Introduction

The topics of this thesis are generalizations of classical power series to planar (= non-commutative, non-associative) power series. That means that we are lifting some classical power series into a planar setting.

An isomorphism class of a finite, planar, reduced, and rooted tree represents, by definition, a planar monomial. By $P$ we denote the set of planar monomials including the empty tree 1. There is a natural $k$-ary grafting operation $\cdot_k : P^k \to P$, for $k \in \mathbb{N}_{\geq 2}$. We also have the grafting over a tree $V$, denoted by $\cdot_V$. It is defined in the following way:

Let $T_1, \ldots, T_m, V \in P$ and $|V| = m$, then $\cdot_V(T_1, \ldots, T_m)$ is the tree $T$, where $T - In(V)$ is a planar rooted forest, which is isomorphic to $(T_1, \ldots, T_m)$.

$T^V := \cdot_V(T, \ldots, T)$, which means that we replace every leaf $k \in L(V)$ with the tree $T$. If we observe a $K$-linear combination over $P$, then we get a planar polynomial over a field $K$. We denote the set of all planar polynomials by $K\{x\}$.

We also have a $K$-multi linear extension of the grafting product and consider $K\{x\}$ as an algebra over an operad $\mathcal{P}$. $\mathcal{P}$ is the free operad generated by the system $(\cdot_k)_{k \geq 2}$, see [MSS] and [Hol1]. Here we can take the $V$-th power of a polynomial $f \in K\{x\}$, which is defined as $f^V := \cdot_V(f, \ldots, f)$. A planar power series is an infinite formal expression over $P$ with coefficients in $K$. If $x$ denotes the unique tree with a single vertex, then any planar power series is an infinite sum over $K$-multiples of planar products in $x$. We call the $K$-vector space $K\{\{x\}\}$ together with the system of the $K$-multi linear extension of the grafting product the algebra of planar power series. Thus $K\{x\}$ is a subalgebra of the algebra of planar power series $K\{\{x\}\}$. On the algebra of planar power series we have a natural topology induced by the order function. The series are continuous with respect to this topology and for a continuous power series $f \in K\{\{x\}\}$ we can consider the $V$-th power $f^V$, which is defined in the same way as for polynomials. Furthermore, there is a canonical, continuous $K$-algebra-homomorphism $\eta : K\{\{x\}\} \to K[[x]]$, which maps a tree $T \in P$ with $|T| = n$ onto $x^n$. This map is the projection of the lifted power series onto the classical algebra.

One of the main results is concerned with planar exponential series. We consider a field $K$ with $\text{char}(K) = 0$ and define these series with respect to all planar rooted trees. We investigate the properties, which these series have, and compare them to the classical case. The recursion formulas for the computations of the coefficients are presented and proved. We state the following

**Proposition** Let $V \in P$ and $k = |V|$ with $k \in \mathbb{N}_{\geq 2}$. Then there is a unique
power series $\exp_V(x) \in K\{x\}$, such that

(i) $\text{ord}(\exp_V(x) - (1 + x)) \geq 2$,

(ii) $(\exp_V(x))^V = \exp_V(kx)$.

Moreover $\frac{d}{dx}(\exp_V(x)) = \exp_V(x)$.

$\exp_V(x)$ is called the planar exponential series with respect to $V$.

Then for the coefficient $a_V(T)$ of $\exp_V(x)$ we get the recursion formula

$$a_V(x) = 1$$

and

$$a_V(T) = \sum_S \frac{\binom{V}{S}}{k^n - k} \cdot a_V(T - S),$$

where $S \in O(T), S \neq x, 2 \leq m = |S|, n = |T|$ and $a_V(T - S) = \prod_{i=1}^m a_V(T_i)$, if $T - S = (T_1, ..., T_m)$.

