot f' + (f')^2,
\]
a \cdot f = (1 + a') \cdot (1 + f') = 1 + a' + f' + a' \cdot f'
and
\[
f \cdot b = (1 + f') \cdot (1 + b') = 1 + f' + b' + f' \cdot b'.
\]

\[\square\]

Proposition 5.3.7. It is also possible to compute the coefficients of \( f \) by using the planar quadratic extension. There we get:
\[
f = x^2 \sqrt{f \cdot (a - b) + a^2 + c} - a.
\]
Then we can use the substitution endomorphism and the planar root resp. \( x^2 \).

Proof. It holds
\[
f^2 + a \cdot f + f \cdot b = c \iff (f + a)^2 - f \cdot a - a^2 + f \cdot b = c \iff (f + a)^2 = f \cdot (a - b) + a^2 + c \iff f = x^2 \sqrt{f \cdot (a - b) + a^2 + c} - a.
\]
\[\square\]

Remark 5.3.8. If \( a = b \), then we better use the planar quadratic extension, because it holds that
\[
f = x^2 \sqrt{a^2 + c} - a,
\]
and thus the recursion formula is not needed. We can easily compute the above term since the coefficients of \( x^2 \sqrt{1 + x} \), \( a \) and \( c \) are known.
Chapter 6

Generic exponential and logarithm series

6.1 Sequences of rooted trees and $q$-functions

Let $K(q)$ be the field of rational functions in one variable $q$ over $K$. For any $n \geq 1$ define the $q$-analogue $[n]$ of $n$ to be

$$[n] = \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}.$$ 

Moreover the $q$-factorial $[n]!$ is defined to be $\prod_{i=1}^{n} [i]$. Let $r \in \mathbb{N} \geq 1$, $r \leq n$. Then we define the $q$-binomial coefficient $\left[ \begin{array}{c} n \\ r \end{array} \right]$ by $\left[ \begin{array}{c} n \\ r \end{array} \right] := \frac{[n]!}{[r]![n-r]!}$. For a partition $n_1, \ldots, n_k$ of $n$ define the $q$-multinomial coefficient $\left[ \begin{array}{c} n \\ n_1,\ldots,n_k \end{array} \right]$ inductively on $k$ to be $\left[ \begin{array}{c} n \\ n_1,\ldots,n_k \end{array} \right] = \left[ \begin{array}{c} n \\ n_1 \end{array} \right] \left[ \begin{array}{c} n-n_1 \\ n_2,\ldots,n_k \end{array} \right]$.

**Remark 6.1.1.** It is well known, that $\left[ \begin{array}{c} n \\ r \end{array} \right] \in \mathbb{Z}(q)$ is a primitive polynomial of degree $r(n-r)$ (see [KC]). It follows that $\left[ \begin{array}{c} n \\ n_1,\ldots,n_k \end{array} \right]$ is a polynomial with integer coefficients, because $q$-binomials are in $\mathbb{Z}(q)$. There are various relations between $q$-analogous, here we fix $1 + q [n-1] = [n], [m] + [n] + (q-1)[m][n] = [m+n]$ and $1 + q[m] + q[n] + q(q-1)[m][n] = [m+n+1]$, for $m, n \geq 1$ (see [KC]).

Let $T, V \in \mathcal{P}$. We define $[T]_V(q)$ inductively over the degree $|T|$ as an element in $K$. If $|T| = 1$, then $T = x$ and $[x]_V := 1$. Equally if $T$ is the empty tree. Let
now $n = |T| > 1$, $q = |V|$, $S \in P(V) \cap O(T)$ and $2 \leq m = |S|$, then we define

$$[T]_V(q) := \sum_{s} \frac{[n-1]!}{\prod_{i=1}^{m}[n_i-1]!} \cdot \binom{V}{s} \cdot [T-S]_V(q)$$

$$= \sum_{s} \frac{[n-2]!}{\prod_{i=1}^{m}[n_i-1]!} \cdot \frac{\binom{V}{s}}{q(q-1)} \cdot [T-S]_V(q),$$

where $[T-S]_V(q) = \prod_{i=1}^{m}[T_i]_V(q)$, if $T - S = (T_1, ..., T_m)$ is a planar rooted forest and $n_i = |T_i|$.

**Proposition 6.1.2.** $[T]_V(q)$ is an element in $K$.

**Proof.** It holds $\forall m \geq 2$

$$\frac{[n-2]!}{[n_1-1]! \cdot \ldots \cdot [n_m-1]!} = \prod_{i=n-m+1}^{n-2} \binom{i}{m} \cdot \binom{n-m}{i-1},$$

where $\binom{k}{k_1, \ldots, k_m} := \frac{\binom{k}{k_1! \cdot \ldots \cdot k_m!}}{k_1! \cdot \ldots \cdot k_m!} = \frac{k!}{k_1! \cdot \ldots \cdot k_m!}$. By induction over $m$ it follows that $\binom{k}{k_1, \ldots, k_m}$ is a polynomial in $\mathbb{Z}[q]$, because the $q$-binomial coefficients are in $\mathbb{Z}[q]$.

$(\frac{V}{S})$ is a planar binomial coefficient and thus it is also a polynomial in $\mathbb{Z}[q]$.

As $q = |V|$ is a fixed element in $K$ all factors in the sum are elements in $K$, thus it follows $[T]_V(q) \in K$. \hfill \Box

**Definition 6.1.3.** A map $f : \mathbb{N} \rightarrow \mathbb{Q}$ is called a rational function if and only if there are two coprime polynomials $p(i)$ and $q(i)$, such that $f(i) = \frac{p(i)}{q(i)}$, $\forall i \in \mathbb{N}$.

**Definition 6.1.4.** Let $V = (V_q)_{q \geq 0} \in \mathbb{P}$ be a sequence of planar rooted trees. Then $V$ is called admissible, if

(i) $V_i \subseteq P(V_{i+1})$, $\forall i \in \mathbb{N}$, with $\rho(V_i) = \rho(V_{i+1})$, where $\rho(V)$ is the root of $V$.

(ii) $f : \mathbb{N} \rightarrow \mathbb{Q}$ with $i \mapsto \frac{V_i}{S}$ is a rational function for all $S \in \mathbb{P}$.

**Remark 6.1.5.** (i) If we have an admissible sequence of planar rooted trees, then it is possible to define a generic exponential series with respect to this sequence by using $[T]_V$, which we then call the $q$-functions.

(ii) There are some classes (sequences) of rooted trees, where the $q$-functions are $q$-polynomials, e.g. if $V_0$ is a corona or a binary comb. In these cases the term $\frac{1}{q(q-1)}$ is reduced by the planar binomial coefficient. Then we get a sum over a product of polynomials which is also a polynomial in $K[q]$. 
(iii) There is another possibility that in the denominator there is only the $q$ because by the planar binomial coefficient the term $q - 1$ is reduced. This happens if $V$ has the form $V = x \cdot x^r$, for all $r \in \mathbb{N}$.

**Example 6.1.6.**

(i) If $V = (V_q)_{q \geq 0}$ is a sequence of coronas with $|V_q| = q$, then

$$[T]_V := \frac{[n-2]!}{\prod_{i=1}^m [n_i - 1]!} \cdot \frac{1}{m(m-1)} \cdot \left(\frac{q - 2}{m - 2}\right) \cdot \prod_{i=1}^m [T_i]_V,$$

where $n = |T| > 1$, $T = T_1 \cdot \ldots \cdot T_m$, $n_i = |T_i|$ and $m \geq 2$.

(ii) If $V = (V_q)_{q \geq 0}$ is a sequence of binary combs with $|V_q| = q$, then

$$[T]_V = \sum S \frac{[n-2]!}{\prod_{i=1}^m [n_i - 1]!} \cdot \frac{1}{m(m-1)} \cdot \left(\frac{q - 2}{m - 2}\right) \cdot [T - S]_V,$$

where $S \in P(V) \cap O(T), S \neq x, 2 \leq m = |S|$, $n = |T|$, $[T - S]_V$ is defined as above and $n_i = |T_i|$.

(iii) If $V$ is of the form $x \cdot x^r$ with $r \in \mathbb{N}$ and $q = r + 1 = |V_q|$ then

$$[T]_V = \frac{[n-2]!}{[n_1 - 1]! \cdot [n_2 - 1]!} \cdot \frac{1}{2} \cdot [T_1]_V \cdot [T_2]_V$$

$$+ \sum_{S \neq \text{Corona}} \frac{[n-2]!}{\prod_{i=1}^m [n_i - 1]!} \cdot \frac{1}{q(m-1)} \cdot \left(\frac{q - 2}{m - 2}\right) \cdot [T - S]_V$$

$$+ \sum_{S = \text{Corona} \setminus \{x^2\}} \frac{[n-2]!}{\prod_{i=1}^m [n_i - 1]!} \cdot \frac{1}{q \cdot m} \cdot \left(\frac{q - 2}{m - 1}\right) \cdot [T - S]_V,$$

where $S \in P(V) \cap O(T), S \neq x$, $n = |T|$, $[T - S]_V$ is defined as above and $n_i = |T_i|$.

Now we observe the logarithm series and there we have an other functional equation.

Let $T, V \in \mathbb{P}$. We define $[T]^V(q)$ inductively on the degree $|T|$ as an element in $K$. If $|T| = 1$ then $T = x$ and $[x]^V := 1$. This holds also if $T$ is the empty tree.
Let now $n = |T| > 1$, $q = |V|$, $S \in O(T)$, $S \neq T$ and $1 \leq m = |S|$, then we define

$$[T]^V(q) := \sum_{S \in O(T)} \frac{n - 1}{m - 1} \cdot \frac{\prod_{i=1}^{m} (V_{T_i})}{q - q^n} \cdot [S]^V(q)$$

$$= \sum_{S} \frac{\prod_{i=1}^{m} (V_{T_i})}{q(1-q) \cdot [m-1]} \cdot [S]^V(q),$$

where $(T_1, ..., T_m) = T - S$.

**Proposition 6.1.7.** $[T]^V(q)$ is a rational function in $K(q)$.

**Proof.** The proof is analogous to the proof of Proposition 6.1.2. \hfill \qed

**Remark 6.1.8.**

(i) In this case we may also define a generic logarithm series, if we have an admissible sequence of planar rooted trees.

(ii) Be aware that the sum often consists of only one summand because the summand is equal to 0 if a rooted tree $T_i$ in the rooted forest $T - S$ is not a subtree of $V$, this means if $T_i \notin P(V)$, for any $T_i$.

**Example 6.1.9.**

(i) If $V = (V_q)_{q \geq 0}$ is a sequence of coronas with $|V_q| = q$, then we have

$$[T]^V = \sum_{S \in O(T)} \frac{\prod_{i=1}^{m} (q)}{q \cdot (1-q) \cdot [m-1]} \cdot [S]^V,$$

if $T_i$ is a corona $\forall i$, where $T - S = (T_1, ..., T_m)$, $|S| = m$ and $n_i = |T_i|$. If there is a $T_i$ with $T_i \neq$ corona, then the summand is equal to 0.

(ii) If $V = (V_q)_{q \geq 0}$ is a sequence of binary left (resp. right) combs with $|V_q| = q$, then we have

$$[T]^V = \sum_{S \in O(T)} \frac{\prod_{i=1}^{m} (q)}{q \cdot (1-q) \cdot [m-1]} \cdot [S]^V,$$

if $T_i$ is a binary left (resp. right) comb $\forall i$, where $T - S = (T_1, ..., T_m)$, $|S| = m$ and $n_i = |T_i|$. If there is a $T_i$ with $T_i \neq$ a left (resp. right) comb, then the summand is equal to 0.
6.2 Generic exponential series

Let $V \in P$ with $k = |V|$. Then we can define

$$\exp_V(x) := \sum_{T \in P} [T]_V \frac{[T]_V}{|[T] - 1|!} (k) \cdot T.$$ 

This is a power series in $K\{x\}$. It means that we can compute the coefficients of $\exp_V(x)$ in the following way by using the so called $q$-function $[T]_V$ with respect to $\exp_V(x)$

$$a_V(T) = \frac{[T]_V}{|[T] - 1|!} (k).$$

In this case the recursion formula also holds, such that one gets

$$(k^n - k) \cdot \frac{[T]_V}{|[T] - 1|!} (k) = \sum_S \binom{V}{S} \cdot \frac{[T - S]_V}{|[T - S] - 1|!} (k),$$

where $S \in P(V) \cap O(T), S \neq x, |S| = m \geq 2$ and $T - S = (T_1, \ldots, T_m)$ is the planar rooted forest and $\frac{[T - S]_V}{|[T - S] - 1|!} (k) = \prod_{i=1}^m \frac{[T_i]_V}{|[T_i] - 1|!} (k)$.

This follows immediately by the definitions of $[T]_V(k)$ and $[n - 1]!(k)$ and the recursive definition of the coefficients of $\exp_V(x)$. There it holds for $a_V(T) = \text{coeff}_T(\exp_V(x))$:

$$a_V(T) = \sum_S \frac{\binom{V}{S}}{k^n - k} \cdot a_V(T - S),$$

where $n = |T|, T - S = (T_1, \ldots, T_m), 2 \leq m = |S|, a_V(T - S) = \prod_{i=1}^m a_V(T_i)$ and $|T_i| = n_i$.

If $T = x$, then the proposition is true, because

$$a_V(x) = 1 = \frac{[x]_V}{[1 - 1]!}.$$
Let us assume that \(\frac{[T_i]_V}{[n-1]!} (k) = a_V(T_i)\) is already true. Then
\[
\frac{[T]_V}{[n-1]!} (k) = \frac{1}{[n-1]!} \sum_s \frac{[n-1]!}{\prod_{i=1}^m [n_i - 1]!} \cdot \frac{V_s}{k^n - k} \cdot [T - S]_V
\]
\[
= \sum_s \frac{V_s}{k^n - k} \cdot \prod_{i=1}^m \frac{T_i}{[n_i - 1]!} (k)
\]
\[
= \sum_s \frac{V_s}{k^n - k} \cdot \prod_{i=1}^m a_V(T_i)
\]
\[
= \sum_s \frac{V_s}{k^n - k} \cdot a_V(T - S)
\]
\[
= a_V(T).
\]

**Proposition 6.2.1.** Let \(V = (V_q)_{q \geq 0} \in \mathbb{P}\) be an admissible sequence of rooted trees. Then we can define a generic exponential series, such that
\[
\exp_V(q, x) := \sum_{T \in \mathbb{P}} \frac{[T]_V}{||T||-1}! \cdot T
\]
and
\[
\exp_V(k, x) = \exp_{V_k}(x), \quad \text{for a } k \in \mathbb{N}^2.
\]

**Proof.** The proof follows by the computations above the proposition. \(\square\)

**Proposition 6.2.2.** Let \(V = (V_q)_{q \geq 0} \in \mathbb{P}\) be a sequence of binary right combs. Then
\[
\lim_{q \to \infty} \exp_V(q, x) = \sum_{m=0}^{\infty} \frac{S_m}{m!},
\]
where \(S_m\) is a binary right comb with \(m = |S_m|\).

**Proof.** It holds
\[
a_V(q, T) = \sum_s \frac{q^n}{q^n - q} \cdot a_V(q, T - S),
\]
where \(n = |T|, S \in P(V) \cap O(T), S \neq x\) and \(m = |S|\).

