Measuring and Signal Processing Techniques for Ultrasound Computed Tomography
Acknowledgments

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M.A.
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Nomenclature

General Symbolic Notation

\[
\begin{align*}
f(t) & \quad \text{Scalar function in time domain} \\
F(\omega) & \quad \text{Scalar function in temporal frequency domain} \\
m(x, y) & \quad \text{Scalar function of spatial coordinates} \\
M(\omega_x, \omega_y) & \quad \text{Scalar function in spatial frequency domain} \\
v_s, v_{ss} \text{ etc.} & \quad \frac{\partial v}{\partial s}, \frac{\partial^2 v}{\partial s^2} \\
F_{y^k} & \quad \frac{\partial F}{\partial y^k} \\
A, b & \quad \text{Matrices and vectors.} \\
\nabla_z & \quad \text{Gradient in z direction } \frac{\partial}{\partial z}.
\end{align*}
\]

Variables in Latin Notation

\[
\begin{align*}
b_{\text{comp}}(x, y) & \quad \text{Compound image} \\
B_n(x) & \quad \text{Spline basis function} \\
d, d_1, d_2 & \quad \text{Distances} \\
c & \quad \text{Intercept of a straight line} \\
c_0 & \quad \text{Speed of sound in reference medium (usually water)} \\
c_1, c_2, \text{ etc.} & \quad \text{Speed of sound in medium 1, 2 etc.} \\
C_k(x, y) & \quad k\text{th Circular lesion} \\
E_{\text{ext}} & \quad \text{External energy of snake} \\
E_{\text{int}} & \quad \text{Internal energy of snake} \\
E_{\text{obj}} & \quad \text{Energy of a signal transmitted through object}
\end{align*}
\]
<table>
<thead>
<tr>
<th>Nomenclature</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{\text{curvature}}$</td>
<td>Curvature term of snake internal energy</td>
</tr>
<tr>
<td>$E_{\text{slope}}$</td>
<td>Slope term of snake internal energy</td>
</tr>
<tr>
<td>$E_{\text{total}}$</td>
<td>Sum of internal and external snake energies</td>
</tr>
<tr>
<td>$E_{\text{water}}$</td>
<td>Energy of signal transmitted through water</td>
</tr>
<tr>
<td>$F$</td>
<td>Functional in a variational problem</td>
</tr>
<tr>
<td>$f$</td>
<td>Frequency (temporal)</td>
</tr>
<tr>
<td>$f_{\text{c, obj}}$</td>
<td>Center frequency of a signal through object</td>
</tr>
<tr>
<td>$f_{\text{c, water}}$</td>
<td>Center frequency of a signal through water</td>
</tr>
<tr>
<td>$f_s$</td>
<td>Sampling frequency</td>
</tr>
<tr>
<td>$h_{\text{bmode}}(x, y)$</td>
<td>PSF of B mode imaging system</td>
</tr>
<tr>
<td>$h_{\text{comp}}(x, y)$</td>
<td>PSF of compound imaging</td>
</tr>
<tr>
<td>$H_{\text{inv}}(\omega_r)$</td>
<td>Inverse Filter</td>
</tr>
<tr>
<td>$H_{\text{window}}(\omega_r)$</td>
<td>Low pass window filter</td>
</tr>
<tr>
<td>$H_{\text{scatter}}(\omega)$</td>
<td>Overall Transfer characteristics due to forward scattering</td>
</tr>
<tr>
<td>$H_{t}(\omega)$</td>
<td>Transfer characteristics of an object relative to water</td>
</tr>
<tr>
<td>$I_i$</td>
<td>Intensity of incident wave</td>
</tr>
<tr>
<td>$k_{\text{amp}}$</td>
<td>Amplitude weighting factor in external energy</td>
</tr>
<tr>
<td>$k_{\text{grad}}$</td>
<td>Gradient weighting factor in external energy</td>
</tr>
<tr>
<td>$k_{\text{ph}}$</td>
<td>Scalar constant denoting phase cancelation effect</td>
</tr>
<tr>
<td>$L, L_1, L_2$</td>
<td>Line segment identifiers</td>
</tr>
<tr>
<td>$m$</td>
<td>Slope of a straight line</td>
</tr>
<tr>
<td>$n$</td>
<td>Index of refraction</td>
</tr>
<tr>
<td>$n_{\text{beams}}$</td>
<td>Number of beams</td>
</tr>
<tr>
<td>$n_{\text{proj}}$</td>
<td>Number of projections</td>
</tr>
<tr>
<td>$P$</td>
<td>Fourier Transform of Projection</td>
</tr>
<tr>
<td>$p$</td>
<td>Projection as a function of spatial coordinates</td>
</tr>
<tr>
<td>$pf$</td>
<td>Filtered projection as a function of spatial coordinates</td>
</tr>
<tr>
<td>$P(r)$</td>
<td>Harmonic pressure field</td>
</tr>
<tr>
<td>$P_i(r)$</td>
<td>Incident harmonic pressure field</td>
</tr>
<tr>
<td>$P_s(r)$</td>
<td>Scattered harmonic pressure field</td>
</tr>
<tr>
<td>$P_s$</td>
<td>Scattered power</td>
</tr>
<tr>
<td>$R_j(x, y)$</td>
<td>$j$th Linear segment of a ray</td>
</tr>
<tr>
<td>$r_p$</td>
<td>Reflection coefficient w.r.t. pressure</td>
</tr>
<tr>
<td>$r_v$</td>
<td>Reflection coefficient w.r.t. particle velocity</td>
</tr>
<tr>
<td>$T$</td>
<td>Transformation matrix</td>
</tr>
</tbody>
</table>
$T$  
Frequency independent transfer characteristics

$T_s$  
Sampling interval

$t$  
Power transmission coefficient

$t$  
Parameter of ray equation

$t_{A\rightarrow B}$  
TOF between point A and B

$t_p$  
Transmission coefficient w.r.t. pressure

$t_v$  
Transmission coefficient w.r.t. particle velocity

$v(s)$  
Parametric representation of a curve

$w_{ext}$  
Weight of external snake energy

$y_{obj}$  
Received signal after propagation through object and transduction

$y_{water}$  
Received signal after propagation through water and transduction

$Z$  
Acoustic impedance

### Variables in Greek Notation

$\alpha$  
Attenuation coefficient

$\alpha_{n_i}, \beta_{n_i}$  
Direction cosines of a ray at point $P_i$

$\alpha(s)$  
Weight of slope term of snake internal energy

$\beta$  
Phase coefficient

$\beta(s)$  
Weight of curvature term of snake internal energy

$\gamma$  
Propagation coefficient

$\gamma$  
Regularization parameter

$\delta f$  
A frequency band

$\delta l$  
Lateral resolution

$\Delta t$  
Time delay

$\delta_{TOF}$  
Time delay caused by an object in comparison to water

$\kappa$  
Compressibility

$\rho$  
Mass density

$\omega$  
Circular frequency (temporal)