We also deform these series by weighting the leaves of the planar rooted trees $V$ by $\lambda \in K^n$, where $n = \sharp(L(V)) = |V|$, thus we get an other power series with deformed functional equations. The main result is, that for all planar deformed exponential series $\exp_{V,\lambda}(x)$ we also get the derivative property

$$\frac{d}{dx}(\exp_{V,\lambda}(x)) = \exp_{V,\lambda}(x).$$

By using the exponential series we can define the hyperbolic sine and cosine as well as the ordinary sine and cosine. There we get even and odd power series with same derivative properties like in the classical case. See also [Con].

We define the composition of two power series by substituting $g(x)$ for $x$ in a power series $f(x)$ if the order of $g(x)$ is greater than 0. If $\text{ord}(g) = 1$ then $f(g)$ is an automorphism. We can find the compositional inverse to the exponential series $\exp_V(x)$ and denote it by $\log_V(1 + x)$, which we call the planar logarithm series with respect to $V$. We state the

**Proposition** Let $V \in P$. Then there is a unique power series $\log_V(1 + x) \in K\{x\}$, such that

(i) $\log_V(\exp_V(x)) = x$ and $\exp_V(\log_V(1 + x)) = 1 + x$.

(ii) $\log_V((1 + x)^V) = k \cdot \log_V(1 + x)$, for $k = |V|$.

From the second property, we get a functional equation for these series and a recursion formula for the coefficients $c_V(T)$ of $\log_V(1 + x)$, such that

$$c_V(x) = 1$$
and

\[ c_V(T) = \sum_{S \in O(T) \setminus T} \left( c_V(S) \cdot \prod_{i=1}^{m} \binom{V_{T_i}}{V} \right) \]

\[ \frac{k - k^n}{c_V(T)} \]

where \( n = |T|, m = |S| \) and \( T - S = (T_1, \ldots, T_m) \).

There is a unique derivation such that \((1 + x) \frac{d}{dx} (x) = 1 + x \). For a power series \( g \in K \{ \{ x \} \} \) with \( g' = 1 + g \) it holds

\[ ((1 + x) \frac{d}{dx}) (f)(g(x)) = f'(g(x)) + \left( x \frac{d}{dx} (f)(g(x)) \right) \]

We call this the special planar chain rule.

We find out that for the logarithm we have to use this special planar chain rule to determine a derivative property. Then we get

\[ \left( (1 + x) \frac{d}{dx} \right) (\log_V(1 + x)) = 1, \quad \forall \ V \in \mathbb{P}. \]

There is also a unique power series \( f(x) \), such that

\[ (f(x))^V = 1 + x, \]

which we denote by \( \sqrt[\mathbb{V}]{1+x} \) and call the \( V \)-th root series of \( 1 + x \). It follows that \( \sqrt[\mathbb{V}]{1+x} = \exp_V \left( \frac{1}{k} \cdot \log_V (1 + x) \right) \) as in the classical case. For the derivative property we use again the special planar chain rule and get the result that

\[ \left( (1 + x) \frac{d}{dx} \right) (\sqrt[\mathbb{V}]{1+x}) = \frac{1}{k} \cdot \sqrt[\mathbb{V}]{1+x}. \]

The following theorem gives us the recursion formula for the coefficients.

**Theorem** Let \( V \in \mathbb{P} \) and \( k = |V| \). Then for the coefficients \( b_V(T) \) of \( \sqrt[\mathbb{V}]{1+x} \) it is true that

\[ b_V(x) = \frac{1}{k} \]

and

\[ b_V(T) = -\sum \binom{V}{S} \cdot \prod_{i=1}^{m} b_V(T_i) \]

\[ \frac{k}{k} \]

where \( 2 \leq m = |S| \), \( T - S = (T_1, \ldots, T_m) \) is a planar rooted forest and \( \binom{V}{S} \) the planar binomial coefficient.

If \( S \notin P(V) \cap O(T) \), then the summand is equal to 0.
We state that planar root series are planar algebraic in a similar sense as in the classical theory. There is a planar polynomial \( F(x, y) = y^V - (1 + x) \) in \( K\{x, y\} \), such that \( F(x, \sqrt{1 + x}) = 0 \).