For all \(f \in K(q), f \neq 0\), we denote by
\[
(q - \text{ord}_\infty)(f)
\]
the order of \(f\) at \(\infty\). If \(f = \frac{f_2}{f_1}\) and \(f_i\) are polynomials in \(K[q]\), then
\[
(q - \text{ord}_\infty)(f) = \deg(f_2) - \deg(f_1).\]
So in this case it holds
\[
(q - \text{ord}_\infty) \left( \frac{\sum_{|S| = m} \frac{q^n}{q^n - q}}{m} \right) = n - \max(|S|).
By induction over \( n = |T| \) it follows that
\[
(q - \text{ord}_\infty)(a_V(q, T)) \geq 0,
\]
for all \( T \in \mathbb{P} \), because
\[
(q - \text{ord}_\infty)(a_V(q, T)) = (q - \text{ord}_\infty) \left( \frac{\sum_{|S|=m} \binom{q}{m}}{q^m - q} \right) \cdot (q - \text{ord}_\infty) \left( \prod_{i=1}^{m} a_V(q, T_i) \right),
\]
where \( S \in P(V) \cap O(T) \) and it holds
\[
(q - \text{ord}_\infty)(a_V(q, T)) > 0,
\]
if \( n > \text{max}(|S|) \) with \( S \in P(V) \cap O(T), S \neq x \).
It is always \( n \geq \text{max}(|S|) \), except if \( T \) is also a binary right comb, because then \( \text{max}(|S|) = n \). Thus
\[
\lim_{q \to \infty} a_V(q, T) = 0, \forall T \in \mathbb{P} - \{S_m = x \cdot S_{m-1} : m \geq 2\}.
\]
Moreover
\[
a_V(q, S_m) = \left( \sum_{n=2}^{m} \binom{q}{n} \right) / (q^m - q)
\]
and
\[
a_V(q, S_m) = \left( \text{leading coefficient of } \binom{q}{m} \right) / \left( \text{leading coefficient of } (q^m - q) \right)
= \frac{1}{m!}.
\]

**Proposition 6.2.3.** Let \( V = (V_q)_{q \geq 0} \in \mathbb{P} \) be a sequence of binary left combs. Then
\[
\lim_{q \to \infty} \exp_V(q, x) = \sum_{m=0}^{\infty} \frac{x^m}{m!},
\]
where \( x^m \) is a binary left comb with \( m = |m^S| \).

**Proof.** Analogous to the above proof for right combs. \( \square \)

**Proposition 6.2.4.** Let \( V = (V_q)_{q \geq 0} \in \mathbb{P} \) be a sequence of coronas. Then
\[
\lim_{q \to \infty} \exp_V(q, x) = \sum_{m=0}^{\infty} \frac{x^m}{m!},
\]
where \( x^m \) is a corona of degree \( m \).

**Proof.** See [Ger1]. \( \square \)
6.3 Generic logarithm series

Let $V \in \mathbb{P}$ with $k = |V|$. Here we can define

$$\log_V (1 + x) := \sum_{T \in \mathbb{P}} \frac{[T]^V}{|[T] - 1|}(k) \cdot T.$$ 

This is also a power series in $K\{x\}$. From it follows that for the coefficients of $\log_V (1 + x)$ it holds

$$c_V(T) = \frac{[T]^V}{|[T] - 1|}(k).$$

The recursion formula is true, so we have

$$(k - k^n) \cdot \frac{[T]^V}{|[T] - 1|}(k) = \sum_{S \in O(T) \setminus \mathbb{T}} \frac{[S]^V}{|m - 1|}(k) \cdot \prod_{i=1}^{m} \left( \frac{V}{T_i} \right),$$

where $n = |T|$, $m = |S|$ and $T - S = (T_1, ..., T_m)$. By the definitions of $[T]^V(k)$ and $[n - 1](k)$ and the recursive definition of the coefficients of $\log_V (1 + x)$, one gets

$$c_V(T) = \sum_{S \in O(T) \setminus \mathbb{T}} \frac{\prod_{i=1}^{m} \left( \frac{V}{T_i} \right)}{k - k^n} \cdot c_V(S),$$

where $n = |T|$, $m = |S|$ and $T - S = (T_1, ..., T_m)$. If $T = x$ then the proposition is true, as

$$c_V(x) = 1 = \frac{[T]^V}{[1 - 1]}(k).$$

Let already be true that $\frac{[S]^V}{[m - 1]}(k) = c_V(S)$. Then

$$\frac{[T]^V}{[n - 1]}(k) = \frac{1}{[n - 1]}(k) \sum_{S} \frac{[n - 1]}{|m - 1|}(k) \cdot \prod_{i=1}^{m} \left( \frac{V}{T_i} \right) \cdot \frac{[S]^V}{q - q^n}$$

$$= \sum_{S} \frac{\prod_{i=1}^{m} \left( \frac{V}{T_i} \right)}{q - q^n} \cdot \frac{[S]^V}{|m - 1|}(k)$$

$$= \sum_{S} \frac{\prod_{i=1}^{m} \left( \frac{V}{T_i} \right)}{q - q^n} \cdot c_V(S)$$

$$= c_V(T).$$
Proposition 6.3.1. Let $V = (V_q)_{q \geq 0} \in \mathbb{P}$ be an admissible sequence of rooted trees. Then we can define a generic logarithm series, such that

$$\log_V(q, 1 + x) := \sum_{T \in \mathbb{P}} \left[ T \right]^V \left( \frac{[T]}{|T| - 1} \right) \cdot T$$

and

$$\log_V(k, 1 + x) = log_{V_k}(1 + x), \quad \text{for } k \in \mathbb{N}^2.$$  

Proof. See the computations above.

Similar to the case of the generic exponential series it is possible to analyze and compute the limit of the generic logarithm series. Unfortunately this limit is not that “considerable” like the one of the associated exponential series, because in this case most coefficients do not vanish if $q \to \infty$.

Proposition 6.3.2. Let $V \in \mathbb{P}$ be a sequence of binary combs or coronas. Then for the coefficients of $T \in \mathbb{P}$ in the limit of the associated logarithm series $log_V(q, 1 + x)$ it holds

$$\lim_{q \to \infty} c_V(q, x) = 1$$

and

$$\lim_{q \to \infty} c_V(q, T) = - \sum_{S \in O(T)} \prod_{i=1}^m \frac{1}{n_i} \cdot \lim_{q \to \infty} (c_V(q, S)),$$

where $m = |S|$ and $n_i = |T_i|$, if $T - S = (T_1, ..., T_m)$.

Proof. It holds

$$c_V(q, T) = \sum_{S \in O(T)} \prod_{i=1}^m \frac{\binom{V}{T_i}}{q - q^n} \cdot c_V(q, S),$$

where $n = |T|$, $m = |S|$ and $T - S = (T_1, ..., T_m)$. For any $f \in K(q), f \neq 0$, we denote by

$$(q - ord_\infty)(f)$$

the order of $f$ at $\infty$. If $f = \frac{f_2}{f_1}$ and $f_i$ are polynomials in $K[q]$ then it is true that

$$(q - ord_\infty)(f) = deg(f_2) - deg(f_1).$$

Here

$$\left( q - ord_\infty \right) \left( \prod_{i=1}^m \frac{\binom{V}{T_i}}{q - q^n} \right) = n - n = 0, \quad \forall T \in \mathbb{P},$$

as $\binom{V}{T_i} = \binom{q}{n_i}$ for $T_i \in P(V)$ and $n_i = |T_i|$, if - as in this case $V$ - is a binary comb or a corona, moreover $\sum_{i=1}^m n_i = n$ and thus $deg \left( \prod_{i=1}^m \frac{q}{n_i} \right) = n$, because
\[ \deg \left( \left( \frac{q}{n_i} \right) \right) = n_i. \]

It follows
\[ (q - \text{ord}_\infty) (c_V(q, T)) = 0, \quad \forall \, T \in \mathbb{P} \]

and thus
\[
c_V(q, T)(\infty) = \sum_{S \in O(T) \setminus \{T\}} \lim_{q \to \infty} \left( \prod_{i=1}^{m} \left( \frac{V_{T_i}}{q - q^n} \right) \right) \cdot \lim_{q \to \infty} (c_V(q, S))
\]
\[
= \sum_{S \in O(T) \setminus \{T\}} \frac{\left( \text{leading coefficient of } \prod_{i=1}^{m} \left( \frac{V_{T_i}}{q - q^n} \right) \right)}{\left( \text{leading coefficient of } (q - q^n) \right)} \cdot \lim_{q \to \infty} (c_V(q, S))
\]
\[
= \sum_{S \in O(T) \setminus \{T\}} \prod_{i=1}^{m} \left( -\frac{1}{n_i!} \right) \cdot \lim_{q \to \infty} (c_V(q, S)).
\]

Example 6.3.3. By using the proposition above, we can now compute the limit series for trees with the degree \( \leq 4 \).

(i) Let \( V \) be the sequence of binary right combs, then

\[
\lim_{q \to \infty} (\log_V(q, 1 + x)) = x - \frac{1}{2} x^2 + \frac{1}{3} \left( \frac{1}{4} x \cdot x^2 + \frac{3}{4} x^2 \cdot x \right)
\]
\[- \frac{1}{4} \cdot \left( \frac{1}{6} (x \cdot x^2) \cdot x + \frac{1}{6} x \cdot (x^2 \cdot x) + \frac{1}{2} (x^2 \cdot x) \cdot x + \frac{1}{6} x \cdot x^2 \right) + \text{higher terms.}
\]

(ii) Let \( V \) be the sequence of binary left combs, then

\[
\lim_{q \to \infty} (\log_V(q, 1 + x)) = x - \frac{1}{2} x^2 + \frac{1}{3} \left( \frac{3}{4} x \cdot x^2 + \frac{1}{4} x^2 \cdot x \right)
\]
\[- \frac{1}{4} \cdot \left( \frac{1}{2} x \cdot (x \cdot x^2) + \frac{1}{6} (x \cdot x^2) \cdot x + \frac{1}{6} x \cdot (x^2 \cdot x) + \frac{1}{6} x \cdot x^2 \right) + \text{higher terms.}
\]
(iii) Let now $V$ be the sequence of coronas, then

$$\lim_{q \to \infty} (\log_V(q,1+x)) = x - \frac{1}{2} x^2 + \frac{1}{3} \cdot \left( \frac{3}{4} x \cdot x^2 + \frac{3}{4} x^2 \cdot x - \frac{1}{2} x^3 \right)$$

$$- \frac{1}{4} \cdot \left( \frac{1}{2} x \cdot (x \cdot x^2) + \frac{1}{2} (x \cdot x^2) \cdot x + \frac{1}{2} x \cdot (x^2 \cdot x) \right)$$

$$+ \frac{1}{2} (x^2 \cdot x) \cdot x - \frac{1}{3} x \cdot x^3 - \frac{1}{3} x^3 \cdot x - \frac{1}{3} x \cdot x \cdot x^2$$

$$- \frac{1}{3} x \cdot x^2 \cdot x - \frac{1}{3} x^2 \cdot x \cdot x + \frac{1}{2} x^2 \cdot x^2 + \frac{1}{6} x^4$$

+ higher terms.

### 6.4 Tupel-presentation of rooted trees

**Definition 6.4.1.** Let $W$ be a rooted tree with the root $\rho$ and the set of leaves $L(W)$. Then

$$H(W) := \max\{\text{dist}_W(\rho, a) : a \in L(W)\}.$$  

$H(W)$ is called the height of $W$.

By help of coronas, we can write every rooted tree $W$ of the height 2 in the following way:

$$W = (w_1, w_2, ..., w_r) = C_{w_1} \cdot ... \cdot C_{w_r},$$

where $w_i \in \mathbb{N}^{\geq 1}$, $w_i = j = x^j$ is the corona $C_j$ of degree $j$, with $C_1 = x$, $C_2 = x^2$, etc. Moreover $|W| = \sum_{i=1}^{r} w_i$ and $ar(W) = r$.

If all $w_i = 1$, then $W$ is a corona itself of degree $r$ and it has the height 1 and not 2.

Now for any rooted tree $V$ of height 2 and the form $V = (k_1, ..., k_r)$ with $ar(V) = r$ and $|V| = k = \sum_{i=1}^{r} k_i$ we can compute the planar binomial coefficient as follows

$$\binom{V}{S} = \begin{cases} 
\sum_{i=1}^{r} \left( \sum_{1 \leq i_1 < ... < i_m \leq r} k_{i_j} \right) & : \text{if } S = x^m \\
\sum_{1 \leq i_1 < ... < i_m \leq r} \prod_{j=1}^{m} k_{i_j} \left( \prod_{j=1}^{m} s_{i_j} \right) & : \text{if } S = (s_1, ..., s_m) \text{ and } m \leq r \\
0 & : \text{else.}
\end{cases}$$
Special case: Let $V = (k_1, k_2)$ with $ar(V) = 2$ and $|V| = k = k_1 + k_2$. Then

$$
\begin{align*}
(V) &= \\
&S = \begin{cases}
\binom{k}{2} : & \text{if } S = x^2 \\
\binom{k_1}{m} + \binom{k_2}{m} : & \text{if } S = x^m \\
\binom{k_1}{s_1} \cdot \binom{k_2}{s_2} : & \text{if } S = (s_1, s_2) \\
0 : & \text{else.}
\end{cases}
\end{align*}
$$

Now by this presentation we can compute the coefficients of the exponential series resp. the rooted trees $V$ of height 2 using the following recursion formulas. Let $V = (k_1, ..., k_r)$, $|V| = k = \sum_{i=1}^{r} k_i$ and $ar(V) = r$. Then for the coefficients of $exp_V(x) = exp((k_1, ..., k_r))(x)$ it holds:

$$a_V(x) = a_{(k_1, ..., k_r)}(x) = 1$$

and

$$a_V(T) = a_{(k_1, ..., k_r)}(T) = \frac{\sum_{i=1}^{r} \binom{k_i}{m} + \sum_{1 \leq i_1 < \ldots < i_m \leq r} \prod_{j=1}^{m} k_{i_j}}{k^n - k} \cdot \prod_{j=1}^{m} a_{(k_1, ..., k_r)}(T_{i_j})$$

$$+ \sum_{S = (s_1, ..., s_m) \in O(T)} \frac{\sum_{1 \leq i_1 < \ldots < i_m \leq r} \prod_{j=1}^{m} \binom{k_{i_j}}{s_{i_j}}}{k^n - k} \cdot a_{(k_1, ..., k_r)}(T - S),$$

where $n = |T|, m = ar(T), T - x^m = (T_1, ..., T_m), T - S = (T_1, ..., T_{|S|})$ is a planar forest and $a_{(k_1, ..., k_r)}(T - S) = \prod_{j=1}^{\frac{|S|}{|V|}} a_{(k_1, ..., k_r)}(T_{i_j})$.

Special case: Let now $V = (k_1, k_2)$, $ar(V) = 2$ and $|V| = k = k_1 + k_2$. Then for the coefficients of the exponential series resp. $V exp_V(x) = exp((k_1, k_2))(x)$ it holds:

$$a_V(x) = a_{(k_1, k_2)}(x) = 1$$

and

$$a_V(T) = a_{(k_1, k_2)}(T) = \frac{\binom{k_1}{2}}{k^n - k} \cdot a_{(k_1, k_2)}(T_1) \cdot a_{(k_1, k_2)}(T_2)$$

$$+ \sum_{S = (s_1, s_2) \in O(T)} \frac{\binom{k_1}{s_1} \cdot \binom{k_2}{s_2}}{k^n - k} \cdot a_{(k_1, k_2)}(T - S),$$
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where \( n = |T|, ar(T) = 2 \), i.e. \( T = T_1 \cdot T_2 \), \( T - S \) and \( a_{(k_1,k_2)}(T - S) \) defined as above and

\[
a_v(T) = a_{(k_1,k_2)}(T) = \frac{(k_1) + (k_2)}{k^n - k} \cdot \prod_{j=1}^{m} a_{(k_1,k_2)}(T_j),
\]

if \( ar(T) = m \) and \( T = T_1 \cdot \ldots \cdot T_m \).