$\sigma_s$  
Scattering cross section

$\theta$  
Angle of rotation

$\theta_1, \theta_2, \text{etc.}$  
Angles of incidence and refraction

$\theta_{\text{critical}}$  
Critical angle
Nomenclature

\( \theta_i \), Angle of incidence
\( \theta_r \), Angle of refraction
\( \theta_{ij} \), Angle of incidence at \( j \)th refracting interface
\( \theta_{rj} \), Angle of refraction at \( j \)th refracting interface
\( \lambda \), Wavelength

Abbreviations

CT Computed Tomography
FFT Fast Fourier Transform
ODE Ordinary Differential Equation
TOF Time Of Flight
PC Personal Computer
PDE Partial Differential Equation
PML Perfectly Matched Layer
PID Proportional-Integral-Differential
PSF Point Spread Function
RF Radio-Frequency
SDK Software Development Kit
UCT Ultrasound Computed Tomography
URI Ultrasound Research Interface
US Ultrasound
w.r.t. With respect to
1 Introduction

This work addresses some selective aspects of the Ultrasound Computed Tomography (UCT). UCT encompasses a vast field of research ranging from oceanography, seismography and mineral exploration to non-destructive testing and evaluation and finally to diagnostic medical imaging. While a few of the topics of research overlap in all the fields, each of them is distinguished by strongly different goals pursued. Diagnostic imaging application of the UCT has one or several of the following goals.

- The morphological imaging of internal organs, for example with the help of one of the several approaches towards reflection mode UCT.

- Reflection mode UCT to rescue the information that remains hidden due to the inherent limitations of conventional pulse echo imaging.

- Standardization of the scan mechanism to help increase the reproducibility and to reduce operator dependence.

- Quantitative reconstruction of acoustic parameters like speed of sound and attenuation with transmission mode UCT in orders to facilitate tissue characterization.

- Quantitative reconstruction of acoustic speed with time of flight UCT to correct the refraction artifacts of the underlying pulse echo images and those of the reflection mode UCT.

Almost all these points will be addressed in this work with varying degrees of attention. The work concentrates on the formulation and modeling of the problems related to the objectives outlined above with a special regard to the investigation of the validity and limitations of the underlying assumptions.
1 Introduction

1.1 A Brief Synopsis of Previous Work

Following is a brief account of the work done by different researchers in the reconstructive ultrasonic imaging and allied fields so far. A more comprehensive annotated survey of the research till early nineties may be found in [59]. The earliest publication on UCT dates back to as early as 1974 where the possibility of utilizing ultrasound instead of x-rays was explored [47]. In the initial efforts large apertures comprising a single transducer were used to receive the transmitted data which led to phase cancelation thus limiting the accuracy of reconstructing attenuation coefficient. The signal model was modified by [71] to include the phase cancelation by separating the effects of frequency independent phenomena from frequency dependent ones. The problem of specular reflection was addressed in [31] and the shift in center frequency caused by the propagation was shown to be proportional to the attenuation coefficient. This technique was adopted and simplified for the case of echo data in [42]. A curved ray reconstruction method was proposed in [98] in conjunction with a matched filter based time of flight estimator. The reported reconstruction results of an excised porcine heart, however, showed scarcely any improvement over straight line inversion. In vivo results of reflection mode tomography were reported in [38, 45, 55]. The attenuation due to scattering was modeled in [99] and the applicability of the ray approximation and Born and Rytov approximations was discussed. The same author reported in [100] a spatial compound imaging methodology with adaptive histogram equalization of the individual B-mode data to enhance the contrast of the soft tissue surrounding bones. Extension of the tomographic reconstruction to flow measurement was reported in [49, 63].

Improvements in the previous tomographic techniques regarding signal modeling and reconstruction techniques were proposed by several research groups in the nineties. Some clinical trials were carried out to establish the histological relevance of the reconstructive techniques. Attenuation and acoustic speed was measured from the excised breast specimens in [35] to substantiate the relevance of the acoustic speed in discriminating between the malignant and benign breast tumors. A custom designed time of flight CT system was reported in [59, 60, 61] to reconstruct the speed of sound to be used in the correction of time delays in the reflection CT. A compression apparatus similar to mammography device was used to compress the breast and estimate the one dimensional time of flight by looking at the lateral course of the compression plate in B-mode image[96]. Relevance of speed of sound and attenuation in differentiating various breast
tissue types was also investigated in vivo. This concept was further developed in [74, 75] into a limited angle tomography scheme with an inversion algorithm based on singular value decomposition of the time of flight data. The concept of the synthetic aperture and phase abberation correction schemes were combined to show the feasibility of ultrasound in detecting microcalcifications in excised breast tissue from patients with dense parenchyma in [4, 5, 6]. The concepts of matched filter and wavelet transforms were combined in [94] to avoid the difficulties posed by the dispersion on the precise estimation of the delay times. A time domain diffraction CT algorithm was propounded and simulation results presented in [81] and a phase abberation correction scheme using time delays estimated from a straight line inversion model for the same was reported in [82]. A 3D qualitative reconstruction concept based on spiral CT was presented in [24].

The recent years witnessed a revival of the reconstructive techniques mainly in the wake of the revolutionary developments in the transducer and data acquisition technology. A multi band approach to transmission tomography for a semi-quantitative reconstruction of attenuation was reported in [62, 68, 69, 79]. Several results were presented for the case of excised and cadaver tissue. It may however be noted that the approach is valid only for very small sized objects possessing very weak variations in acoustic parameters. Compound data acquisition technique was employed in [104] to recover the attenuation coefficient of phantoms from echo data. The reported attenuation estimates of a phantom were however inadequate in their accuracy partly due to the limited angle restriction. In a similar approach average speed of sound of phantoms was estimated in a registration based approach in [73] to minimize registration errors in spatial compounding. A 3D hemispherical array was simulated under a weak inhomogeneous medium assumption for a tomographic reconstruction concept in [41]. Results of B-mode compounding in mice were reported in [56].