Another interesting result in this work is that we can define a generic exponential and logarithm series not only for the coronas but for other admissible sequences of planar rooted trees. Some of the admissible sequences are coronas, the binary left and right combs and the sequence \( V = (V_q)_{q \geq 0} \) of the form \( V_r = x \cdot x^{r-1}, r \in \mathbb{N} \). In the case that we have such a sequence \( V \) we denote the generic exponential series by \( \exp_V(q, x) \), which is an element in \( K(q)\{x\} \). For a fixed \( k \in \mathbb{N}_{\geq 2} \) we get the equation \( \exp_V(k, x) = \exp_{V_k}(x) \) and it is possible to compute the limit of this generic series. Analogously for the admissible sequences we get a generic logarithm series with the property \( \log_V(k, 1 + x) = \log_{V_k}(1 + x) \) for a fixed \( k \in \mathbb{N}_{\geq 2} \).

Furthermore, we find another possibility to present planar rooted trees of height 2 and 3 and to compute the planar binomial coefficient of these trees.

In the past Lazard in [Laz] already regarded generalizations of the classical exponential series \( \exp(x) \) and logarithm series \( \log(x) \) to non-commutative, non-associative and other types of algebras. Drensky, Gerritzen in [DG] and Gerritzen in [Ger1] also consider power series in one non-commutative, non-associative variable. They have already shown that there is a unique planar exponential series \( \exp_V(x) \) if \( V \) is a corona and for this case there is also a unique planar logarithm \( \log_k(1 + x) \), for \( k \in \mathbb{N}_{\geq 2} \).

In Chapter 1 we collect definitions and theorems for trees and define the planar binomial coefficient and a new multiplication on the set of planar trees.
In Sections 1.1-1.3 we recall definitions and theorems, together with some examples, for rooted trees, planar rooted trees and forests, open and closed subtrees, reduction and contraction of trees onto leaves sets.
In Sections 1.4 and 1.5 the planar binomial coefficients and the \( \lambda \)-deformation of the same are introduced. The planar binomial coefficient is the number of contractions of a tree \( V \) onto an other tree \( T \). Some relations and examples are given.
In Section 1.6 a new \( * \)- multiplication on \( \mathbb{P} \) is defined and some properties of \( \mathbb{P} \) with this multiplication and inverse trees are given.

In Chapter 2 we define polynomials and power series. At first the classical and then the planar ones.
In Sections 2.1-2.3 some information about classical algebras of polynomials and power series are collected. We briefly recall some definitions and theorems of these algebras and of the derivation in this case.
In Sections 2.4-2.5 we introduce the planar algebras of polynomials and power
series and show the connection between these and the associated classical algebra by using the map $\eta$. Even and odd series are also defined.

Chapter 3 deals with substitution homomorphisms, the special planar chain rule and the $k$-th derivative. We show in Sections 3.1 and 3.2 some properties and formulas of the planar substitution homomorphisms, see also [Ger4]. These formulas and theorems are used in the following Chapters to compute the inverse series. In Sections 3.3 and 3.4 the special planar chain rule and the $k$-th derivative are given. For the derivations of the planar logarithm series and the planar roots the chain rule will be used. By the $k$-th derivative some new coherence of the planar binomial coefficients are shown.

In Chapter 4 we state some properties of the planar exponential and logarithm series as well as the $\lambda$-deformations of them. Furthermore we define the planar hyperbolic sine and cosine as well as the ordinary planar sine and cosine. In Sections 4.1-4.4 the planar exponential series, planar logarithm series and the $\lambda$-deformations of these are introduced. The recursion formulas for the computation of the coefficients are given and proved. Furthermore the derivation properties, relations and examples of these series are described. The planar exponential series and the planar logarithm series are inverse to each other, but the $\lambda$-deformations are not. The derivation properties remain true in the case of the exponential series, but the planar $\lambda$-deformed logarithm series does not show any visible derivation property. We also show that there are some classes of planar rooted trees which have the same planar exponential and logarithm series, for all members of this class. These classes are of the form $V, V_n \in \mathbb{P}$, $V_1 = V$ and $V_{n+1} = V \ast V_n$, for $n \in \mathbb{N}^\geq 1$.