For a rooted tree \( V = (k_1, \ldots, k_r) \) of the height 2 with \( ar(V) = r \) and \( |V| = k = \sum_{i=1}^{r} k_i \) it holds:

- **Degree 1:**

  \[
a_{(k_1,\ldots,k_r)}(x) = 1
\]

- **Degree 2:**

  \[
a_{(k_1,\ldots,k_r)}(x^2) = \frac{(k)}{k^2 - k} = \frac{1}{2}
\]

- **Degree 3:**

  \[
a_{(k_1,\ldots,k_r)}(x \cdot x^2) = \sum_{i=1}^{r} \binom{k_i}{2} + \sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot k_{i_2} \cdot \frac{1}{2} + \sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2}
\]

  \[
  = \frac{(k)}{k^3 - k} \cdot \frac{1}{2} + \frac{\sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2}}{k^3 - k}
\]

  \[
a_{(k_1,\ldots,k_r)}(x^2 \cdot x) = \frac{(k)}{k^3 - k} \cdot \frac{1}{2} + \frac{\sum_{1 \leq i_1 < i_2 \leq r} \binom{k_{i_1}}{2} \cdot k_{i_2}}{k^3 - k}
\]

  \[
  a_{(k_1,\ldots,k_r)}(x^3) = \sum_{i=1}^{r} \binom{k_i}{3} + \sum_{1 \leq i_1 < i_2 < i_3 \leq r} k_{i_1} \cdot k_{i_2} \cdot k_{i_3}
\]

  \[
  \frac{1}{k^3 - k}
\]
Generic exponential and logarithm series

• Degree 4:

\[
\begin{align*}
  a_{(k_1,\ldots,k_3)}(x \cdot (x^2 \cdot x^2)) &= \frac{\sum_{i=1}^r \binom{k_i}{2} + \sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot k_{i_2}}{k^4 - k} \cdot a_{(k_1,\ldots,k_3)}(x \cdot x^2) \\
  &+ \frac{\sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2}}{k^4 - k} \cdot \frac{1}{2} \\
  &= \frac{k}{k^4 - k} \cdot \left( \frac{\binom{k}{2}}{k^3 - k} \cdot \frac{1}{2} + \frac{\sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2}}{k^3 - k} \right) \\
  a_{(k_1,\ldots,k_r)}((x \cdot x^2) \cdot x) &= \frac{\sum_{i=1}^r \binom{k_i}{2} + \sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot k_{i_2}}{k^4 - k} \cdot a_{(k_1,\ldots,k_r)}(x \cdot x^2) \\
  &+ \frac{\sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2}}{k^4 - k} \cdot \frac{1}{2} \\
  &= \frac{k}{k^4 - k} \cdot \left( \frac{\binom{k}{2}}{k^3 - k} \cdot \frac{1}{2} + \frac{\sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2}}{k^3 - k} \right) \\
  a_{(k_1,\ldots,k_r)}((x \cdot x^2) \cdot x) &= \frac{\sum_{i=1}^r \binom{k_i}{2} + \sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot k_{i_2}}{k^4 - k} \cdot a_{(k_1,\ldots,k_r)}(x \cdot x^2) \\
  &+ \frac{\sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2}}{k^4 - k} \cdot \frac{1}{2} \\
  &= \frac{k}{k^4 - k} \cdot \left( \frac{\binom{k}{2}}{k^3 - k} \cdot \frac{1}{2} + \frac{\sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2}}{k^3 - k} \right) \\
  &+ \frac{k}{k^4 - k} \cdot \left( \frac{\binom{k}{2}}{k^3 - k} \cdot \frac{1}{2} + \frac{\sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2}}{k^3 - k} \right) \\
  &+ \frac{\sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2}}{k^4 - k} \cdot \frac{1}{2}
\end{align*}
\]
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\[
a_{(k_1,\ldots,k_r)}(x^2 \cdot x) = \sum_{i=1}^{r} \left( \binom{k_i}{2} \right) + \sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot k_{i_2} \cdot a_{(k_1,\ldots,k_r)}(x^2 \cdot x) + \sum_{1 \leq i_1 < i_2 \leq r} \left( \binom{k_{i_1}}{2} \cdot \binom{k_{i_2}}{2} \right) \cdot \frac{1}{2}
\]

\[
a_{(k_1,\ldots,k_r)}(x^2 \cdot x^2) = \frac{\binom{k}{2}}{k^4 - k} \cdot \frac{1}{2} \cdot \frac{1}{2} + \sum_{1 \leq i_1 < i_2 \leq r} \frac{k_{i_1} \cdot \binom{k_{i_2}}{2}}{k^4 - k} \cdot \frac{1}{2} + \sum_{1 \leq i_1 < i_2 \leq r} \frac{k_{i_1} \cdot k_{i_2} \cdot \binom{k_{i_3}}{2}}{k^4 - k} \cdot \frac{1}{2}
\]

\[
a_{(k_1,\ldots,k_r)}(x \cdot x^3) = \sum_{i=1}^{r} \left( \binom{k_i}{2} \right) + \sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot k_{i_2} \cdot \frac{1}{2} + \sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{3} \cdot k_{i_3} \cdot \frac{1}{k^3 - k} + \sum_{1 \leq i_1 < i_2 \leq r} \frac{k_{i_1} \cdot \binom{k_{i_2}}{3}}{k^4 - k} \cdot \frac{1}{k^3 - k}
\]

\[
a_{(k_1,\ldots,k_r)}(x^3 \cdot x) = \sum_{i=1}^{r} \left( \binom{k_i}{2} \right) + \sum_{1 \leq i_1 < i_2 \leq r} \frac{k_{i_1} \cdot k_{i_2}}{k^4 - k} \cdot \frac{1}{k^4 - k} + \sum_{1 \leq i_1 < i_2 \leq r} \frac{k_{i_1} \cdot \binom{k_{i_2}}{3} \cdot \binom{k_{i_3}}{2}}{k^4 - k} \cdot \frac{1}{k^3 - k} + \sum_{1 \leq i_1 < i_2 \leq r} \frac{k_{i_1} \cdot \binom{k_{i_2}}{3} \cdot \binom{k_{i_3}}{2}}{k^4 - k} \cdot \frac{1}{k^3 - k}
\]
By this presentation we can also compute the coefficients of the logarithm series resp. $V$ as follows:

- **Degree 1:**
  \[ c_{(k_1, \ldots, k_r)}(x) = 1 \]

- **Degree 2:**
  \[ c_{(k_1, \ldots, k_r)}(x^2) = \frac{k}{k - k^n} = -\frac{1}{2} \]

- **Degree 3:**
  \[
  c_{(k_1, \ldots, k_r)}(x \cdot x^2) = \frac{\sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot (\frac{k_{i_2}}{2})}{k - k^3} + \frac{k \cdot \frac{k}{2}}{k - k^3} \cdot \left(-\frac{1}{2}\right)
  \]
  \[
  c_{(k_1, \ldots, k_r)}(x^2 \cdot x) = \frac{\sum_{1 \leq i_1 < i_2 \leq r} (\frac{k_{i_1}}{2}) \cdot k_{i_2}}{k - k^3} + \frac{k \cdot \frac{k}{2}}{k - k^3} \cdot \left(-\frac{1}{2}\right)
  \]
  \[
  c_{(k_1, \ldots, k_r)}(x^3) = \frac{\sum_{i=1}^{r} \left(\frac{k_i}{3}\right) + \sum_{1 \leq i_1 < i_2 < i_3 \leq r} k_{i_1} \cdot k_{i_2} \cdot k_{i_3}}{k - k^3}
  \]
6.4 Tuple-presentation of rooted trees

- **Degree 4:**

\[
c_{(k_1, \ldots, k_r)}(x \cdot (x \cdot x^2)) = \frac{k \cdot \left( \sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2} \right)}{k - k^4} \cdot \left( -\frac{1}{2} \right) \\
+ \frac{k \cdot k \cdot \binom{k}{2}}{k - k^4} \cdot c_{(k_1, \ldots, k_r)}(x \cdot x^2)
\]

\[
= \frac{k \cdot \left( \sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2} \right)}{k - k^4} \cdot \left( -\frac{1}{2} \right) \\
+ \frac{k^2 \cdot \binom{k}{2}}{k - k^4} \cdot \left( \sum_{1 \leq i_1 < i_2 \leq r} \frac{k_{i_1} \cdot \binom{k_{i_2}}{2}}{k - k^3} \right)
\]

\[
+ \frac{k \cdot k \cdot \binom{k}{2}}{k - k^4} \cdot \left( -\frac{1}{2} \right)
\]

\[
c_{(k_1, \ldots, k_r)}(x \cdot (x^2 \cdot x)) = \frac{k \cdot \left( \sum_{1 \leq i_1 < i_2 \leq r} \binom{k_{i_1}}{2} \cdot k_{i_2} \right)}{k - k^4} \cdot \left( -\frac{1}{2} \right) \\
+ \frac{k \cdot k \cdot \binom{k}{2}}{k - k^4} \cdot c_{(k_1, \ldots, k_r)}(x \cdot x^2)
\]

\[
= \frac{k \cdot \left( \sum_{1 \leq i_1 < i_2 \leq r} \binom{k_{i_1}}{2} \cdot k_{i_2} \right)}{k - k^4} \cdot \left( -\frac{1}{2} \right) \\
+ \frac{k^2 \cdot \binom{k}{2}}{k - k^4} \cdot \left( \sum_{1 \leq i_1 < i_2 \leq r} \frac{k_{i_1} \cdot \binom{k_{i_2}}{2}}{k - k^3} \right)
\]

\[
+ \frac{k \cdot k \cdot \binom{k}{2}}{k - k^4} \cdot \left( -\frac{1}{2} \right)
\]

\[
c_{(k_1, \ldots, k_r)}((x \cdot x^2) \cdot x) = \frac{k \cdot \left( \sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2} \right)}{k - k^4} \cdot \left( -\frac{1}{2} \right) \\
+ \frac{k \cdot k \cdot \binom{k}{2}}{k - k^4} \cdot c_{(k_1, \ldots, k_r)}(x^2 \cdot x)
\]

\[
= \frac{k \cdot \left( \sum_{1 \leq i_1 < i_2 \leq r} k_{i_1} \cdot \binom{k_{i_2}}{2} \right)}{k - k^4} \cdot \left( -\frac{1}{2} \right) \\
+ \frac{k^2 \cdot \binom{k}{2}}{k - k^4} \cdot \left( \sum_{1 \leq i_1 < i_2 \leq r} \frac{k_{i_1} \cdot \binom{k_{i_2}}{2}}{k - k^3} \right)
\]

\[
+ \frac{k \cdot k \cdot \binom{k}{2}}{k - k^4} \cdot \left( -\frac{1}{2} \right)
\]
\[ c(k_1, \ldots, k_r)((x^2 \cdot x) \cdot x) = \frac{k \cdot \left( \sum_{1 \leq i_1 < i_2 < r} \left( \frac{k_{i_1}}{2} \right) \cdot k_{i_2} \right)}{k - k^4} \cdot (-\frac{1}{2}) \]

\[ + \frac{k \cdot k \cdot \left( \frac{k}{2} \right)}{k - k^4} \cdot c(k_1, \ldots, k_r)(x^2 \cdot x) \]

\[ = \frac{k \cdot \left( \sum_{1 \leq i_1 < i_2 < r} \left( \frac{k_{i_1}}{2} \right) \cdot k_{i_2} \right)}{k - k^4} \cdot (-\frac{1}{2}) \]

\[ + \frac{k^2 \cdot \left( \frac{k}{2} \right)}{k - k^4} \cdot \left( \frac{1}{k - k^3} \sum_{1 \leq i_1 < i_2 < r} \left( \frac{k_{i_1}}{2} \right) \cdot k_{i_2} \right) + \frac{k \cdot \left( \frac{k}{2} \right)}{k - k^3} \cdot \left( -\frac{1}{2} \right) \]

\[ c(k_1, \ldots, k_r)(x^2 \cdot x^2) = \frac{\sum_{1 \leq i_1 < i_2 < r} \left( \frac{k_{i_1}}{2} \right) \cdot \left( \frac{k_{i_2}}{2} \right)}{k - k^4} + \frac{k \cdot \left( \frac{k}{2} \right)}{k - k^4} \cdot \left( -\frac{1}{2} \right) \]

\[ + \frac{k \cdot k \cdot \left( \frac{k}{2} \right)}{k - k^4} \cdot \left( c(k_1, \ldots, k_r)(x \cdot x^2) + c(k_1, \ldots, k_r)(x^2 \cdot x) \right) \]

\[ c(k_1, \ldots, k_r)(x \cdot x^3) = \frac{\sum_{1 \leq i_1 < i_2 < i_3 < r} k_{i_1} \cdot k_{i_2} \cdot \left( \frac{k_{i_3}}{3} \right)}{k - k^4} + \frac{k \cdot \left( \sum_{1 \leq i_1 < i_2 < i_3 < r} k_{i_1} \cdot k_{i_2} \cdot k_{i_3} \right)}{k - k^4} \cdot \left( -\frac{1}{2} \right) \]

\[ c(k_1, \ldots, k_r)(x^3 \cdot x) = \frac{\sum_{1 \leq i_1 < i_2 < i_3 < r} k_{i_1} \cdot k_{i_2} \cdot \left( \frac{k_{i_3}}{2} \right)}{k - k^4} + \frac{k \cdot k \cdot \left( \frac{k}{2} \right)}{k - k^4} \cdot c(k_1, \ldots, k_r)(x^2) \]

\[ = \frac{\sum_{1 \leq i_1 < i_2 < i_3 < r} k_{i_1} \cdot k_{i_2} \cdot \left( \frac{k_{i_3}}{2} \right)}{k - k^4} \]

\[ + \frac{k \cdot k \cdot \left( \frac{k}{2} \right)}{k - k^4} \cdot \left( \sum_{i=1}^{r} \left( \frac{k_i}{3} \right) + \sum_{1 \leq i_1 < i_2 < i_3 < r} k_{i_1} \cdot k_{i_2} \cdot k_{i_3} \right) \]

\[ c(k_1, \ldots, k_r)(x \cdot x^2 \cdot x) = \frac{\sum_{1 \leq i_1 < i_2 < i_3 < r} k_{i_1} \cdot \left( \frac{k_{i_2}}{2} \right) \cdot k_{i_2}}{k - k^4} + \frac{k \cdot k \cdot \left( \frac{k}{2} \right)}{k - k^4} \cdot c(k_1, \ldots, k_r)(x^2) \]

\[ c(k_1, \ldots, k_r)(x \cdot x \cdot x^2) = \frac{\sum_{1 \leq i_1 < i_2 < i_3 < r} \left( \frac{k_{i_1}}{2} \right) \cdot k_{i_2} \cdot k_{i_3}}{k - k^4} + \frac{k \cdot k \cdot \left( \frac{k}{2} \right)}{k - k^4} \cdot c(k_1, \ldots, k_r)(x^2) \]

\[ = \sum_{i=1}^{r} \left( \frac{k_i}{4} \right) + \sum_{1 \leq i_1 < \ldots < i_4 \leq r} k_{i_1} \cdot \ldots \cdot k_{i_4} \]

\[ c(k_1, \ldots, k_r)(x^4) = \frac{\sum_{i=1}^{r} \left( \frac{k_i}{4} \right)}{k - k^4} \]
Moreover it is also possible to write the rooted tree of height 3 in the following way using coronas:

\[ W = ( (w_{1,1}, \ldots, w_{1,r_1}), \ldots, (w_{l,1}, \ldots, w_{l,r_l}) ) \]

with \( ar(W) = l, |W| = \sum_{i=1}^{l} \sum_{j=1}^{r_i} w_{i,j} \), where \( W = W_1 \cdots W_l \) and \( W_i = (w_{i,1}, \ldots, w_{i,r_i}) \) are trees of height 2 with \( ar(W_i) = r_i, |W_i| = \sum_{j=1}^{r_i} w_{i,j} \).

It also holds that \( w_{i,j} \in \mathbb{N} \geq 1 \) with \( w_{i,j} = k = x^k \) is the \( k \)-corona.

Thus if \( r_i = 1, \forall i = 1, \ldots, l \), then \( W \) is a rooted tree of height 2.