1.2 Preview of the Following Chapters

The following chapters are organized as under. Some important aspects of the imaging with ultrasound are treated in chapter 2. Only the topics relevant to the succeeding chapters are touched upon to facilitate a smooth reading. Chapter 3 describes the modeling of ultrasound tomography with a special emphasis on difficulties peculiar to ultrasound CT. Solutions to some of the difficulties are proposed and a very brief description of the
alternative approaches towards ultrasound CT is included to illustrate their limitations. The time of flight estimation forms the core of a transmission tomography system. Different approaches towards time of flight estimation are discussed in chapter 4 and a novel approach based on deformable splines is treated in detail. Furthermore, different factors affecting the achievable accuracy will be dealt with. Chapter 5 will discuss the effects of refraction on the speed of sound reconstruction. These effects are quantified with respect to speed of sound variations. Effects of speed of sound distribution are discussed in detail. Some strategies to limit the extent of adverse effects in case of transmission as well as reflection CT are given. The work concludes with chapter 6 which present the UCT system realized in this work and the achieved experimental results.
2 Imaging with Ultrasound

Ultrasound is one of the most commonly employed diagnostic tools in medicine to image soft tissue. Purpose of diagnostic imaging is to help visualize ‘some aspect’ of an organ not visible to naked eye. The visualized aspect may either be of morphological, functional or histological nature. Morphological and functional imaging attempts to reveal hidden form and function of an organ respectively; an example of function being blood flow. While histology deals with tissue structure, histological aspects of interest are the ones which may help differentiate various tissue types on the basis of some quantitative information about them. Barring functional imaging which does not form the subject matter of this work, attention will be devoted to the process of extracting morphological and histological details through insonification of biological tissue. Subjecting a target to some form of energy, measuring its response and extracting the required information about the target from the measured response is the basic idea behind all active imaging modalities. Limiting ourselves to the case of ultrasound, insonification of an object gives rise to several physical effects inside it as a result of mutual interaction of the acoustic field and the object. Separation of useful information about the insonified object encoded in the measured response involves identification of material parameters of interest regarding their relevance, modeling of field-medium interaction and finally the formulation of an algorithm either to directly display the required information or to reconstruct it from the measured data. In the following sections, some relevant fundamental concepts of ultrasound physics with specific emphasis on ultrasound imaging systems will be considered without any effort to cover the subject in detail. Some of the material presented here is to facilitate a smoother comprehension of the following chapters. A comprehensive model for pulse echo imaging system is out of the scope of this work and may be found in several standard works, e.g. in [8, 9, 70].
2 Imaging with Ultrasound

2.1 Pulse echo imaging

The most prevalent of the ultrasonic imaging techniques is pulse echo technique. The basic idea may be sketched as follows. A short ultrasonic pulse focused along a preferred direction is coupled into a medium. A part of the transient field thus coupled will be sent back by the medium as a result of scattering and reflection, which will be detected and transduced to an electrical signal for further processing. Further processing includes envelope detection and displaying of the echo signal on some suitable scale. This echo signal from the object is known as an A-scan, where \( A \) stands for Amplitude. If several such coplanar adjacent A-scans are acquired by some means and displayed on a suitable gray scale, a cross-sectional image will result. This image is known as a B-scan or a B-mode image, where \( B \) abbreviates Brightness. The displayed image thus formed rests on a number of assumptions which are known to be inaccurate. Several adverse effects arise depending on the degree of deviation from these assumptions. Some examples of such effects are image artifacts like speckle and shadowing of details, false registration and limitations in objective image interpretation. The pulse echo imaging itself does not form the major focus of this work, it will, however, be considered in the following discussion of the physical fundamentals for two reasons. Firstly it is useful to know the limitations of this ‘state of the art’ modality to appreciate the need for alternative modalities, the other more important reason is the use of a commercial ultrasound system driven in B-mode to implement a transmission ultrasound system in this work.

The discussion in this and the following sections will be limited to the propagation of longitudinal waves in soft tissue in usual ultrasonic range of frequencies which for customary fields of application span from 1 MHz to 15 MHz, though ultrasonic imaging with much higher frequencies is also carried out for specific applications [92, 105]. Broadband signals having a relative bandwidth of up to 100% are usually employed. The wave propagation is customarily described in terms of field quantities pressure \( p \) and particle velocity \( v \) and some mechanical properties of the medium. As the soft tissue consisting primarily of water is usually modeled as a liquid, the material properties considered in the modeling are usually compressibility \( \kappa \) and density \( \rho \). Essential elements of the transmit-receive part of a modern pulse echo ultrasound system using an array of transducers is shown in the schematic of figure 2.1. A short pulse is applied to the transmit beamformer whose output is connected to the ultrasound array over a switch. Due to relatively slow speed of propagation of ultrasonic waves, it is possible to realize elec-
2.1 Pulse echo imaging

Figure 2.1: A simplified schematic depiction of a pulse echo ultrasound system emphasizing the use of array transducer and beamformer.

Electronic focussing using a beamformer. A part of the transient wave reaching the array after being reflected and scattered by the medium will be focussed dynamically by the receive beamformer after being converted to an electrical signal. If the acoustic speed though the medium is known the exact depth from which an echo comes back can be found out on the basis of its time of arrival. Since acoustic speed in a medium is generally unknown, a reasonable assumption has to be made to be able to assign appropriate depths to received echoes.

Assumption 2.1 (Constancy of acoustic speed). In the context of medical ultrasound, it is customary to assume that the speed of sound does not vary for different tissue types.

This is one of the several assumptions made in the process. Several important factors have to be taken into account in order to comprehend the image formation as well as the limitations underlying it. The most important of them are focussing, transduction
2 Imaging with Ultrasound

and propagation of acoustic waves in addition to processing of received signals.

2.2 Transduction, Focussing and Spatial Resolution

The key function of transmit unit consisting of a wave generator, a beamformer and a transducer is to send transient acoustic waves into a medium along a desired direction in the form of a narrow beam. The transducer array of figure 2.1 consists of one row of transducers used to convert the electrical waves to acoustic ones and vice versa. The distance $d$ between two adjacent transducer elements is known as pitch. The accuracy of the axial coordinate $z$ of the received data depends directly on the degree of validity of assumption 2.1. The focussing of the transient wave into a narrow beam along a desired direction is realized with the help of an electronic beamformer by delaying the transmit pulses in some suitable way before feeding them to individual transducer elements to realize an ‘electronic lens’. The advantage of such a focussing scheme vis-à-vis an acoustic lens is its versatility due to variably selectable time delays and aperture size, the latter being limited by the size of the array.

- The beam width along the lateral direction is limited by the directional characteristics of individual transducer elements, size of the active aperture and focussing performed with beamformer.

- The elevational focussing is usually carried out with the help of an acoustic lens. The other way is to add a few parallel transducer rows and to carry out the elevational focussing, also, with the help of an electronic beamformer.