In Sections 4.5 and 4.6 both the planar hyperbolic sine and cosine and the planar sine and cosine are defined. For these series we use the coefficients of the planar exponential series. Some relations and properties are given. One new property in the planar algebra is that the equation

$$\sin^2_V(x) + \cos^2_V(x) = 1$$

is not true, because there are higher terms, which do not vanish. But most other properties remain true.

Chapter 5 contains formulas and examples of planar root series and the planar algebraic power series are also defined in this chapter. We describe the planar root functions in Sections 5.1 and 5.2. We show properties in connection with the planar exponential and logarithm series and give some examples. The recursion formulas for the coefficients are proved. In Section 5.3 the planar algebraic power series are defined and we see that the planar root series
are algebraic. We analyze some other algebraic series and prove the recursion formulas.

In Chapter 6 the main results are the definitions of admissible sequences of rooted trees and $q$-functions which help us define the generic exponential and logarithm. The tupel-presentation of rooted trees is also introduced. In Section 6.1 the $q$-polynomials $[T]$ of a rooted tree $T$ are given and the admissible sequences of rooted trees are defined. Some examples of admissible sequences are presented. The generic exponential and logarithm are defined in Sections 6.2 and 6.3, moreover limits of some examples of these generic power series are computed. Another presentation for trees is described in Section 6.4. We show, that it is possible to present every tree of height 2 or 3 only by using the coronas and that we can then compute the planar binomial coefficient only by using the classical one. From this we can also use this presentation for the planar exponential and logarithm series and check what happens if we let one corona run towards infinity.

In Chapter 7 we consider further aspects and applications of the planar theory. In Section 7.1 we introduce a new map $\Delta$, which is a co-additive $K$-algebra homomorphism and the completion of it $\hat{\Delta}$. This maps are also regarded by Holtkamp in [Hol1] and by Schiller in [Sch]. We state that if we apply the map $\hat{\Delta}$ to the planar exponential series we get the equation

$$\hat{\Delta}(\exp_V(x)) = \exp_V(x) \otimes \exp_V(x), \quad \forall V \in \mathbb{P}.$$  

Furthermore from this equation we get some identities for the coefficients of the exponential series and we define the planar binomial coefficients $\binom{R}{S,T}$ of the second kind, which are equal to the coefficients of $R$ in the shuffle of two trees $S$ and $T$.

We define left- and right-inverse series with respect to the grafting in Section 7.2. An interesting result is that $\exp_V(-x)$ is not the inverse of $\exp_V(x)$, but both the left- and right-inverse of $\exp_V(x)$ have the same derivative property like $\exp_V(-x)$, namely if $f \in \{\exp_V(-x), \left(\frac{1}{\exp_V(x)}\right)_R, \left(\frac{1}{\exp_V(x)}\right)_L\}$ then

$$\frac{d}{dx}(f) = -f.$$ 

For these series we can define a planar quotient rule, which we describe and prove in Section 7.3.

In Section 7.4 we introduce the planar Hermite polynomials. We define new sets that we need and show some properties of these polynomials, which are similar to the classical ones. In this case we have three possible definitions of the Hermite polynomials, because we have two variables $x$ and $z$ for which we can choose weather they are commutative and associative or not.
The last Section 7.5 describes planar modules $E_\mu$. The elements of these modules are planar power series $f$, which have the property

\[
\frac{d}{dx}(f) = \mu \cdot f.
\]

We state that the planar exponential series $exp_V(x)$ and $exp_{(V,\lambda)}(x)$ are elements of $E_1$, resp. $exp_V(\mu \cdot x) \in E_\mu$, but the question if these series are a basis remains open.