Now we can compute the planar binomial coefficients for rooted trees \( V = ((k_{1,1}, \ldots, k_{1,r_1}), \ldots, (k_{l,1}, \ldots, k_{l,r_l})) \) of height 3 with \( ar(V) = l, |V| = \sum_{i=1}^{l} \sum_{j=1}^{r_i} k_{i,j} = k \) and \( V = V_1 \cdots V_l \) with \( ar(V_i) = r_i, |V_i| = \sum_{j=1}^{r_i} k_{i,j} = k_i \) as follows:

\[
\binom{V}{S} = \begin{cases} 
\sum_{i=1}^{l} \sum_{j=1}^{r_i} \binom{k_{i,j}}{m} + \sum_{i=1}^{l} \sum_{1 \leq i_1 < \ldots < i_m \leq r_i} \prod_{j=1}^{m} k_{i,j} + \sum_{1 \leq i_1 < \ldots < i_m \leq l} \prod_{j=1}^{m} |V_j| & : \text{if } H(S) = 1 \\
\sum_{i=1}^{l} \sum_{1 \leq i_1 < \ldots < i_m \leq r_i} \prod_{j=1}^{m} \binom{k_{i,j}}{s_j} + \sum_{1 \leq i_1 < \ldots < i_m \leq l} \prod_{j=1}^{m} (V_j)_{s_j} & : \text{if } H(S) = 2 \\
\sum_{1 \leq i_1 < \ldots < i_m \leq l} \prod_{j=1}^{m} \left( \sum_{1 \leq b_1 < \ldots < b_{q_j} \leq r_{i,j}} \prod_{t=1}^{q_j} \binom{k_{i,j}}{s_j} \right) & : \text{if } H(S) = 3 \\
0 & : \text{else.}
\end{cases}
\]

There it holds

- If \( H(S) = 1 \), then \( S = x^m \).
- If \( H(S) = 2 \), then \( S = (s_1, \ldots, s_m) \).
- If \( H(S) = 3 \), then \( S = ((s_{1,1}, \ldots, s_{1,q_1}), \ldots, (s_{m,1}, \ldots, s_{m,q_m})) \).

Moreover in the case \( H(S) = 2 \) it is necessary to use the formula for the trees of height 2 in the computation of \( \binom{V_j}{s_j} \).

Let \( V_j = (k_{j,1}, \ldots, k_{j,r_j}) \) with \( ar(V_j) = r_j, |V_j| = \sum_{p=1}^{r_j} k_{i,j,p} = k_j \) and \( s_j = x^{q_j} \) be a corona with \( ar(s_j) = q_j \). Then

\[
\binom{V_j}{s_j} = \sum_{p=1}^{r_j} \binom{k_{i,j,p}}{q_j} + \sum_{1 \leq b_1 < \ldots < b_{q_j} \leq r_{i,j}} \prod_{t=1}^{q_j} k_{i,j,b_t}.
\]
By this presentation it is possible to compute the coefficients of the exponential series resp. the rooted trees of height 3 with the following recursion formula.

Let \( V = (((k_1,1,\ldots,k_1,r_1)),\ldots,(k_l,1,\ldots,k_l,r_l)) \) be a rooted tree of height 3 with \( ar(V) = l, |V| = \sum_{i=1}^l \sum_{j=1}^{r_i} k_{i,j} = k \) and \( V = V_1 \cdot \ldots \cdot V_l \) with \( ar(V_l) = r_i, |V_l| = \sum_{j=1}^{r_i} k_{i,j} = k_i \).

Then for the coefficients of \( \exp_V(x) = \exp((k_1,1,\ldots,k_1,r_1),\ldots,(k_l,1,\ldots,k_l,r_l))(x) \) it holds

\[
a_V(x) = 1
\]

and

\[
a_V(T) = \frac{\sum_{i=1}^l \sum_{j=1}^{r_i} \binom{k_{i,j}}{m}}{k^{n-k}} + \frac{\sum_{i=1}^l \sum_{q_1<\ldots<q_{m}\leq r_i} \prod_{j=1}^{m} k_{i,q_j}}{k^{n-k}} + \frac{\sum_{1\leq i_1<\ldots<i_{m}\leq l} \prod_{j=1}^{m} V_{i_j}}{k^{n-k}} \cdot a_V(T_S)
\]

where \( n = |T|, m = ar(T), T - x^m = (T_1,\ldots,T_m), T - S = (T_1,\ldots,T_{|S|}) \) is a planar rooted forest and \( a_V(T_S) = \prod_{j=1}^{\frac{|S|}{m}} a_V(T_j) \).

**Proposition 6.4.2.** Let \( V = (k_1,\ldots,k_r) \in P \) be a rooted tree of height 2, \( |V| = \sum_{i=1}^r k_i = k, k_i = x^{k_i} \) coronas of degree \( k_i \), \( \forall i = 1,\ldots,r \) and \( \exp_V(x) = \exp(k_1,\ldots,k_r)(x) = \exp_V(k,x) \) the exponential series resp. \( V \).

Let \( r-1 \) entries in \( V \) be constant and one entry equal to a corona run to infinity, then

(i) \( |V| = k \to \infty \), if \( k_i \to \infty \) for one \( i = 1,\ldots,r \).

(ii) \[ \lim_{k_i \to \infty} \exp_V(k,x) = \lim_{k \to \infty} \exp_V(k,x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}. \]

**Proof.** 1) \( k := \sum_{i=1}^r k_i \implies k \to \infty \), if \( k_i \to \infty \), for one \( i = 1,\ldots,r \), is trivial by the definition of \( k \).
2) It holds
\[
a_V(k, T) = \sum_{i=1}^{r} \frac{(k_i)}{m} + \sum_{1 \leq i_1 < \ldots < i_m \leq r} \prod_{j=1}^{m} k_{ij} \cdot \prod_{j=1}^{m} a_V(k, T_j) \\
+ \sum_{S = (s_1, \ldots, s_m) \in O(T)} \frac{\prod_{j=1}^{m} (k_{s_j})}{k^n - k} \cdot a_V(k, T - S),
\]

where \( n = |T|, m = ar(T), T - x^m = (T_1, \ldots, T_m), T - S = (T_1, \ldots, T_{|S|}) \) is the planar rooted forest and \( a_{(k_1, \ldots, k_r)}(T - S) = \prod_{j=1}^{s} a_{(k_1, \ldots, k_r)}(T_j). \)

For any \( f \in K(k), f \neq 0 \), we denote by
\[
(k - ord_{\infty})(f)
\]
the order of \( f \) at \( \infty \). If \( f = \frac{f_2}{f_1} \) and \( f_i \) are polynomials in \( K[k] \) then: \( (k - ord_{\infty})(f) = deg(f_2) - deg(f_1) \). In this case it holds:
\[
(k - ord_{\infty}) \left( \sum_{i=1}^{r} \frac{(k_i)}{m} + \sum_{1 \leq i_1 < \ldots < i_m \leq r} \prod_{j=1}^{m} k_{ij} \cdot \prod_{j=1}^{m} a_V(k, T_j) \right) = n - m = n - ar(T)
\]

and
\[
(k - ord_{\infty}) \left( \sum_{S = (s_1, \ldots, s_m) \in O(T), s_j \neq 1, \forall j} \frac{\prod_{j=1}^{m} (k_{s_j})}{k^n - k} \right) = n - \max\{s_j\},
\]

where \( \max\{s_j\} \) is only picked out of \( s_j \) if in the sum \( i_j \) is equal to the \( i \) which runs towards infinity.

Thus
\[
(k - ord_{\infty})(a_V(k, T)) \geq 0,
\]

\( \forall T \in \mathbb{P} \), because
\[
(k - ord_{\infty})(a_V(k, T))
\]
\[
= (k - ord_{\infty}) \left( \sum_{i=1}^{r} \frac{(k_i)}{m} + \sum_{1 \leq i_1 < \ldots < i_m \leq r} \prod_{j=1}^{m} k_{ij} \cdot \prod_{j=1}^{m} a_V(k, T_j) \right) \cdot (k - ord_{\infty}) \left( \prod_{j=1}^{m} a_V(k, T_j) \right)
\]
\[
+ (k - ord_{\infty}) \left( \sum_{S = (s_1, \ldots, s_m) \in O(T), s_j \neq 1, \forall j} \frac{\prod_{j=1}^{m} (k_{s_j})}{k^n - k} \right) \cdot (k - ord_{\infty})(a_V(k, T - S))
\]
\[ (k - \text{ord}_\infty)(a_V(k, T)) > 0, \ \forall \ T \in \mathbb{P} - \{x^m : m \geq 2\}, \]
and for \( T = x^m \)
\[ (k - \text{ord}_\infty)(a_V(k, T)) = 0, \]
because only then \(|T| = n = m = ar(T)\) and as in the case \(H(T) = 2, \max\{s_j\} < n = \sum_{j=1}^{m} s_j\). Thus
\[ \lim_{k \to \infty} (a_V(k, T)) = 0, \ \forall \ T \in \mathbb{P} - \{x^m : m \geq 2\}. \]

Moreover
\[ a_V(k, x^m) = \frac{\sum_{i=1}^{m} (\binom{k}{m}) + \sum_{1 \leq i_1 < \ldots < i_m \leq r} \prod_{j=1}^{m} k_{i_j}}{k^m - k} \]
\[ \Longrightarrow \]
\[ a_V(k, x^m)(\infty) = \frac{1}{m!}, \]
because the second sum in the numerator does not have an influence. This is because these polynomials either have the degree 0 or 1 with regard to \(k_i\) for the chosen \(i\) and hence they disappear in the limit. So the leading coefficient of the first sum in the numerator is \(\frac{1}{m!}\) and in the denominator 1.

\textbf{Remark 6.4.3.} By this presentation it is of course also possible to compute the coefficients of the logarithm series \(\log V(1 + x)\) for trees \(V\) of height 3 in a similar way as in the case of the exponential series.
Chapter 7

Further aspects and applications

7.1 Co-additive Co-multiplication $\Delta$

Proposition 7.1.1. Let $K\{x\}$ be the planar $K$-algebra of polynomials. Then there is a co-additive $K$-algebra homomorphism $\Delta : K\{x\} \rightarrow K\{x\} \otimes K\{x\}$ defined by

$$\Delta(1) = 1 \otimes 1 \quad \text{and} \quad \Delta(x) = x \otimes 1 + 1 \otimes x.$$  

Let $K\{(x)\}$ be the $x$-adic completion of $K\{x\}$. Similar considerations as in Remark 2.5.4 we apply to tensor products and we denote by

$$K\{(x)\} \hat{\otimes} K\{(x)\},$$

the $(x \otimes 1, 1 \otimes x)$-adic completion of $K\{x\} \otimes K\{x\}$ with respect to the topology induced by

$$ord(f) = \min\{|S| + |T| : \text{coeff}_{S \otimes T}(f), S, T \in \mathbb{P}\},$$

for all $f \in K\{x\} \otimes K\{x\}$. The co-additive $K$-algebra homomorphism extends continuously to a $K$-algebra homomorphism

$$\hat{\Delta} : K\{(x)\} \rightarrow K\{(x)\} \hat{\otimes} K\{(x)\}$$

of formal power series completions of $K\{x\}$ and $K\{x\} \otimes K\{x\}$, see [Sch], [Hol1] and [Hol2].

The exponential series $\exp_V(x)$ have the following important property.

Proposition 7.1.2. Let $\exp_V(x) \in K\{(x)\}$ be the planar exponential series with respect to $V$ and $\hat{\Delta} : K\{(x)\} \rightarrow K\{(x)\} \hat{\otimes} K\{(x)\}$ a $K$-algebra homomorphism. Then

$$\hat{\Delta}(\exp_V(x)) = \exp_V(x) \hat{\otimes} \exp_V(x), \quad \forall \ V \in \mathbb{P}.$$  

Proof. 1) Let $V \in \mathbb{P}$ with $|V| = k \geq 2$ and $\exp_V(x) \in K\{(x)\}$ be the planar exponential series resp. $V$. We will show that

$$\hat{\Delta}(\exp_V(x)) = \exp_V(x) \hat{\otimes} \exp_V(x).$$
Denote the right side of the equation with $f$ and the left side with $g$.

2) Let $E(x) \in K\{\{x\}\}$ be a power series with the constant term 1, such that $(E(x))^V = E(kx)$ for one $V$ and $|V| = k \geq 2$. Moreover write $E(x) = \sum_{n=0}^{\infty} F_n$, where $F_n = F_n(x)$ is the homogeneous component of $E$ of degree $n$; then $F_0 = 1$.

Let now $F = E - 1 = \sum_{n=1}^{\infty} F_n$. The homogeneous component of $E^V$ of degree $n \geq 1$ is

$$
((E(x))^V)_n = \left( \sum_{m=1}^{k} \sum_{\substack{s \in P(V) \setminus \{S\} = m}} \binom{V}{S} \cdot (F(x))^S \right)_n
$$

$$
= \sum_{m=1}^{k} \sum_{\substack{s \in P(V) \setminus \{S\} = m}} \binom{V}{S} \cdot ((F(x))^S)_n
$$

$$
= k \cdot (F(x))_n + \sum_{m=2}^{k} \sum_{\substack{s \in P(V) \setminus \{S\} = m}} \binom{V}{S} \cdot ((F(x))^S)_n.
$$

Further $(E(kx))_n = k^n \cdot (F(x))_n$ and because of the functional equation of $E(x)$ it holds

$$(k^n - k) \cdot (F(x))_n = \sum_{m=2}^{k} \sum_{\substack{s \in P(V) \setminus \{S\} = m}} \binom{V}{S} \cdot ((F(x))^S)_n. \quad (*)$$

Now for the homogeneous component of $(F(x))^S$ of degree $n$ it is true that

$$
((F(x))^S)_n = \sum_{\nu \in M_S(m,n)} F_{\nu},
$$

where $m = |S|$, $M_S(m,n) = \{ \nu = (\nu_1, \ldots, \nu_m) : \nu_i \in \mathbb{N}^{\geq 1}, |\nu| = \nu_1 + \ldots + \nu_m = n \}$ and $F_{\nu} := \cdot_S(F_{\nu_1}, \ldots, F_{\nu_m})$ for $\nu = (\nu_1, \ldots, \nu_m) \in M_S(m,n)$. Thus $(F(x))_n$ is defined uniquely by $F_0$ and $F_1$ because of $(*)$.

3) From the property $\exp_V(kx) = (\exp_V(x))^V$ of the exponential series it follows that

$$f(kx) = \hat{\Delta}(\exp_V(kx)) = \hat{\Delta}((\exp_V(x))^V) = \left( \hat{\Delta}(\exp_V(x)) \right)^V = (f(x))^V$$

and obviously

$$(g(x))^V = (\exp_V(x))^V \otimes (\exp_V(x))^V = \exp_V(kx) \hat{\otimes} \exp_V(x) = g(kx).$$

If we write $f$ and $g$ as a sum over the homogeneous components $f = \sum_{n \geq 0} f_n$ and $g = \sum_{n \geq 0} g_n$ then we get

$$f_0 = g_0 = 1 \otimes 1,$$
7.1 Co-additive Co-multiplication \( \Delta \)

\[ f_1 = g_1 = 1 \otimes x + x \otimes 1 \]

and thus \( f = g \) with the step 2).

**Proposition 7.1.3.** Let \( \exp_{(V, \lambda)}(x) \in K\{\{x\}\} \) be the planar \( \lambda \)-deformed exponential series resp. \( V \) with \( k = |V| \), \( \lambda = (\lambda_1, \ldots, \lambda_k) \) with \( \delta_r(\lambda) \neq 0 \), \( \forall r \geq 2 \) and \( \hat{\Delta} : K\{\{x\}\} \to K\{\{x\}\} \otimes K\{\{x\}\} \) a \( K \)-algebra homomorphism, defined as above. Then

\[ \hat{\Delta}(\exp_{(V, \lambda)}(x)) = \exp_{(V, \lambda)}(x) \otimes \exp_{(V, \lambda)}(x). \]

**Proof.** 1) Let \( V \in \mathbb{P} \) with \( |V| = k \geq 2 \) and \( (\lambda_1, \ldots, \lambda_k) = \lambda \) with \( \delta_r(\lambda) \neq 0 \), \( \forall r \geq 2 \).