In order to be able to assign a reasonably accurate lateral coordinate $x$ to the received echo signal, one has to focus the transmitted beam as good as possible. The focussing itself can be understood as an effect of diffraction. Far field may be shown to be proportional to the spatial fourier transform of the aperture in case of a mono-frequent field [77]. This simplified model can be used to estimate the position and width of the main lobe as well as the position and level of side and grating lobes. The exact modeling of the focussing phenomena is quite complex for the case of the broad band signals [74]. However, two of the important results obtained by this simplified model retain their validity in most practical cases and may be summarized as follows.
The lateral resolution is directly proportional to the wavelength at center frequency and inversely proportional to the width of the active aperture $D$.

In order to effectively suppress the side lobes, pitch $d$ has to be chosen not larger than the shortest wavelength of the transient wave. The first grating lobes will be orthogonal to the main lobe in either direction, which will be of little consequence in the context of linear arrays.

The actual transduction expressing acoustic field as a function of the applied electrical signal as well as the material properties of the transducer also play an important role in the focussing in the board band case [74]. The transduction may be modeled quite accurately using the concept of Krimholtz, Leedom and Matthaei (KLM) [72], but it is usual to simplify transfer characteristics of the transducer to that of proportionality.

The transmit beamformer feeds each individual array element of the active sub-aperture with suitably delayed and weighted signals so as to get the focal point at the desired depth along a desired line. The weights are used to realize an apodization to minimize the side lobes resulting from the finite aperture size. The delays are computed on the basis of the speed of sound under assumption 2.1. There are several factors limiting the efficacy of the focussing. For example, it would be desirable to increase the size of the active aperture in order to get a better focussing. The active aperture size above a certain limit may be ineffective alone due to the directional characteristics of individual elements. In addition, depending on the degree of the inhomogeneity of the speed of sound defocussing will occur which may be understood as phase abberation that results in image artifacts [89]. The purpose of the receive beamformer is to focus the received signals dynamically to achieve an optimal focussing for each depth independent of the depth of the transmit focal point. Again, one has to invoke assumption 2.1 in order to calculate the depths from arrival times. The transmit focus is usually fixed at a point for a given transmit event. It may also be carried out dynamically as is the case with receive focussing. The data needed for that case will have to be acquired in a multi-static way, which means one has to transmit and receive with single elements respectively in such a way that all possible combinations of transmitter and receiver are exhausted [75].
2.3 Field-Medium Interaction

As the transient field propagates through the medium, it interacts with the medium thus encoding the properties of the medium into it. The objective of an imaging system is to decode these signals to extract the useful information about the medium. Unfortunately, it’s not only the useful information that is encoded into the received signal. A host of undesirable effects that get encoded into the signal makes it more difficult to separate the useful information from noise. An understanding of the mechanisms at play is necessary to be able to formulate the signal processing algorithms effectively. Following is a brief account of some of the most relevant of them in present context.

2.3.1 Attenuation

One of the most evident changes the ultrasonic field undergoes during propagation through a biological tissue is its attenuation. The attenuation of acoustic waves is caused mainly by absorption and scattering. Absorption of ultrasonic energy is caused by the lossy nature of compression and rarefaction of the medium. In thermodynamical sense the compression and rarefaction phenomena are irreversible. This implies that there is a certain hysteresis between the two extremities when plotted on a $p-V$ diagram. The

![Hysteresis Diagram](image)

Figure 2.2: One cycle of the hysteresis between compression and rarefaction causing phase lag

area bounded by the closed hysteresis curve is equal to the energy absorbed in one cycle. This is similar to the case of a lossy dielectric which is modeled by complex permittivity.
The ultrasonic absorption can similarly be modeled on the basis of a complex compressibility \( \kappa(\omega) = \kappa'(\omega) - j\kappa''(\omega) \). The propagation constant \( \gamma(\omega) = j\frac{\omega}{c_0} = j\omega\sqrt{\rho\kappa} \) will thus become complex whose real part will account for the absorption losses.

\[
\alpha(\omega) = j\omega\sqrt{\rho}(\sqrt{\kappa'(\omega)} - j\kappa''(\omega)) - \sqrt{\kappa'(\omega)} + j\kappa''(\omega) \tag{2.1}
\]

The attenuation coefficient is not directly proportional to the frequency \( \omega \) as equation 2.1 may seem to suggest at first glance. A close inspection of the equation will reveal that \( \kappa'(\omega) \) and \( \kappa''(\omega) \) are also functions of frequency. This complicates the frequency dependence of the attenuation coefficient. Experimental results show that the attenuation coefficient varies with around first to second power of frequency depending on the type of the tissue [53, 54].

\[
\alpha(\omega) = \alpha_1\omega^x, \text{ where } 1 \leq x \leq 2 \tag{2.2}
\]

For the current specific case of soft tissue the exponential term \( x \) varies between 1 and 1.5 [76, 91]. It is however a usual practice to approximate the frequency dependence with a straight line within the bandlimits of the transducer [97] given by

\[
\alpha(f) = \alpha_1(f - f_0) + \alpha_0 \tag{2.3}
\]

where \( \alpha_0 \) is the attenuation at \( f_0 = \omega_0/2\pi \). The attenuation caused by scattering is dependent on the distribution and the shape of the scatterers. Though equation 2.1 does not take scattering into effect, yet it is usually used to model the cumulative attenuation.

Apart from absorption and scattering there are some other phenomena like reflection and refraction that are also responsible for the local attenuation of the ultrasonic signal in the current specific context. Attenuation due to specular reflection and refraction will be addressed in the next section. Thermal conduction and viscosity also cause relaxation phenomena which contribute to the total attenuation. Their effect is however assumed to be negligible.

**Assumption 2.2 (Attenuation).** The attenuation coefficient \( \alpha \) is usually assumed to be the effect of absorption alone discarding the contribution of other phenomena like scattering, reflection and refraction.
2 Imaging with Ultrasound

2.3.2 Scattering, Reflection and Refraction

If the acoustic impedance $Z = \sqrt{\rho / \kappa}$ or the acoustic speed $c = 1 / \sqrt{\rho \kappa}$ of a medium is not constant, the wave propagation can no longer be described by a homogeneous wave equation. The variations in the acoustic impedance give rise to varying reflection factors resulting either in reflection or scattering of the incident wave, depending on the physical size of the inhomogeneity compared to the wavelength of the acoustic wave. These are the two mechanism giving rise to echoes of the incident field making the pulse echo imaging possible.

If the size of an object having an acoustic impedance different from that of the surrounding medium is comparable to the wavelength of the field incident on it, the latter will be scattered in all possible directions including the original direction of propagation of the incident field. The directional characteristics of scattering depends on the shape and the size of the scatterer. The difference in the acoustic impedance may be due to a variation in either or both of the acoustic parameters compressibility $\kappa$ and density $\rho$.