Denote the right side of the equation in the proposition with \( f \) and the left side with \( g \).

2) Let \( E(x) = \sum_{r=0}^{\infty} E_r(x) \in K\{\{x\}\} \) with the constant term 1, such that

\[ \cdot_{\cdot}(E(\lambda_1 x), \ldots, E(\lambda_k x)) = E((\lambda_1 + \ldots + \lambda_k) x), \]

where \( E_r(x) \) is the homogeneous component of \( E(x) \) of degree \( r \) and it holds

\[ E_r(x) = \delta_r^{-1} \cdot \sum_{m=2}^{k} \sum_{\substack{S \in \mathcal{P}(V) \setminus \emptyset \neq x \ \text{and} \ S \neq \emptyset \ \text{and} \ 1 \leq i_1 < \ldots < i_m \leq k}} \lambda_{i_1}^{r_1} \cdots \lambda_{i_m}^{r_m} \cdot s(E_{r_1}(x), \ldots, E_{r_m}(x)) \]

where \( m = |S| \) and \( (i_1, \ldots, i_m) \in I_S \); then \( E_0 = 1 \). Thus

\[ (\cdot_{\cdot}(E(\lambda_1 x), \ldots, E(\lambda_k x)))_r = \sum_{S} \sum_{\substack{r_1 + \ldots + r_m = r \ \text{and} \ \delta_{r_1}(\lambda) \neq 0 \ \text{and} \ \delta_{r_m}(\lambda) \neq 0 \ \text{and} \ 1 \leq i_1 < \ldots < i_m \leq k \}} \lambda_{i_1}^{r_1} \cdots \lambda_{i_m}^{r_m} \cdot s(E_{r_1}(x), \ldots, E_{r_m}(x)) \]

Moreover \( (E((\lambda_1 + \ldots + \lambda_k) x))_r = (\lambda_1 + \ldots + \lambda_k)^r \cdot E_r(x) \) and because of the functional equation of \( E(x) \) it holds that

\[ ((\lambda_1 + \ldots + \lambda_k)^r - (\lambda_1^r + \ldots + \lambda_k^r)) \cdot E_r(x) = \sum_{S, S \neq x} \sum_{\substack{r_1 + \ldots + r_m = r \ \text{and} \ 1 \leq i_1 < \ldots < i_m \leq k \ \text{and} \ \delta_{r_1}(\lambda) \neq 0 \ \text{and} \ \delta_{r_m}(\lambda) \neq 0 \}} \lambda_{i_1}^{r_1} \cdots \lambda_{i_m}^{r_m} \cdot s(E_{r_1}(x), \ldots, E_{r_m}(x)) \]

\[ \iff \delta_r(\lambda) \cdot E_r(x) = \sum_{S, S \neq x} \sum_{\substack{r_1 + \ldots + r_m = r \ \text{and} \ 1 \leq i_1 < \ldots < i_m \leq k \ \text{and} \ \delta_{r_1}(\lambda) \neq 0 \ \text{and} \ \delta_{r_m}(\lambda) \neq 0 \}} \lambda_{i_1}^{r_1} \cdots \lambda_{i_m}^{r_m} \cdot s(E_{r_1}(x), \ldots, E_{r_m}(x)). \] (*)

Thus

\[ ((E(x))^S)_r = \sum_{S, S \neq x} \sum_{\substack{r_1 + \ldots + r_m = r \ \text{and} \ 1 \leq i_1 < \ldots < i_m \leq k \ \text{and} \ \delta_{r_1}(\lambda) \neq 0 \ \text{and} \ \delta_{r_m}(\lambda) \neq 0 \}} \cdot s(E_{r_1}(x), \ldots, E_{r_m}(x)). \]
Further aspects and applications

$E_r(x)$ is uniquely defined by ($E_0$ and) $E_1$ since (*) holds for all $r$ and thus the uniqueness follows by induction on $r$.

3) From the property $\exp_{(V,\lambda)}((\lambda_1+\ldots+\lambda_k)x) = \cdot V(\exp_{(V,\lambda)}(\lambda_1x),\ldots,\exp_{(V,\lambda)}(\lambda_kx))$ of the exponential series it follows

$$f((\lambda_1 + \ldots + \lambda_k)x) = \hat{\Delta}(\exp_{(V,\lambda)}((\lambda_1 + \ldots + \lambda_k)x))$$
$$= \hat{\Delta}(\cdot V(\exp_{(V,\lambda)}(\lambda_1x),\ldots,\exp_{(V,\lambda)}(\lambda_kx)))$$
$$= \cdot V(\hat{\Delta}(\exp_{(V,\lambda)}(\lambda_1x)),\ldots,\hat{\Delta}(\exp_{(V,\lambda)}(\lambda_kx)))$$
$$= \cdot V(f(\lambda_1x),\ldots,f(\lambda_kx))$$

and obviously

$$\cdot V(g(\lambda_1x),\ldots,g(\lambda_kx))$$
$$= \cdot V(\exp_{(V,\lambda)}(\lambda_1x),\ldots,\exp_{(V,\lambda)}(\lambda_kx)) \hat{\otimes} \cdot V(\exp_{(V,\lambda)}(\lambda_1x),\ldots,\exp_{(V,\lambda)}(\lambda_kx))$$
$$= \exp_{(V,\lambda)}((\lambda_1 + \ldots + \lambda_k)x) \hat{\otimes} \exp_{(V,\lambda)}((\lambda_1 + \ldots + \lambda_k)x)$$
$$= g((\lambda_1 + \ldots + \lambda_k)x).$$

If $f = \sum_{n \geq 0} f_n$ and $g = \sum_{n \geq 0} g_n$ are written as a sum over the homogeneous components then one gets

$$f_0 = g_0 = 1 \otimes 1,$$
$$f_1 = g_1 = 1 \otimes x + x \otimes 1$$

and thus with step 2) it follows that $f = g$. \hfill \Box

**Corollary 7.1.4.** Let $S, T, V \in \mathbb{P}$ with $|S| + |T| = n$ and $|V| = k$, $\lambda = (\lambda_1,\ldots,\lambda_k)$ with $\delta_r(\lambda) \neq 0, \forall r \geq 2$ and $a_{(V,\lambda)}(R)$ be the exponential coefficient of $R \in \mathbb{P}$ with respect to $V$ and resp. $a_{(V,\lambda)}(R)$ the exponential coefficient of $R$ in the series $\exp_{(V,\lambda)}$. Then

(i) $$a_{(V,\lambda)}(S) \cdot a_{(V,\lambda)}(T) = \sum_{R \in \mathbb{P}} a_{(V,\lambda)}(R) \cdot \langle R, S \shuffle T \rangle.$$  

(ii) $$a_{(V,\lambda)}(S) \cdot a_{(V,\lambda)}(T) = \sum_{R \in \mathbb{P}} a_{(V,\lambda)}(R) \cdot \langle R, S \shuffle T \rangle,$$

where $\langle R, S \shuffle T \rangle$ is the coefficient of $R$ in the shuffle of $S$ and $T$.

**Proof.** These identities follow immediately from Propositions 7.1.2 and 7.1.3. \hfill \Box
Example 7.1.5. Let \( V \in \mathbb{P} \) with \(|V| = k\), \( S = x^{n-2} \) and \( T = x^2 \) with \( n > 5 \). Then
\[
\langle R, S \uplus T \rangle = \langle \Delta(R), S \otimes T \rangle = \left( \begin{array}{c} R \\ x^{n-1}, x^2 \end{array} \right).
\]
Here
\[
\left( \begin{array}{c} (n) \\ 2 \\
1 \\
1 \\
1 \\
3 \\
3 \\
n-1 \\
2n-3 \\
2 \\
4 \\
0 \\
\end{array} \right) = \left( \begin{array}{c} R \\ x^{n-1}, x^2 \end{array} \right)
\]
(1 possibility)
(2 possibilities)
(3 possibilities)
(4 possibilities)
(n - 2 possibilities)
(n - 2 possibilities)
(n - 1 possibilities)
(2 possibilities)
(2 \cdot (n - 2) possibilities)
\((n-2)\) possibilities
else.
For the exponential coefficients it holds
\[
a_V(R) = \sum_S \frac{\binom{V}{S}}{k^n - k} \cdot a_V(T - S),
\]
where \( S \in P(V) \cap O(T) \), \( S \neq x \), \( 2 \leq m = |S|, n = |T| \) and \( a_V(T - S) = \prod_{i=1}^{m} a_V(T_i) \), if \( T - S = (T_1, \ldots, T_m) \). Thus
\[
a_V(S) \cdot a_V(T) = \frac{1}{2} \cdot \frac{(V \setminus x^{n-2})}{k^{n-2} - k}
\]
and
\[
\sum_{\substack{R \in \mathbb{P} \\ |R| = n}} a_V(R) \cdot \langle R, S \uplus T \rangle
= \left( \begin{array}{c} n \\
2 \\
1 \\
1 \\
1 \\
3 \\
3 \\
2n-3 \\
2 \\
4 \\
(n-1) \\
\end{array} \right) \cdot a_V(x^n) + a_V(x^{n-2} \cdot x^2) + a_V(x^2 \cdot x^{n-2})
+ a_V(x \cdot x \cdot x^{n-2}) + a_V(x \cdot x^{n-2} \cdot x) + a_V(x^{n-2} \cdot x \cdot x)
+ a_V(x \cdot (x \cdot x^{n-2})) + a_V((x \cdot x^{n-2}) \cdot x) + a_V(x \cdot (x^{n-2}) \cdot x) + a_V((x^{n-2} \cdot x) \cdot x)
+ a_V(x^3 \cdot x \cdot x \cdot x) + a_V(x \cdot x^3 \cdot x \cdot x)
+ a_V((x^2 \cdot x) \cdot x \cdot x \cdot x) + a_V(x \cdot (x^2 \cdot x) \cdot x \cdot x)
+ a_V((x \cdot x^2) \cdot x \cdot x \cdot x) + a_V(x \cdot x \cdot x \cdot x \cdot x)
+ a_V((x^2 \cdot x) \cdot x \cdot x \cdot x) + a_V(x \cdot (x^2 \cdot x) \cdot x \cdot x)
+ (n-1) \cdot (a_V(x \cdot x^{n-1}) + a_V(x^{n-1} \cdot x))
\]
Further aspects and applications

\[
ar_{(R) = n}^{n-1} + (2n + 3) \cdot (a_{V}(x^2 \cdot x \cdot \ldots \cdot x) + \ldots + a_{V}(x \cdot \ldots \cdot x \cdot x^2))
+ 2 \cdot (a_{V}(x \cdot (x^2 \cdot x \cdot \ldots \cdot x)) + \ldots + a_{V}(x \cdot (x \cdot \ldots \cdot x \cdot x^2)))
+ 2 \cdot (a_{V}((x^2 \cdot x \cdot \ldots \cdot x) \cdot x) + \ldots + a_{V}((x \cdot \ldots \cdot x \cdot x^2) \cdot x))
= a_{V}(x^{n-2}) \cdot \left( \frac{(k)}{k^n-k} + 2 \cdot \frac{(2)}{(k^n-k) \cdot (k^{n-1}-k)} \right)
+ V \cdot \left( \frac{1}{k^n-k} + \frac{2 \cdot (n-1) \cdot (n^2-1)}{(k^n-k) \cdot (k^{n-1}-k)} \right)
+ \left( \frac{V}{x \cdot x^n-1} \right) \cdot \left( \frac{1}{k^n-k} + \frac{2 \cdot (n-1) \cdot (n^2-1)}{(k^n-k) \cdot (k^{n-1}-k)} \right)
+ \left( \frac{V}{x \cdot x \cdot x^n-2} \right) \cdot \left( \frac{1}{k^n-k} + \frac{2 \cdot (n-1) \cdot (n^2-1)}{(k^n-k) \cdot (k^{n-1}-k)} \right)
+ \left( \frac{V}{x \cdot x \cdot x \cdot x^n-2} \right) \cdot \left( \frac{1}{k^n-k} + \frac{2 \cdot (n-1) \cdot (n^2-1)}{(k^n-k) \cdot (k^{n-1}-k)} \right)
+ \left( \frac{V}{x \cdot x \cdot x \cdot x \cdot x} \right) \cdot \left( \frac{1}{k^n-k} + \frac{2 \cdot (n-1) \cdot (n^2-1)}{(k^n-k) \cdot (k^{n-1}-k)} \right)
+ \left( \frac{V}{x \cdot x \cdot x \cdot x \cdot x \cdot x} \right) \cdot \left( \frac{1}{k^n-k} + \frac{2 \cdot (n-1) \cdot (n^2-1)}{(k^n-k) \cdot (k^{n-1}-k)} \right)
\]
Because of Corollary 7.1.4 it holds

\[ a_V(S) \cdot a_V(T) - \sum_{R \in \mathcal{P}} a_V(R) \cdot \langle R, S \llcorner \llcorner T \rangle = 0 \]

and thus

\[
\left( \begin{array}{c}
V \\
(x^{n-2})
\end{array} \right) \cdot \frac{1}{k^{n-2} - k} \cdot \left( \frac{1}{2} - \frac{k}{2} + \frac{3}{k} - \frac{4 \cdot (k/2)^2}{(k^n - k) \cdot (k^{n-1} - k)} \right) - \left( \begin{array}{c}
V \\
x^n
\end{array} \right) \cdot \frac{\binom{n}{2}}{k^n - k} \\
- \left( \begin{array}{c}
V \\
x^{n-2}
\end{array} \right) \cdot \left( \frac{1}{2} \cdot (n - 2) + \frac{(n-2)^2}{2} \right) + \frac{2 \cdot (n-2) \cdot \binom{k}{2}}{(k^n - k) \cdot (k^{n-1} - k)} \\
- \left( \begin{array}{c}
V \\
x^{n-1}
\end{array} \right) \cdot \left( \frac{1}{2} \cdot (n - 1) \cdot (2n - 3) \right) + \frac{2 \cdot (n-1) \cdot \binom{k}{2}}{(k^n - k) \cdot (k^{n-1} - k)} \\
- \left( \begin{array}{c}
V \\
x^n \cdot x^{n-2}
\end{array} \right) + \left( \begin{array}{c}
V \\
x^{n-2} \cdot x
\end{array} \right) \cdot \left( \frac{1}{2} + n - 2 \cdot \frac{1}{k^n - k} + \frac{2 \cdot \binom{k}{2}}{(k^n - k) \cdot (k^{n-1} - k)} \right) \\
- \left( \begin{array}{c}
V \\
x \cdot x^n
\end{array} \right) + \left( \begin{array}{c}
V \\
x^{n-1} \cdot x
\end{array} \right) \cdot \frac{n-1}{k^n - k} + \left( \begin{array}{c}
V \\
x \cdot x^{n-2}
\end{array} \right) \cdot \frac{1}{k^n - k} \\
- \left( \begin{array}{c}
V \\
x \cdot x \cdot x^{n-2}
\end{array} \right) + \left( \begin{array}{c}
V \\
x \cdot x^{n-2} \cdot x
\end{array} \right) + \left( \begin{array}{c}
V \\
x^n \cdot x^n
\end{array} \right) \cdot \frac{1}{k^n - k} \\
- \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x^{n-2}
\end{array} \right) + \left( \begin{array}{c}
V \\
x \cdot x^{n-2} \cdot x \cdot x
\end{array} \right) + \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x
\end{array} \right) \cdot \frac{1}{k^n - k} \\
- \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) + ... + \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) \cdot \frac{3}{k^n - k} \\
- \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) + ... + \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) \cdot \frac{3}{k^n - k} \\
- \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) + ... + \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) \cdot \frac{3}{k^n - k} \\
- \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) + ... + \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) \cdot \frac{4}{k^n - k} \\
- \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) + ... + \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) \cdot \frac{2}{k^n - k} \\
- \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) + ... + \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) \cdot \frac{2}{k^n - k} \\
- \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) + ... + \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) \cdot \frac{2n-3}{k^n - k} \\
- \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) + ... + \left( \begin{array}{c}
V \\
x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x
\end{array} \right) \cdot \frac{2n-3}{k^n - k} + \frac{4 \cdot \binom{k}{2}}{(k^n - k) \cdot (k^{n-1} - k)} \right)
\]
1) If $V$ does not contain an $x^{n-2}$-corona then the equation is trivial: $0 = 0$.