The problem has been solved for a number of object of different regular shapes [7, 39], though a general solution for an object of arbitrary shape becomes unmanageable. The characteristics of a scatterer is sometimes given by its scattering cross-section $\sigma$ which is defined as the ratio of the scattered power $P_s$ and the intensity $I_i$ of the incident wave.

$$\sigma = \frac{P_s}{I_i}$$  \hfill (2.4)

The scattering cross-section may be visualized as the effective area intercepting the wave.

The field scattered from a scatterer will be incident on other scatterers and will also be scattered subsequently and will be further attenuated. This situation is usually taken care of by assuming the variations in both the parameters $\rho$ and $\kappa$ to be negligible so that the scattered pressure has a negligible effect on further scattering. This assumption is known as the first Born approximation and may be stated somewhat more formally as follows.

Assumption 2.3 (Born Approximation). It is assumed that the scattered field can be ignored in the context of further scattering. If this assumption is invoked, the scattered field can be calculated alone from the incident field. This is roughly equivalent to saying, the effect of the varying acoustic parameters is negligible.

It, hence, implies that all the approaches based on Born approximation essentially ignore
the effect of refraction, the latter being caused by variations in the acoustic speed which is a function of the the compressibility and density of a compressible medium.

Another implication of this approximation is that the total change in scattered phase is negligible. This does not hold for most practical cases of interest here. Another lesser restrictive approximation sometimes used in the modeling of scattering is the Rytov approximation, which implies that the change in scattered phase per wavelength is negligible. For large scale problems involving several hundreds of wavelengths, this approximation yields marginally better results compared to its Born counterpart. Both these approximations are capable of being extended to higher approximations. A second Born approximation will thus calculate the field scattered by a scatterer on the basis of the incident field only in a first step. In a second step, however, the scattered field will be calculated on the basis of the summation of the incident field and the scattered field calculated in the first step. The advantage promised by higher approximations is usually not commensurate with complexity they bring in.

If the object size is much greater than the wavelength of the acoustic wave, specular reflection will take place. Determined by the definition and orientation of the interface and its reflectance and transmittance, a part of the incident wave will be reflected back while a part of it will travel in the forward direction. The reflection, whether strictly specular or diffuse, always obeys the law of reflection. Specular reflection is, in fact, one of the causes of attenuation besides absorption. The reflection and refraction are illustrated in figure 2.3.

Inhomogeneities in the acoustic speed result in refraction. Different biological tissue types possess different acoustic speeds thus causing a refraction which is usually ignored in the wake of assumption 2.1.

**Assumption 2.4 (Refraction).** *Refraction of acoustic waves is assumed to play no important role. This assumption is mainly born out of necessity, as underlying speed of sound distribution is generally unknown.*

As pointed out above, this assumption is implied implicitly by the Born approximation. In case of a discrete boundary between two media each having a uniform speed of sound, the bending of the acoustic ray is governed by the Snell’s law

\[
\frac{\sin \theta_1}{c_1} = \frac{\sin \theta_2}{c_2}
\]  

(2.5)
where $\theta_1$ and $\theta_2$ are the angles the incident and the refracted beams subtend to the normal to boundary at the point of incidence, and $c_1$ and $c_2$ are the speeds of sound of the two media respectively. This result is a special case of the Fermat’s principle which states that sound waves will take the path of minimum time in traveling from one point to another. In fact Fermat’s principle is the more useful alternative, as the speed of sound usually does not change abruptly in the case of biological tissue, so that the bending takes place gradually. Fermat’s principle will be used in section 5.2 to derive a ray tracing algorithm for a medium with known speed of sound distribution.

The reflection and refraction also cause attenuation, which is dependent on the angles of incidence and refraction as well as the acoustic properties of the two tissue types [37]. It is usual to describe the effects of reflection on the amplitude of the signal in terms of the transmission and the reflection coefficients $t_p$ and $r_p$ defined as the ratios of the amplitudes of the transmitted and the reflected waves respectively and that of the incident wave, given by the following equations

$$t_p = \frac{2Z_{F2} \cos \theta_1}{Z_{F2} \cos \theta_1 + Z_{F1} \cos \theta_2} \quad (2.6)$$

$$r_p = \frac{Z_{F2} \cos \theta_1 - Z_{F1} \cos \theta_2}{Z_{F2} \cos \theta_1 + Z_{F1} \cos \theta_2} \quad (2.7)$$

where $Z_{F1}$ and $Z_{F2}$ are the acoustic impedances of the two mediums respectively.
Similarly the two coefficients relating to the particle velocity amplitude are given by
\[ t_v = \left( \frac{Z_{F1}}{Z_{F2}} \right) t_p \] and \( r_v = r_p \). It is evident that the attenuation caused by reflection
depends not only on the acoustic impedance of the two mediums but also on the
angle of incidence and subsequently the angle of refraction. The reported values of
the acoustic impedance in soft tissue vary approximately between \( 1.4 \times 10^5 \) g/cm\(^2\) and
\( 1.6 \times 10^5 \) g/cm\(^2\) [8], which results in worst case transmission and reflection coefficients
\( (r_p = r_v = 0.067, t_p = 1.067, t_v = 0.93) \) in case of normal incidence. It may be noted
that the power transmission coefficient defined by
\[ t = \sqrt{t_p t_v} \]
is the correct measure for
the fraction of the power being transmitted across the boundary, which for this example
turns out to be \( t = 0.988 \). In order to estimate the variation of the transmission coefficient
for inclined incidence, a uniform mass density of the tissue is assumed and the acoustic
speed is supposed to vary between 1450 m/s and 1560 m/s for different tissue types.
Defining the refractive index of the interface as \( n = c_2/c_1 \) and incorporating \( n \) and the
law of refraction of equation 2.5 into equations 2.6, equation 2.8 is obtained.

\[ t = \frac{2 \sqrt{n \cos \theta_1}}{n \cos \theta_1 + \sqrt{1 - n^2 \sin^2 \theta_1}} \]  

(2.8)

It is however visually convenient to depict the percent reflected power calculated on the
basis of 2.8 as shown in figure 2.4. Though this figure depicts the changes in the reflection
factor with changing angle of incidence only qualitatively due to the assumption of
uniform tissue density, it is evident that the deviation from normal incidence changes
the reflection factor drastically.

The bending of the ultrasonic beams also cause selective attenuation in the signal. It
may be intuitively visualized that only a part of the transmitted field will be converted
to electrical signal on reception due to two major factors.

- Spreading of the beam in elevational direction due to diffraction and out of plane refraction.
- Bending and defocusing of the beams inside the imaging plane due to refraction

While the effects of elevational spreading and bending may be more severe, yet it will
remain an unknown and irremediable source of error. The bending inside the imaging plane will be covered in chapter 5.
2.3.3 Speckle

Speckle is an inherent artifact in pulse echo ultrasound technique. Partially coherent nature of ultrasonic imaging leads to the occurrence of speckle in pulse echo images. Resulting from the constructive and destructive interference of the scattered acoustic waves, the speckle exhibits itself as a granular structure spread all over the B-mode images. As described earlier in section 2.2, the spatial resolution of the pulse echo ultrasonic imaging is limited by the pulse width axially and beam width laterally; it can be inferred that the maximum achievable resolution is insufficient to resolve each scatterer individually, the latter being much smaller than the achievable resolution. The echo signal is thus composed of superposition of the signal scattered from individual scatterers lying roughly in one resolution cell. Scattered signal from neighboring regions will also be superposed to the received signal, depending on the level of the side lobes. The received signal is, therefore, a complex combination of several signals having random phases. The bright and dark granular structures present in the image correspond to constructive and destructive interference respectively. The speckle texture does not correspond to that of the underlying structure, though the average local brightness
corresponds roughly to the local scattering cross section, the density of the scatters and the attenuation coefficient of the underlying structure. The combined effect of the three parameters is sometimes termed as echogeneity. In the granular structure, the size of the grain generally corresponds to the beam width in focus [70]. As a result the underlying anatomical details are buried under the speckle. Understandably, the speckle degrades the spatial resolution of the image in addition to marring the contrast resolution of the B-mode images. Due to the stochastic nature of the speckle its characteristics are usually described statistically. The intensity of a ‘fully developed speckle’ is normally assumed to possess a Raleigh distribution. The deviations from this behavior observed in practical situations are ascribed to low scatterer densities and nonlinearities of the medium on one hand and limitations imposed by the electronic components on the other [106].

2.4 Chapter Summary

The most important aspects of the imaging with ultrasound were presented in this chapter. Fundamental concepts of transduction, focusing and different field-medium interaction phenomena were described. Factors such as bandwidth, active element width and center frequency limiting the axial and lateral resolution were discussed. Some of the assumptions underlying the pulse echo imaging systems were highlighted in order to point out the problems associated with it. Special attention was devoted to differentiate between different origins of the ultrasonic field attenuation. Beside the absorption resulting from different relaxation mechanisms, scattering, reflection and even refraction was also illustrated to contribute to the resultant attenuation. The bending effect of refraction were also discussed.
2 Imaging with Ultrasound
3 Ultrasound Computed Tomography Models

Derived from the Greek words *tomos* and *graphein* meaning *to cut* and *to write* respectively, the term tomography is literally applicable to any system capable of imaging a cross section of an object. The usage of the term *computed tomography* (CT) is, however, restricted to the sectional imaging of an object from either transmission or reflection field data collected by illuminating the object from many different directions [66]. In the current specific context, illumination stands for insonification with ultrasound waves, though in the most prevalent form of computed tomography the illumination is done by exposing the target to x-rays. Initially, computed tomography was understood as a mathematical proposition of reconstructing the distribution of a parameter whose line integrals are known [66]. In fact the x-ray computed tomography is based on the (largely valid) assumption that the measured transmission data forms line integrals of the attenuation coefficient of the illuminated object. This assumption is only loosely valid in the case of UCT, as will be discussed later. It was therefore necessary to look for some alternative means to formulate tomographic approaches incorporating more specific physical and practical aspects of the ultrasound wave propagation. In a broader sense the term computed tomography thus refers to an imaging technique which consists of measuring the field scattered or reflected by the object to be imaged from multiple angles of incidence and accordingly reconstructing parameters of interest from the measured field. In the following sections, starting from the classical notion of the CT, several other approaches will be outlined. In the following discussion special attention will be devoted to the case of ultrasound CT.
3.1 The Notion of CT

As with all other active imaging techniques, the underlying physical concept is to apply a known field to an object and to measure the field resulting from the interaction of the field and the object. The objective of a CT system is to extract this information from the multiplicity of field measurements. The two essential components of every CT system are

- a mathematical model describing the relation between the measured field quantities and the spatial boundary conditions imposed by the employed measurement setup and the parameter of interest, and
- a reconstruction procedure suitable to extract the parameter from the measured data.

As alluded to above, computed tomography is distinguished from other imaging systems in the fact that it necessitates some form of multiplicity in the measurement. The multiplicity is often provided by letting the field interact with the object from several different angles of incidence and measuring the resultant field. The other underlying concept of all the tomographic systems is to furnish a mathematical model of the object and the field in such a way that the interaction between the two may be formulated in a manner suitable to arrive at a mathematical rule for the reconstruction of a desired parameter from the measured data. The different approaches towards CT differ from each other either in the modeling itself or in the data measurement strategy or both.

3.1.1 Radon Transform as a Mathematical Model

The most straightforward model is essentially subject to the condition that the field interaction with the object may be reduced to a decoupled form, in such a way that the effect of the parameter of interest of the object on the field as the latter passes through the former, may be decoupled from that of the other phenomena taking place at the same time. Though this simplifying assumption is often not fulfilled strictly in reality, the simplest model is arrived at under this assumption. In that case it is assumed that the line integral of a parameter along any arbitrary straight line in the imaging plane may be ‘measured’. Let $xy$-plane be the imaging plane and let the parameter of interest
of the object be a scalar function \( m(x, y) \) of the spatial coordinates \( x \) and \( y \) inside this plane. Assuming the possibility of illuminating the object with an appropriate field in such a way that the line integral along the line \( L \) shown in Figure 3.1, can be calculated on the basis of measured field. Let the line \( L \) subtend an angle \( \theta \) to the \( y \)-axis. A new Cartesian coordinates system \( x'y'z \) has its \( y' \)-axis parallel to \( L \). If the \( x' \)-intercept of the straight line through \( L \) is \( x' \), the parametric equation of the straight line \( L \) may be written down as 
\[
x \cos \theta + y \sin \theta - x' = 0.
\]
Now the two dimensional delta function 
\[
\delta(x \cos \theta + y \sin \theta - x')
\]
will be zero everywhere except on the line \( L \), so that sifting property of the delta function may be invoked to rewrite the line integral
\[
\int_L m(x, y) dy'.
\]
as the following double integral.
\[
p(\theta, x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(x, y) \delta(x \cos \theta + y \sin \theta - x') dx dy.
\]
This double integral \( p(\theta, x') \) is the line integral of the parameter \( m(x, y) \) along the line \( L \) and is formally known as the Radon transform of the function \( m(x, y) \). Mathematically speaking the imaging section has to be a compact support of \( m(x, y) \) which simply means that \( m(x, y) \) has a zero value everywhere outside the imaging section. In other words, one has to ignore the effect of neighboring sections of the object and that of
the surrounding medium. Though the former condition is hardly fulfilled in the case of ultrasound, the latter is realized, in practice, by carrying out a reference measurement through the medium itself which may arguably be supposed to possess a known uniform parameter distribution. A projection is usually a set of line integrals measured from an illumination of the object from a given source position, parallel projection being the simplest one formed by line integrals along equidistant straight lines. If sufficient number of data projections are available from different aspect angles around the object, a reconstruction may be performed as follows.