2) Let $V = x^k$, $k = |V| \geq n - 2$. Then

$$
\begin{align*}
&\left( \binom{k}{n-2} \cdot \frac{1}{k^{n-2} - k} \right) \cdot \left( \frac{1}{2} - \frac{k \cdot (k-1)^2}{2 \cdot (k^n - k)} - \frac{k^2 \cdot (k-1)^2}{(k^n - k) \cdot (k^{n-1} - k)} \right) \\
&- \left( \binom{k}{n} \cdot \frac{n \cdot (n-1)}{2 \cdot (k^n - k)} \right) \\
&- \left( \binom{k}{n-1} \cdot \frac{1}{k^n - k} \right) \cdot \left( (n-1) \cdot \left( \frac{n-3}{2} + \frac{n-1 \cdot k \cdot (k-1)}{k^{n-1} - k} \right) \right) \\
&- \left( \binom{k}{n-2} \cdot \frac{1}{k^n - k} \right) \cdot \left( (n-2) \cdot \left( \frac{n-2}{2} + \frac{(n-2) \cdot k \cdot (k-1)}{k^{n-1} - k} \right) \right) \\
&= 0. \quad \text{(with the help of MuPAD, see c1 in the Appendix)}
\end{align*}
$$

3) Let $V = x \cdot x^{k-1}$, $k = |V| \geq n - 1$. Then

$$
\begin{align*}
&\left( \binom{k-1}{n-2} \cdot \frac{1}{k^{n-2} - k} \right) \cdot \left( \frac{1}{2} - \frac{k \cdot (k-1)^2}{2 \cdot (k^n - k)} - \frac{k^2 \cdot (k-1)^2}{(k^n - k) \cdot (k^{n-1} - k)} \right) \\
&- \left( \binom{k-1}{n-1} \cdot \frac{1}{k^n - k} \right) \cdot \left( (n-1) \cdot \left( \frac{n-3}{2} + \frac{(n-1) \cdot k \cdot (k-1)}{k^{n-1} - k} \right) \right) \\
&- \left( \binom{k-1}{n-2} \cdot \frac{1}{k^n - k} \right) \cdot \left( (n-2) \cdot \left( \frac{n-2}{2} + \frac{(n-2) \cdot k \cdot (k-1)}{k^{n-1} - k} \right) \right) \\
&- \left( \binom{k-1}{n} \cdot \frac{n \cdot (n-1)}{2 \cdot (k^n - k)} \right)
\end{align*}
$$
7.1 Co-additive Co-multiplication \( \Delta \)

\[
\begin{align*}
- & \left( \frac{k-1}{n-2} \right) \cdot \frac{1}{k^n - k} \cdot \left( \frac{n-3}{2} + \frac{k \cdot (k-1)}{k^{n-1} - k} \right) - \left( \frac{k-1}{n-1} \right) \cdot \frac{n-1}{k^n - k} \\
= & \left( \frac{k-1}{n-2} \right) \cdot \frac{1}{(k^n - k) \cdot (k^{n-1} - k) \cdot (k^{n-2} - k)} \cdot \left[ \frac{(k^n - k) \cdot (k^{n-1} - k)}{2} - \frac{k \cdot (k-1)^2 \cdot (k^{n-1} - k)}{2} - k^2 \cdot (k-1)^2 \right. \\
& \left. - (k^{n-1} - k) \cdot (k^{n-2} - k) \cdot \left( \frac{(k-n+1) \cdot (k+n-3)}{2} + \frac{(n-2)^2 + 2k-1}{2} \right) \right] \\
= & 0. \quad \text{(with the help of MuPAD, see c2 in the Appendix)}
\end{align*}
\]

4) If \( V = x^{k-1} \cdot x, k = |V| \geq n - 1 \), then the computation is analogous to 3).

5) Let \( V = x^{k-2} \cdot x^2, k = |V| \geq n \). Then

\[
\begin{align*}
- & \left( \frac{k-2}{n-2} \right) \cdot \frac{1}{k^{n-2} - k} \cdot \left( \frac{1}{2} - \frac{k \cdot (k-1)^2}{2 \cdot (k^n - k)} - \frac{k^2 \cdot (k-1)^2}{(k^n - k) \cdot (k^{n-1} - k)} \right) \\
= & \left( \frac{k-2}{n-1} \right) \cdot \frac{1}{k^n - k} \cdot \left( (n-1) \cdot \left( \frac{n-3}{2} + \frac{(n-1) \cdot k \cdot (k-1)}{k^{n-1} - k} \right) \right) \\
- & \left( \frac{k-2}{n-2} \right) \cdot \frac{1}{k^n - k} \cdot \left( \frac{(n-2)^2}{2} + \frac{(n-2) \cdot k \cdot (k-1)}{k^{n-1} - k} \right) \\
- & \left( \frac{k-2}{n} \right) \cdot \frac{n \cdot (n-1)}{2 \cdot (k^n - k)} \\
- & \left( \frac{k-2}{n-2} \right) \cdot \frac{1}{k^n - k} \cdot \left( 2n-2 + \frac{2 \cdot k \cdot (k-1)}{k^{n-1} - k} \right) - \left( \frac{k-2}{n-1} \right) \cdot \frac{2n-2}{k^n - k} \\
= & \left( \frac{k-2}{n-2} \right) \cdot \frac{1}{(k^n - k) \cdot (k^{n-1} - k) \cdot (k^{n-2} - k)} \cdot \left[ \frac{(k^n - k) \cdot (k^{n-1} - k)}{2} - \frac{k \cdot (k-1)^2 \cdot (k^{n-1} - k)}{2} - k^2 \cdot (k-1)^2 \right. \\
& \left. - (k^{n-1} - k) \cdot (k^{n-2} - k) \cdot \left( \frac{(k-n) \cdot (k+n-4)}{2} + \frac{(n-2)^2}{2} + 2 \cdot (k-1) \right) \right] \\
= & 0. \quad \text{(with the help of MuPAD, see c3 in the Appendix)}
\end{align*}
\]

6) The computation is analogous if \( V = x^2 \cdot x^{k-2}, V = x \cdot x \cdot x^{k-2}, \)
\( V = x \cdot x^{k-2} \cdot x, V = x^{k-2} \cdot x \cdot x, k = |V| \geq n. \)
7) Let \( V = x \cdot \ldots \cdot x \cdot x^2, |V| = k \geq n - 1 \) and \( ar(V) = k - 1 \). Then

\[
\begin{align*}
&\left( \binom{k-2}{n-2} + 2 \cdot \binom{k-2}{n-3} \right) \cdot \frac{1}{k^{n-2} - k} \\
&\cdot \left( \frac{1}{2} - \frac{k \cdot (k-1)^2}{2 \cdot (k^n - k)} - \frac{k^2 \cdot (k-1)^2}{(k^n - k) \cdot (k^{n-1} - k)} \right) \\
&= \binom{k-2}{n-3} \cdot \frac{1}{(k^n - k) \cdot (k^{n-1} - k) \cdot (k^{n-2} - k)} \cdot \frac{k + n - 3}{n - 2} \\
&\cdot \left( \frac{(k^n - k) \cdot (k^{n-1} - k)}{2} - \frac{(k^{n-1} - k) \cdot k \cdot (k-1)^2}{2} - k^2 \cdot (k-1)^2 \right) \\
&= \binom{k-2}{n-3} \cdot \frac{1}{k^{n-2} - k} \cdot \left[ \frac{(k + n - 1) \cdot (k - n) \cdot (k - n + 1)}{2 \cdot (n - 2)} \right. \\
&\left. + \frac{k - n + 1}{n - 2} \cdot \frac{(2n - 3) \cdot (k + n - 2)}{2} + \frac{k \cdot (k-1) \cdot (k + n - 2)}{k^{n-1} - k} \right] \\
&\left. + \frac{(k + n - 3) \cdot (n - 2)}{2} + 2n - 3 + \frac{(2n - 3) \cdot (k - n + 1)}{n - 2} \right)
\[
\left. \quad + \frac{k \cdot (k-1) \cdot (k + n - 1)}{k^{n-1} - k} \right]
\]

\[
= 0. \quad \text{(with the help of MuPAD, see c4 in the Appendix)}
\]

8) The computation follows analogously if \( V = x^2 \cdot x \cdot \ldots \cdot x \cdot x^2, k = |V| \geq n - 1 \) and \( ar(V) = k - 1 \).
9) Let $V = x \cdot \ldots \cdot x \cdot x^3$, $|V| = k \geq n$ and $ar(V) = k - 2$. Then

\[
\left(\frac{k - 3}{n - 2} + 3 \cdot \frac{k - 3}{n - 3}\right) \cdot \frac{1}{k^{n-2} - k} \\
\cdot \left(1 - \frac{k \cdot (k - 1)^2}{2 \cdot (k^n - k)} \cdot \frac{k^2 \cdot (k - 1)^2}{(k^n - k) \cdot (k^{n-1} - k)}\right) \\
- \left(\frac{k - 3}{n} + 3 \cdot \frac{k - 3}{n - 1}\right) \cdot \frac{n \cdot (n - 1)}{2 \cdot (k^n - k)} - \left(\frac{k - 3}{n - 3}\right) \cdot \frac{3}{k^n - k} \\
- \left(\frac{k - 3}{n - 2} + 3 \cdot \frac{k - 3}{n - 2}\right) \cdot \frac{1}{k^n - k} \\
\cdot \left((n - 1) \cdot \left(n - \frac{3}{2}\right) + \frac{(n - 1) \cdot k \cdot (k - 1)}{k^{n-1} - k}\right) \\
- \left(\frac{k - 3}{n - 2} + 3 \cdot \frac{k - 3}{n - 3}\right) \cdot \frac{1}{k^n - k} \cdot \left(\frac{(n - 2)^2}{2} + \frac{(n - 2) \cdot k \cdot (k - 1)}{k^{n-1} - k}\right) \\
- 3 \cdot \left(\frac{k - 3}{n - 2} \cdot \frac{2n - 3}{k^n - k} - 3 \cdot \frac{k - 3}{n - 3}\right) \cdot \frac{1}{k^n - k} \cdot \left(2n - 3 + \frac{2k \cdot (k - 1)}{k^{n-1} - k}\right) \\
= \left(\frac{k - 3}{n - 3}\right) \cdot \frac{(k^n - k) \cdot (k^{n-1} - k)}{2} \cdot \frac{1}{(k^n - k) \cdot (k^{n-2} - k)} \cdot \frac{k + 2n - 6}{n - 2} \\
\cdot \left(\frac{(k^n - k) \cdot (k^{n-1} - k)}{2} - \frac{(k^{n-1} - k) \cdot k \cdot (k - 1)^2}{2} - k^2 \cdot (k - 1)^2\right) \\
- \left(\frac{k - 3}{n - 3}\right) \cdot \frac{1}{k^n - k} \cdot \left[\frac{(k - n) \cdot (k - n - 1) \cdot (k + 2n - 2)}{2 \cdot (n - 2)}\right] \\
+ \frac{k \cdot (k - 1) \cdot (k + 2n)}{k^{n-1} - k} + \frac{(k - n) \cdot (2n - 3) \cdot (k + 2n - 4)}{2 \cdot (n - 2)} \\
+ \frac{k \cdot (k - 1) \cdot (k - n) \cdot (k + 2n - 4)}{(n - 2) \cdot (k^{n-1} - k)} + \frac{(n - 2) \cdot (k + 2n - 6)}{2} \\
+ \frac{(6n - 9) \cdot (k - n)}{n - 2} + 6n - 6 \right]
= 0.
\]
(with the help of MuPAD, see c5 in the Appendix)

10) The computation follows analogously for all $V = x^2 \cdot x \cdot \ldots \cdot x$, $|V| = k \geq n$ and $ar(V) = k - 2$.

11) Let $V = x \cdot (x \cdot \ldots \cdot x \cdot x^2)$, $|V| = k \geq n$ and $ar(V_2) = k - 2$, where $V = x \cdot V_2$. 
Then
\[
\left(\binom{k-3}{n-2} + 2 \cdot \binom{k-3}{n-3}\right) \cdot \frac{1}{k^{n-2}} - \frac{k+1}{n-2} \cdot \left(\frac{1}{2} - \frac{k \cdot (k-1)^2}{2 \cdot (k^n - k)} - \frac{k^2 \cdot (k-1)^2}{(k^n - k) \cdot (k^{n-1} - k)}\right) \\
- \left(\binom{k-3}{n} + 2 \cdot \binom{k-3}{n-1}\right) \cdot \frac{n \cdot (n-1)}{2 \cdot (k^n - k)} - \frac{k+1}{n-2} \cdot \left(\frac{1}{n-1} - \frac{k \cdot (k-1)^2}{2 \cdot (k^n - k)} \cdot \frac{n-1}{k^n - k}\right) \\
- \left(\binom{k-3}{n-2} + 2 \cdot \binom{k-3}{n-3}\right) \cdot \frac{1}{k^n - k} \cdot \left(\binom{n-1}{n-1} \cdot \frac{3}{2} + \frac{n-1 \cdot k \cdot (k-1)}{k^n - k}\right) \\
- \left(\binom{k-3}{n-2} + 2 \cdot \binom{k-3}{n-3}\right) \cdot \frac{1}{k^n - k} \cdot \left(\frac{n-3}{2} + \frac{k \cdot (k-1)}{k^{n-1} - k}\right) \\
- \left(\binom{k-3}{n-2} \cdot \frac{2n-3}{k^n - k} - \binom{k-3}{n-3} \cdot \frac{1}{k^n - k} \cdot \left(\frac{2n-1 + \frac{2k \cdot (k-1)}{k^{n-1} - k}}{k^n - k}\right)\right) \\
= \left(\binom{k-3}{n-3} \cdot \frac{1}{(k^n - k) \cdot (k^{n-1} - k) \cdot (k^{n-2} - k) \cdot n-2} \cdot \left(\frac{(k^n - k) \cdot (k^{n-1} - k) \cdot (k^{n-2} - k)}{2} \cdot \frac{k+n-1}{k^n - k} \cdot \frac{k+1}{n-2}\right) \\
- \left(\binom{k-3}{n-3} \cdot \frac{1}{2} \cdot \left(\frac{(k+n-2) \cdot (k-n-1) \cdot (k-n)}{2 \cdot (n-2)} + 2n-1\right) \right) \\
+ \frac{k \cdot (k-1) \cdot (k+n-3) \cdot (k+n-4) + k \cdot (k-1) \cdot (k+n-4)}{k^{n-1} - k} \\
+ \frac{(n-2) \cdot (k^n - k) \cdot \frac{(k+n-3) \cdot (k+n-4) + (n-2) \cdot (k+n-4)}{2}}{n-2} \right) \\
= 0. \quad \text{(with the help of MuPAD, see \(c6\) in the Appendix)}
\]

12) The computation for the rooted trees \(V = x \cdot (x \cdot ... \cdot x \cdot x^2 \cdot x), ...\), \(V = x \cdot (x^2 \cdot x \cdot ... \cdot x), V = (x \cdot ... \cdot x \cdot x^2) \cdot x, ..., V = (x^2 \cdot x \cdot x \cdot ... \cdot x) \cdot x\) is analogous to 11).

### 7.2 Left- and right-inverse power series resp. the grafting

By the help of the substitution endomorphism, we already know that we can compute the compositional inverse, but in this section we want to compute left-
and right-inverse power series with respect to the grafting operation. Here we have more possibilities for the inverse series which depends on the tree over the grafting and on the position in the grafting operation of the inverse power series. In this work we consider power series with the constant term 1.

**Proposition 7.2.1.** Let \( f \in K\{x\} \) be a power series with the constant term 1. Then there is a unique right-inverse power series \((f\backslash 1)\), such that \(f \cdot (f\backslash 1) = 1\).

**Theorem 7.2.2.** The coefficients of \((f\backslash 1)\) are given by \(f\) in the following way

\[ c_x(f\backslash 1) = -c_x(f) \]

and

\[ c_T(f\backslash 1) = -c_T(f) - c_{T_1}(f) \cdot c_{T_2}(f\backslash 1), \]

if \( T = T_1 \cdot T_2 \), otherwise

\[ c_T(f\backslash 1) = -c_T(f). \]

**Proof.** Let \( f = 1 + \sum_{T \in \mathcal{P}} c_T(f) \cdot T \). Then

\[ c_X(f \cdot (f\backslash 1)) = c_x(f) + c_x(f\backslash 1) = c_x(f) - c_x(f) = 0 \]

and

\[ c_T(f \cdot (f\backslash 1)) = c_T(f) + c_T(f\backslash 1) + c_{T_1}(f) \cdot c_{T_2}(f\backslash 1) \\
= c_T(f) - c_T(f) - c_{T_1}(f) \cdot c_{T_2}(f\backslash 1) + c_{T_1}(f) \cdot c_{T_2}(f\backslash 1) = 0. \]

Thus \((f\backslash 1)\) satisfies the wanted property.

The uniqueness is easy to show, because if there is another power series \(g\) with this property then the coefficients of \(g\) would have to satisfy the same recursion equations.

**Proposition 7.2.3.** Let \( f \in K\{x\} \) be a power series with the constant term 1. Then there is a unique left-inverse power series \((1/f)\), such that \((1/f) \cdot f = 1\).

**Theorem 7.2.4.** The coefficients of \((1/f)\) are given by \(f\) in the following way

\[ c_x(1/f) = -c_x(f) \]

and

\[ c_T(1/f) = -c_T(f) - c_{T_1}(1/f) \cdot c_{T_2}(f), \]

if \( T = T_1 \cdot T_2 \), otherwise

\[ c_T(1/f) = -c_T(f). \]

**Proof.** Proceed as in the proof of Proposition 7.2.1 and Theorem 7.2.2.
Example 7.2.5. (i) Let $f = 1 + x$, then the right-inverse $\left( \frac{1}{f} \right)_R = \left( \frac{1}{1+x} \right)_R$ of $f$ is
\[
\left( \frac{1}{f} \right)_R = \sum_{n=0}^{\infty} (-1)^n \cdot S_n,
\]
where $S_n$ are the binary right combs.

(ii) Let $f = 1 + x$, then the left-inverse $\left( \frac{1}{f} \right)_L = \left( \frac{1}{1+x} \right)_L$ of $f$ is
\[
\left( \frac{1}{f} \right)_L = \sum_{n=0}^{\infty} (-1)^n \cdot S_n,
\]
where $S_n$ are the binary left combs.

(iii) Let $f = \exp_{x^2}(x)$, then the right-inverse $\left( \frac{1}{f} \right)_R = \left( \frac{1}{\exp_{x^2}(x)} \right)_R$ of $f$ is
\[
\left( \frac{1}{f} \right)_R = 1 - x + \frac{1}{2} x^2 + \frac{1}{3!} \left( \frac{-7}{2} x \cdot x^2 + \frac{5}{2} \right) + \frac{1}{4!} \cdot \left( \frac{97}{4} x \cdot (x \cdot x^2) + \frac{13}{4} (x \cdot x^2) \cdot x - \frac{71}{4} x \cdot (x^2 \cdot x) + \frac{13}{4} (x^2 \cdot x) \cdot x - \frac{45}{4} x^2 \cdot x^2 \right) + \text{higher terms}.
\]

(iv) Let $f = \exp_{x^2}(x)$, then the left-inverse $\left( \frac{1}{f} \right)_L = \left( \frac{1}{\exp_{x^2}(x)} \right)_L$ of $f$ is
\[
\left( \frac{1}{f} \right)_L = 1 - x + \frac{1}{2} x^2 + \frac{1}{3!} \left( \frac{5}{2} x \cdot x^2 - \frac{7}{2} \right) + \frac{1}{4!} \cdot \left( \frac{13}{4} x \cdot (x \cdot x^2) - \frac{71}{4} (x \cdot x^2) \cdot x + \frac{13}{4} (x^2 \cdot x) + \frac{97}{4} (x^2 \cdot x) \cdot x - \frac{45}{4} x^2 \cdot x^2 \right) + \text{higher terms}.
\]

Remark 7.2.6. If $f = \exp_V(x)$, then for $\left( \frac{1}{f} \right)_R$ and $\left( \frac{1}{f} \right)_L$ it holds that
\[
\frac{d}{dx} \left( \left( \frac{1}{f} \right)_R \right) = - \left( \frac{1}{f} \right)_R
\]
and
\[
\frac{d}{dx} \left( \left( \frac{1}{f} \right)_L \right) = - \left( \frac{1}{f} \right)_L.
\]
This is true as
\[ f \cdot \left( \frac{1}{f} \right)_R = 1 \implies \frac{d}{dx} \left( f \cdot \left( \frac{1}{f} \right)_R \right) = 0 \]
\[ \iff \frac{d}{dx} (f) \cdot \left( \frac{1}{f} \right)_R + f \cdot \frac{d}{dx} \left( \frac{1}{f} \right)_R = 0. \]

It holds that \( \frac{d}{dx} (f) = f \) thus
\[ 1 + f \cdot \frac{d}{dx} \left( \left( \frac{1}{f} \right)_R \right) = 0 \iff f \cdot \frac{d}{dx} \left( \left( \frac{1}{f} \right)_R \right) = -1 \iff \frac{d}{dx} \left( \left( \frac{1}{f} \right)_R \right) = - \left( \frac{1}{f} \right)_R. \]

Analogous for the left-inverse.
As \( \frac{d}{dx} (\exp V(-x)) = -\exp V(-x) \) it follows that these three series have the same derivative property \( \frac{d}{dx} (f) = -f \).

**Proposition 7.2.7.** Let \( f, g \in K\{x\} \) be power series with the constant term 1. Then there is a unique power series \( h \), such that \( f \cdot h = g \). \( h \) is also denoted by \( f \backslash g \).

**Theorem 7.2.8.** The coefficients of \( h \) are given by \( f \) and \( g \) in the following way
\[ c_x(h) = c_x(g) - c_x(f) \]
and
\[ c_T(h) = c_T(g) - c_T(f) - c_{T_1}(f) \cdot c_{T_2}(h), \]
if \( T = T_1 \cdot T_2 \), otherwise
\[ c_T(h) = c_T(g) - c_T(f). \]

**Proof.** Let \( f = \sum_{T \in \mathbb{F}} c_T(f) \cdot T \) and \( g = \sum_{T \in \mathbb{F}} c_T(g) \cdot T \).
Then
\[ c_x(g) = c_x(f) + c_x(h) = c_x(f) + c_x(g) - c_x(f) = c_x(g). \]
Moreover
\[ c_T(g) = c_T(f) + c_T(h) + c_{T_1}(f) \cdot c_{T_2}(h) \]
\[ = c_T(f) + c_T(g) - c_T(f) - c_{T_1}(f) \cdot c_{T_2}(h) + c_{T_1}(f) \cdot c_{T_2}(h) = c_T(g). \]

The uniqueness is trivial.

**Proposition 7.2.9.** Let \( f, g \in K\{x\} \) be power series with the constant term 1. Then there is a unique power series \( h \), such that \( h \cdot f = g \). \( h \) is also denoted by \( (g/f) \).
Theorem 7.2.10. The coefficients of \( h \) are given by \( f \) and \( g \) in the following way

\[
c_x(h) = c_x(g) - c_x(f)
\]

and

\[
c_T(h) = c_T(g) - c_T(f) - c_{T_1}(h) \cdot c_{T_2}(f),
\]

for \( T = T_1 \cdot T_2 \), otherwise

\[
c_T(h) = c_T(g) - c_T(f).
\]

Proof. Proceed as in the proof of Proposition 7.2.7 and Theorem 7.2.8.

\[\square\]

Notation 7.2.11. We use the following notation

\[(f_1, ..., f_r, ..., f_n) = (f_1, ..., f_{r-1}, f_{r+1}, ..., f_n)\].

Proposition 7.2.12. Let \( f_1, ..., f_{r-1}, f_{r+1}, ..., f_n, g \in K\{x\} \) be power series with the constant term 1 and \( V \in \mathbb{P} \) with \( n = |V| \). Then there is a unique power series \( h \), such that

\[V(f_1, ..., f_{r-1}, h, f_{r+1}, ..., f_n) = g \].

\( h \) is also denoted by

\[
\left( \frac{g}{(f_1, ..., f_r, ..., f_n)} \right)_{(V, r)}.
\]

Theorem 7.2.13. The coefficients of \( h \) are given by \( f_1, ..., f_r, ..., f_n, g \) in the following way

\[
c_1(h) = c_1(g) = 1,
\]

\[
c_x(h) = c_x(g) - \sum_{i=1}^{r-1} c_x(f_i) - \sum_{i=r+1}^{n} c_x(f_i)
\]

and

\[
c_T(h) = c_T(g) - \sum_{i=1}^{r-1} c_T(f_i) - \sum_{i=r+1}^{n} c_T(f_i) - \sum_{1 \leq \lambda_1 < ... < \lambda_m \leq n}^{m} \prod_{i=1}^{m} c_{T_i}(f_{\lambda_i}),
\]

where \( m = |S| \), \( T - S = (T_1, ..., T_m) \), \( f_r = h \) and \( (\lambda_1, ..., \lambda_m) \in I_S \), if \( V|S = \{I_S \subseteq L(V) : V|I_S = S\} \) and \( I_S = \{(\lambda_1, ..., \lambda_m) \in \mathbb{N}^m : \lambda_j \in L(V)\} \).

Proof. Let \( f_i = 1 + \sum_{T \in \mathbb{P}} c_T(f_i) \cdot T, \forall i = 1, ..., r, ..., n \) and \( g = 1 + \sum_{T \in \mathbb{P}} c_T(g) \cdot T \).

Then

\[
c_x(g) = \sum_{i=1}^{r-1} c_x(f_i) + c_x(h) + \sum_{i=r+1}^{n} c_x(f_i)
\]

\[
= \sum_{i=1}^{r-1} c_x(f_i) + c_x(g) - \sum_{i=1}^{r-1} c_x(f_i) - \sum_{i=r+1}^{n} c_x(f_i) + \sum_{i=r+1}^{n} c_x(f_i) = c_x(g).
\]
Moreover
\[ c_T(g) = \sum_{i=1}^{r-1} c_T(f_i) + c_T(h) + \sum_{i=r+1}^{n} c_T(f_i) + \sum_{S \in O(T) \cap P(V)} \prod_{i=1}^{m} c_T(f_{\lambda_i}) \]
\[ = \sum_{i=1}^{r-1} c_T(f_i) + c_T(g) - \sum_{i=1}^{r-1} c_T(f_i) - \sum_{i=r+1}^{n} c_T(f_i) - \sum_{S \in O(T) \cap P(V)} \prod_{i=1}^{m} c_T(f_{\lambda_i}) \]
\[ + \sum_{i=r+1}^{n} c_T(f_i) + \sum_{S \in O(T) \cap P(V)} \prod_{i=1}^{m} c_T(f_{\lambda_i}) \]
\[ = c_T(g). \]

\[ \square \]

Example 7.2.14. Let \( V = x^2 \cdot x, f_1 = f_3 = 1 + x \) and \( g = 1 + x + x^2 \). Then
\[ h = 1 - x + 2x^2 - 2x \cdot x^2 - x^2 \cdot x + 2x \cdot (x \cdot x^2) + x \cdot (x^2 \cdot x) + (x^2 \cdot x) \cdot x + \text{higher terms.} \]

7.3 Planar quotient rule

Proposition 7.3.1. Let \( f, g, h \in K\{\{x\}\} \) with \( h \cdot f = g \). Thus \( h = g/f \). Then
\[ h' = (g' - h \cdot f')/f. \]

This rule is called the planar quotient rule for left-inverse power series.

Proof. Let \( h \cdot f = g \). Then by the product rule
\[ h' \cdot f + h \cdot f' = g' \]
\[ \iff h' \cdot f = g' - h \cdot f' \]
\[ \iff h' = (g' - h \cdot f')/f. \]
\[ \square \]

Proposition 7.3.2. Let \( f, g, h \in K\{\{x\}\} \) with \( f \cdot h = g \). Thus \( h = f \setminus g \). Then
\[ h' = f \setminus (g' - f' \cdot h). \]

This rule is called the planar quotient rule for right-inverse power series.

Proof. Let \( f \cdot h = g \). Then by the product rule
\[ f' \cdot h + f \cdot h' = g' \]
\[ \iff f \cdot h' = g' - f' \cdot h \]
\[ \iff h' = f \setminus (g' - f' \cdot h). \]
\[ \square \]
Proposition 7.3.3. Let \( f_1, \ldots, f_{r-1}, f_{r+1}, \ldots, f_n, g, h \in K\{x\} \) and \( V \in \mathbb{P} \) with \(|V| = n\) and \( \cdot_V(f_1, \ldots, f_{r-1}, h, f_{r+1}, \ldots, f_n) = g \). Thus
\[
h = \left( \frac{g}{(f_1, \ldots, f_r, \ldots, f_n)} \right)_{(V,r)}.
\]
Then
\[
h' = \left( \frac{g' - \sum_{1 \leq i \leq n} \cdot_{V} (f_1, \ldots, f_i', \ldots, f_{r-1}, h, f_{r+1}, \ldots, f_n)}{(f_1, \ldots, f_r, \ldots, f_n)} \right)_{(V,r)}.
\]
This is called the general planar quotient rule with respect to \( V \) at the position \( r \).

Proof. Let \( \cdot_{V}(f_1, \ldots, f_{r-1}, h, f_{r+1}, \ldots, f_n) = g \). Then by the product rule
\[
\sum_{i=1}^{r-1} \cdot_{V}(f_1, \ldots, f_i', \ldots, f_{r-1}, h, f_{r+1}, \ldots, f_n) + \cdot_{V}(f_1, \ldots, f_{r-1}, h', f_{r+1}, \ldots, f_n)
\]
\[\iff\]
\( \cdot_{V}(f_1, \ldots, f_{r-1}, h', f_{r+1}, \ldots, f_n) = g' \)
\[\iff\]
\[\sum_{i=1}^{r-1} \cdot_{V}(f_1, \ldots, f_i', \ldots, f_{r-1}, h, f_{r+1}, \ldots, f_n) - \sum_{i=r+1}^{n} \cdot_{V}(f_1, \ldots, f_i', \ldots, f_{r-1}, h, f_{r+1}, \ldots, f_n)
\]
\[\iff\]
\( h' = \left( \frac{g' - \sum_{1 \leq i \leq n, i \neq r} \cdot_{V} (f_1, \ldots, f_i', \ldots, f_{r-1}, h, f_{r+1}, \ldots, f_n)}{(f_1, \ldots, f_r, \ldots, f_n)} \right)_{(V,r)}\)

\(\square\)

7.4 Planar Hermite polynomials

Let the planar rooted trees be labeled with \( z \). Then define
\[
\Phi(T) := \{S \in O(T) : T - S = (T_1, \ldots, T_m) \text{ is a planar rooted forest, which contains only the rooted trees } z \text{ and } z^2, \text{ thus } T_i = z \text{ or } T_i = z^2\}
\]
\[
n_S(T) := \sharp\{z \in T - S : S \in \Phi(T)\}, 0 \leq n_S(T) \leq m = |S|
\]
\( I_z := \{(i_1, \ldots, i_{n_S(T)}) \in |m| : T_{ij} = z, \ \forall 1 \leq j \leq n_S(T)\} \)

\( W = S|I_z = \) the restriction of \( S \) on the leaves labeled with \( I_z \).