### 3.1.2 Reconstruction Algorithms

**Fourier Slice Theorem**

The reconstruction is based on the Fourier Slice Theorem which states that the Fourier transform $P(\theta, \omega_x')$ of a projection $p(\theta, x')$ is a slice through the two dimensional Fourier transform of the parameter to be reconstructed. The theorem can be demonstrated straightforwardly by considering the one dimensional Fourier transform $P(\theta, \omega_x') = \int_{-\infty}^{\infty} p(\theta, x')e^{-j\omega_x'x'}dx'$ of the projection $p(\theta, x')$ of equation 3.2

$$P(\theta, \omega_x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(x, y)\delta(x \cos \theta + y \sin \theta - x')e^{-j\omega_x'x'}dx'dy$$

(3.3)
rearranging the integrals as \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(x, y) \left( \int_{-\infty}^{\infty} e^{-j\omega_x x'} \delta(x \cos \theta + y \sin \theta - x') dx' \right) dxdy \) and using the sifting property of the delta function and rearranging again, it reduces to.

\[
P(\theta, \omega_x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(x, y)e^{-j(\omega_x' \cos \theta x + (\omega_x' sin \theta)y)} dxdy \tag{3.4}
\]

The transformation matrix \( T_\theta \) for the rotation of \( xy \) - and \( \omega_x \omega_y \)-coordinate systems to \( x'y' \) - and \( \omega_x' \omega_y' \)-coordinate systems is given by.

\[
T_\theta = \begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix} \tag{3.5}
\]

It is straightforward to recognize that the double integral \( P(\theta, \omega_x') \) of equation 3.4 is located at a point on the line \( \omega_y = \omega_x \tan \theta \) shown in figure 3.2. It is evident that the Fourier transform of the rest of the projections \( p(\theta, x') \) for varying \( x' \) will also lie on the same line in frequency domain. The Fourier transform \( P(\theta, \omega_x') \) of the projection \( p(\theta, x') \) is thus a slice along the line \( \omega_y = \omega_x \tan \theta \) of the following two dimensional Fourier transform of the parameter \( m(x, y) \).

\[
M(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(x, y) e^{-j(\omega_x x + \omega_y y)} dxdy \tag{3.6}
\]

This demonstrates the Fourier slice theorem. This theorem forms the basis of the reconstruction of a parameter whose projections from several different directions are known. The reconstruction algorithm in the form of Fourier slice theorem is computationally quiet involved as it is easy to appreciate that the Fourier transform \( M(\omega_x, \omega_y) \) will be known only on discrete points along radial lines \( \omega_y = \omega_x \tan \theta \) which makes the direct calculation of the inverse transform quite difficult. Alternately an interpolation of the points on radial lines to a Cartesian grid can solve this problem. The sampling requirements will be discussed later. Alone the two dimensional interpolation and computation of the two dimensional inverse Fourier transform makes the approach quite unattractive.

**Filtered Back Projection**

To comprehend the other more practical approach the inverse Fourier transform of \( M(\omega_x, \omega_y) \) given by \( m(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} M(\omega_x, \omega_y) e^{j(\omega_x x + \omega_y y)} d\omega_x d\omega_y \) is transformed to polar coordinates using \( \omega_x = \omega \cos \theta \) and \( \omega_y = \omega \sin \theta \).

\[
m(x, y) = \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{\infty} M(\omega, \theta) e^{j\omega(x \cos \theta + y \sin \theta)} \omega d\omega d\theta \tag{3.7}
\]
Splitting the integral \( \int_{0}^{2\pi} \) into \( \int_{0}^{\pi} + \int_{\pi}^{2\pi} \), making use of the property \( M(\omega, \theta + \pi) = -M(\omega, \theta) \), substituting \( x' = x\cos\theta + y\sin\theta \), and collecting the two integrals again, equation 3.7 becomes,

\[
m(x, y) = \frac{1}{2\pi} \int_{0}^{\pi} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} M(\omega, \theta) |\omega| e^{j\omega x'} d\omega \right) d\theta
\]

Recognizing the inner integral to be the one dimensional inverse Fourier transform of \( M(\omega, \theta)|\omega| \) along the line \( x' = x\cos\theta + y\sin\theta \), \( M(\omega, \theta) \) can be replaced by the Fourier transform \( P(\theta, \omega) \) of the projection \( p(\theta, x') \) according to Fourier slice theorem, so that the integral of equation 3.8 is reduced to

\[
m(x, y) = \frac{1}{2\pi} \int_{0}^{\pi} p_f(\theta, x\cos\theta + y\sin\theta) d\theta
\]

where \( p_f(\theta, x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\theta, \omega) e^{j\omega x'} |\omega| d\omega \) is the filtered version of \( p(\theta, x') \), the filter being a ramp shaped high pass filter with frequency response of \( |\omega| \). Equation 3.9 is the reconstruction algorithm known as filtered back projection. This has to be discretized to a summation for a finite number of projections each comprising a finite number of line integrals. The filter response \( |\omega| \) also known as Ramachandaran-Lakshminarayanan filter has the undesirable side effect of amplifying the high frequency noise in case of measured signals possessing a limited signal to noise ratio. It is often replaced by filter of more desirable frequency response for higher frequencies like the one proposed by Shepp and Logan having a constant response for higher frequencies.

### 3.2 A Model for Ultrasound Transmission Tomography

A simplified model will be derived below, thus expressing the line integrals of the two parameters, acoustic attenuation and speed of sound in terms of measured field quantities in a bi-static measuring setup illustrated in figure 3.3. It will be assumed that the two transducers can be rotated unobstructedly about the object to be reconstructed.

#### 3.2.1 Coefficient of Attenuation

The attenuation coefficient, though more difficult to model for tomographic reconstruction, will be discussed first due to its similarities with the x-ray attenuation coefficient
in its simplest formulation. This model [48] for the tomographic reconstruction of the ultrasound attenuation is based on several assumptions outlined below.