In the classical algebra, the Hermite polynomials are defined by

\[
\exp(2xz - z^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} \cdot z^n,
\]

where

\[
H_n(x) = n! \cdot \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^m (2x)^{n-2m}}{m!(n-2m)!}
\]

are the Hermite polynomials.

We consider the tensor-algebra \( K \{\{y\}\} \otimes K \{\{y\}\} \) with \( x = 1 \otimes y \) and \( z = y \otimes 1 \).

In \((K\{\{x\}\})[[z]]\) it holds

\[
\exp_V(2xz - z^2) = 1 + \sum_{T \in P} a_V(T) \cdot \left( \sum_{S \in \Phi(T) \atop |S| = m} (-1)^m \binom{T}{S} \cdot 2^m \cdot x^S \cdot z^{n+m} \right).
\]

\[ \implies H_n(x) = n! \cdot \text{coeff}(z^n) = n! \cdot \sum_{T \in P \atop |T| = n} \left( \sum_{S \in \Phi(T) \atop |S| = m} a_V(S) \cdot (-1)^{m-n_S(T)} \cdot 2^{n_S(T)} \cdot x^W \right). \]

In \((K\{\{x\}\})\{\{z\}\}\) it holds

\[
\exp_V(2xz - z^2) = 1 + \sum_{T \in P} a_V(T) \cdot (2xz - z^2)^T
\]

\[ \implies H_T(x) = n! \cdot \text{coeff}(z^n) = \sum_{S \in \Phi(T) \atop |S| = m} a_V(S) \cdot (-1)^{m-n_S(T)} \cdot 2^{n_S(T)} \cdot x^{n_S(T)}. \]

In \((K\{\{x\}\})\{\{z\}\}\) it holds

\[
H_T(x) = n! \cdot \text{coeff}(z^n) = \sum_{S \in \Phi(T) \atop |S| = m} a_V(S) \cdot (-1)^{m-n_S(T)} \cdot 2^{n_S(T)} \cdot x^W.
\]

**Proposition 7.4.1.** It holds for \( n \leq 4 \)

\[ (i) \]

\[ H_n(x) = \sum_{T \in P \atop |T| = n} \pi(H_T(x)). \]
(ii)
\[ \sum_{T \in \mathbb{P}} \frac{d}{dx}(H_T(x)) = 2n \cdot \left( \sum_{S \subseteq T \mid |S| = n-1} H_S(x) \right). \]

Proof.  (i) \( n = 1; \) \( H_1(x) = \pi(H_x(x)) = \pi(1 \cdot \text{coeff}(z)) = \pi((-1)^0 \cdot 2 \cdot x) = 2x \)
\( n = 2; \) \( H_2(x) = \pi(H_{x^2}(x)) = \pi(2! \cdot \text{coeff}(z^2)) \)
\( = \pi(2(1 \cdot (-1)^1) \cdot 2^0(1) + \frac{1}{2} \cdot (-1)^0 \cdot 2^2 \cdot x^2) = \pi(-2 + 4x^2) = 4x^2 - 2 \)
\( n = 3; \) \( H_3(x) = \pi(H_{x \cdot x^2}(x) + H_{x^2 \cdot x}(x) + H_{x^3}(x)) = \pi(3! \cdot \sum_{T \in \mathbb{P}} \text{coeff}(z^T)) \)
\( = \pi(6(\frac{1}{2} \cdot (-1) \cdot 2 \cdot x + a_v(x \cdot x^2) \cdot (-1)^0 \cdot 2^3 \cdot x \cdot x^2 + \frac{1}{2} \cdot (-1) \cdot 2 \cdot x + a_v(x^2 \cdot x) \cdot (-1)^0 \cdot 2^3 \cdot x^2 + a_v(x \cdot x^2) \cdot (-1)^0 \cdot 2^3 \cdot x^3)) \)
\( = \pi(6(-2x + 8(a_v(x \cdot x^2) \cdot x \cdot x^2 + a_v(x^2 \cdot x) \cdot x^2 \cdot x + a_v(x^3) \cdot x^3))) \)
\( = 8(a_v(x \cdot x^2) + a_v(x^2 \cdot x) + a_v(x^3)) \cdot 6 \cdot x^3 - 12x \)
\( = 8 \cdot x^3 - 12x \)
\( n = 4; \) \( H_4(x) = \pi(4! \cdot \sum_{T \in \mathbb{P}} \text{coeff}(z^T)) \)
\( = \pi(24(\sum_{T \in \mathbb{P}} a_v(T) \cdot T - 2^2 \cdot x^2 \cdot 3(a_v(x \cdot x^2) + a_v(x^2 \cdot x) + a_v(x^3) + \frac{1}{2})) \)
\( = 24 \cdot 2^4 \cdot \sum_{T \in \mathbb{P}} a_v(T) \cdot T - 2^2 \cdot x^2 \cdot 3(a_v(x \cdot x^2) + a_v(x^2 \cdot x) + a_v(x^3) + \frac{1}{2})) \)
\( = 24 \cdot 2^4 \cdot \sum_{T \in \mathbb{P}} a_v(T) \cdot T - 2^2 \cdot x^2 \cdot 3 \cdot (a_v(x \cdot x^2) + a_v(x^2 \cdot x) + a_v(x^3) + \frac{1}{2})) \)
\( = 16 \cdot x^4 - 48 \cdot x^2 + 12. \)

(ii) \( n = 1; \) 2 \( \sum_{S \subseteq T \mid |S| = 0} H_S(x) = 2 \cdot H_1(x) = 2 \)
\( = \frac{d}{dx}(H_x(x)) = \frac{d}{dx}(2x) \)
\( n = 2; \) 4 \( \sum_{S \subseteq T \mid |S| = 0} H_S(x) = 8x = \frac{d}{dx}(H_{x^2}(x)) = \frac{d}{dx}(4x^2 - 2) \)
\( n = 3; \) 6 \( \sum_{S \subseteq T \mid |S| = 0} H_S(x) = 6 \cdot (4x^2 - 2) = 24x^2 - 12 = \sum_{T \in \mathbb{P}} \frac{d}{dx}(H_T(x)) \)
\( = \frac{d}{dx}(48 \cdot (a_v(x \cdot x^2) \cdot x \cdot x^2 + a_v(x^2 \cdot x) \cdot x^2 \cdot x + a_v(x^3) \cdot x^3) - 12x) \)
\( = 48 \cdot 3 \cdot x^2 \cdot (a_v(x \cdot x^2) + a_v(x^2 \cdot x) + a_v(x^3)) - 12 \)
\( n = 4; \) 8 \( \sum_{S \subseteq T \mid |S| = 0} H_S(x) \)
\( = 8 \cdot (48 \cdot (a_v(x \cdot x^2) \cdot x \cdot x^2 + a_v(x^2 \cdot x) \cdot x^2 \cdot x + a_v(x^3) \cdot x^3) - 12x) \)
Remark 7.4.2. The properties of Hermite polynomials in Proposition 7.4.1 are probably also true for \( n > 4 \). Due to its length, the proof, however, is not given here. We have to consider and compute more coefficients of the planar exponential series of a higher degree.

7.5 Planar modules \( E_\mu \)

Definition 7.5.1. Let \( M \) be a set and \( R \) be a ring. We call \( M \) together with a \( R \)-bilinear map \( \cdot : R \times M \to M \) a planar \( R \)-left module if \( \forall a \in R, b \in M \) it holds
\[
a \cdot b \in M.
\]

Analogously \( M \) together with a \( R \)-bilinear map \( \cdot : M \times R \to M \) is called a planar \( R \)-right module if \( \forall a \in R, b \in M \) it holds
\[
b \cdot a \in M.
\]

If \( M \) is an \( R \)-left and a \( R \)-right module, then we say that \( M \) is a planar \( R \)-bi-module.

Definition 7.5.2. For all \( \mu \in \mathbb{N}_0 \) we define
\[
E_\mu := \{ f \in K\{x\} : \frac{d}{dx}(f) = \mu \cdot f \}.
\]

Remark 7.5.3. \( E_0 \) is the set of all constant power series with respect to the derivation. \( E_0 \) is a ring.

Proposition 7.5.4. \( E_\mu \) is a planar \( E_0 \)-bi-module in the sense that \( \forall f \in E_\mu, g \in E_0 \) it holds
\[
g \cdot f \in E_\mu \quad \text{and} \quad f \cdot g \in E_\mu.
\]

Proof. Let \( f \in E_\mu, g \in E_0 \), then
\[
\frac{d}{dx}(f \cdot g) = (f \cdot g)' = f' \cdot g + f \cdot g' = (\mu \cdot f) \cdot g + f \cdot 0 = \mu \cdot (f \cdot g)
\]
\[
\implies f \cdot g \in E_\mu.
\]
Analogously \( g \cdot f \in E_\mu \). \( \square \)
Proposition 7.5.5. Let $f_i \in E_{\mu_i}, \forall 1 \leq i \leq n$, $V \in \mathbb{P}$, $|V| = n$ and $\mu_i \in \mathbb{N}_0$. Then

$$V(f_1, \ldots, f_n) \in E_{\mu_1+\ldots+\mu_n}.$$  

Proof. Let $f_i, \mu_i, V$ be defined as in the proposition. Then

$$\frac{d}{dx}(V(f_1, \ldots, f_n)) = \sum_{i=1}^{n} \cdot V(f_1, \ldots, f_i', \ldots, f_n) = \sum_{i=1}^{n} \cdot V(f_1, \ldots, \mu_i \cdot f_i, \ldots, f_n)$$

$$= \sum_{i=1}^{n} \mu_i \cdot \cdot V(f_1, \ldots, f_i, \ldots, f_n)$$

$$= (\mu_1 + \ldots + \mu_n) \cdot \cdot V(f_1, \ldots, f_n) \in E_{\mu_1+\ldots+\mu_n}.$$  

If for example $\mu_j = 0$ then the $j$-th summand is equal to 0. The same happens if more of $\mu_i = 0$, then the summands in the derivation are equal to 0 and the proposition is true as

$$\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} \mu_i.$$

\[\square\]

Remark 7.5.6. The planar exponential series $exp_V(x)$ with respect to $V \in \mathbb{P}$ and the planar $\lambda$-deformed exponential series $exp_{(V,\lambda)}(x)$ are elements of $E_1$. If we consider these power series with the argument $\mu x$ instead of $x$, then we get elements of $E_{\mu}$.

Open Question: We know that $exp_V(\mu x), exp_{(V,\lambda)}(\mu x) \in E_{\mu}$, for all $\mu \in \mathbb{N}^{>0}$. But there is still an unanswered question if these planar exponential series create a basis of the sets $E_{\mu}$ for $\mu \in \mathbb{N}^{>0}$. 

Appendix A

MuPAD commands

In this Chapter one can find the commands c1,..., c6 from Section 7.1 to check out that they are really equal to 0. If we enter these commands into a MuPAD notebook, then we get as the output at first the files c1,...,c6 and if we simplify these then we get the output 0.

- \( c1 := (k^n - k) * (k^n(n - 1) - k)/2 - k * (k - 1)^3 * (k^n(n - 1) - k)/2 - k^n2 * (k - 1)^3 - 2 - (k - n) * (k - n + 1) * (k^n(n - 1) - k) * (k^n(n - 2) - k)/2 - (k - n + 1) * \frac{n - 3}{2} * (k^n(n - 1) - k) * (k^n(n - 2) - k) \)

- \( c2 := (k^n - k) * (k^n(n - 1) - k)/2 - k * (k - 1)^3 * (k^n(n - 1) - k)/2 - k^n2 * (k - 1)^3 - 2 - (k^n(n - 1) - k) * (k^n(n - 2) - k) * ((k - n + 1) * (k + n - 3)/2 + ((n - 2)^2 + 2 * k - 1)/2) - (k^n(n - 2) - k) * (k - 1) * k^n2 \)

- \( c3 := (k^n - k) * (k^n(n - 1) - k)/2 - k * (k - 1)^3 * (k^n(n - 1) - k)/2 - k^n2 * (k - 1)^3 - 2 - (k^n(n - 1) - k) * (k^n(n - 2) - k) * ((k - n) * (k + n - 4)/2 + (n - 2)^2/2 + 2 * (k - 1)) - (k^n(n - 2) - k) * k^n2 * (k - 1) \)

- \( a1 := \text{binomial}(k - 2, n - 3)/((k^n - k) * (k^n(n - 1) - k) * (k^n(n - 2) - k)) * ((k - 2 - n + 3)/(n - 2) + 2) * ((k^n - k) * (k^n(n - 1) - k)/2 - k * (k - 1)^3 - 2) * (k^n(n - 1) - k)/2 - k^n2 * (k - 1)^2) \)

- \( a2 := 1/(k^n - k) * (\text{binomial}(k - 2, n - 2) * (k + n - 1) * (k - n)/2 + \text{binomial}(k - 2, n - 3) * (k - n + 1)/(n - 2) * ((2 * n - 3) * (k + n - 2)/2 + k * (k - 1) * (k + n - 2)/(k^n(n - 1) - k)) + \text{binomial}(k - 2, n - 3) * ((n -
\[ 2\left(\frac{k+n-3}{2} + k(k-1)(k+n-3)/(k(n-1)-k) + 2(n-3) \right) \frac{(k-n+1)/(n-2) + 2n-3 + 2k(k-1)/(k(n-1)-k))}{}}

- \( c4 := a1 - a2 \)
- \( simplify(expand(c4)) \)

- \( a3 := \text{binomial}(k-3,n-3)/((k-n-k)(k(n-1)-k)(k(n-2)-k)(k+2n-6)/(n-2)((k-n-k)(k(n-1)-k)/2 - (k(n-1)-k)k(k-1))2/2 - k^2(k-1)2) \)

- \( a4 := \text{binomial}(k-3,n-3)/((k-n-k)((k+2n-2)(k-n-1)(k-n)/(2(n-2)) + 2n-3)(k+2n-4)(k-n)/(2(n-2)) + k(k-1)(k+2n-4)(k-n)((n-2)(k(n-1)-k)) + (n-2)(k+2n-6)/2 + k(k-1)(k+2n-6)/(k(n-1)-k) + 3 + 6n-9 + 6k(k-1)/(k(n-1)-k) + (6n-9)(k-n)/(n-2)) \)

- \( c5 := a3 - a4 \)
- \( simplify(c5) \)

- \( a5 := \text{binomial}(k-3,n-3)/((k-n-k)((k(n-2)-k)(k(n-1)-k)(k-n-k)((k-n)/(n-2)+2)((k-n-k)(k(n-1)-k)/2 - (k(n-1)-k)k(k-1))2/2 - k^2(k-1)2) \)

- \( a6 := \text{binomial}(k-3,n-3)/(k-n-k)((k+n-2)(k-n-1)(k-n)/(2(n-2)) + (k-n)(2n-3)(k+n-3)/(2(n-2)) + k(k-1)(k+n-3)(k-n)/(n-2)(k(n-1)-k)) + (n-2)(k+n-4)/2 + (2n-3)(k+n-4)/(2(n-2)) + k(k-1)(k+n-4)/(k(n-1)-k) + k(k-1)(k+n-4)/(n-2)(k(n-1)-k) + (k-n)(k+3n-6)/(n-2) + 2n-1 + 2k(k-1)/(k(n-1)-k)) \)

- \( c6 := a5 - a6 \)
- \( simplify(c6) \)
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