**Assumptions 3.1 (Ultrasound Tomography).**

1. The propagation of ultrasonic energy takes place along straight lines. The bending effect due to refraction, scattering and diffraction are ignored.

2. All the tissue types are supposed to possess the same acoustic impedance, so that there is no reflection and no signal attenuation due to it.

3. The signal attenuation is solely due to the absorption in tissue, the one caused by diffraction and scattering may be neglected.

4. In case of a broad band signal all the frequencies are attenuated with the same coefficient of attenuation.

![Figure 3.3: Schematic depiction of ultrasound transmission CT](image)

If these assumptions hold for some given case, the tissue attenuation may be modeled quite simply as a line integral calculable from the measured field data. Let the object of Figure 3.3 be insonified with a narrow beam of ultrasound along the line $L$ from a *transmitter*. The object is surrounded by the coupling medium water, which attenuates negligibly with a uniform coefficient of attenuation $\alpha_{\text{water}}$. The ultrasonic wave is supposed to travel along straight lines and is measured with a *receiver*. Let the coefficient
of attenuation inside the object be $\alpha(x, y)$. In the following, it will be shown that it is possible to calculate the line integral of the attenuation from the received signals with and without the object under the above assumptions. Let the received signals for the two cases (with and without the object) be $y_{\text{obj}}$ and $y_{\text{water}}$, respectively for the same electrical signal applied to the transmitting transducer. The amplitude of a signal at a distance $l$ from the transmitter will be attenuated by a factor of $e^{-\int_0^l \alpha(x, y)dl}$. Conversely, if the amplitude of the attenuated signal at a distance $l$ from the transmitter is known to be $A$, the amplitude of the signal emitted by the transmitter will be $A e^{\int_0^l \alpha(x, y)dl}$. In the wake of this argument, the signals $y_{\text{obj}}$ and $y_{\text{water}}$ will be related to each other by means of the following equation.

$$|y_{\text{obj}}| e^{\int_{L_1} \alpha_{\text{water}}dl + \int_{L_2} \alpha(x,y)dl} = |y_{\text{water}}| e^{\int_{L_1} \alpha_{\text{water}}dl + \int_{L_2} \alpha_{\text{water}}dl + \int_{L_2} \alpha_{\text{water}}dl}$$

(3.10)

The reference medium is mostly water, whose attenuation coefficient $\alpha_{\text{water}}$ is constant at a given temperature. The above equation thus reduces to

$$\int_{L} \alpha(x,y)dl = \ln \left| k_\alpha \frac{y_{\text{water}}}{y_{\text{obj}}} \right|$$

(3.11)

where $k_\alpha = e^{\int_{L} \alpha_{\text{water}}dl}$ is a known constant and may even be dropped practically as the attenuation through water may be assumed to be negligible. This equation shows that the line integral of the attenuation coefficient along the beam path may be calculated from the measured signals. In addition to the assumptions listed above, there are several other difficulties in using equation 3.11 for the reconstruction of the attenuation coefficient, which will become clear later.

### 3.2.2 Speed of Sound

In contrast to x-rays it is possible, in case of ultrasound, to measure the time of flight of the waves from the transmitter to the receiver due to comparatively much slower speed of propagation. Starting from the definition of the speed as a time derivative of distance, it is straightforward to show that the time of flight (TOF) along a propagation path is the line integral of the reciprocal of speed over that path. Applying the same assumptions 3.1 as in section 3.2.1 above, the following two relationships are arrived at for times of flight in both the cases (measurement with and without the object)

$$TOF_{\text{obj}} = \int_{L_1} \frac{1}{c_w}dl + \int_{L} \frac{1}{c(x,y)}dl + \int_{L_2} \frac{1}{c_w}dl$$

(3.12)
3.2 A Model for Ultrasound Transmission Tomography

\[ \text{TOF}_{\text{water}} = \int_{L_1} \frac{1}{c_w} dl + \int_{L_2} \frac{1}{c} dl + \int_{L_2} \frac{1}{c_w} dl \]  

(3.13)

where \( c_w \) and \( c(x, y) \) are the acoustic speeds through water and the object respectively. Subtracting 3.13 from 3.12 and simplifying yield,

\[ \int_{L} \frac{1}{c(x, y)} dl = \text{TOF}_{\text{obj}} - \text{TOF}_{\text{water}} + k_c \]  

(3.14)

where, again, \( k_c = \int_{L} \frac{1}{c_w} dl \) is a known constant. It is thus possible to calculate the line integrals of the reciprocal of the speed of sound along the direction of propagation from the signals measured in a bi-static transmission measurement scheme under several simplifying assumptions. Equation 3.15 may alternatively be written as

\[ \int_{L} \left( \frac{1}{c(x, y)} - \frac{1}{c_w} \right) dl = \text{TOF}_{\text{obj}} - \text{TOF}_{\text{water}} \]  

(3.15)

which is the more practical form. It is simple to calculate the acoustic speed map from the map of the reconstructed quantity \( \frac{1}{c(x, y)} - \frac{1}{c_w} \).

3.2.3 Sampling Requirements for the Projection Data

It should be pointed out that the following discussion is based on the assumptions 3.1 above. Only the time of flight tomography will be considered here exemplarily. In the framework of assumptions 3.1, a focussed beam of the ultrasound along the line \( L \) is, at first, considered to have a uniform width which can be made arbitrarily narrow. Since the measurement setup used for this work is implemented around a commercial ultrasound scanner, it’s reasonable to assume the distance between two adjacent beams to be predefined. Let this distance be \( \Delta_{\text{lat}} \). The highest spatial frequency along the lateral course is thus determined by Nyquist theorem to be \( \frac{1}{2\Delta_{\text{lat}}} \). Hence it is evident that the lateral course of the time of flight profile for linear array transmitting and receiving transducers will define the ultimately achievable resolution of the speed of sound reconstruction. The radial resolution of the Fourier transform of this projection will be given by \( \frac{1}{2\Delta_{\text{lat}}} + \frac{1}{2\Delta_{\text{lat}}} \) where \( n_{\text{beams}} \) is the number of beams per projection. In order to suppress the aliasing artifacts the number of projections \( n_{\text{proj}} \) has to be selected such that the worst case azimuthal resolution \( \left( \frac{1}{2\Delta_{\text{lat}}} \right) \left( \frac{n_{\text{proj}}}{n_{\text{beams}}} \right) \) is equal to the radial resolution \( \frac{1}{2\Delta_{\text{lat}}} \). Equating the two resolutions gives the following condition for the number of projections \( n_{\text{proj}} \) in terms of the number of beams \( n_{\text{beams}} \)

\[ n_{\text{proj}} = \frac{\pi}{2} n_{\text{beams}} \]  

(3.16)
This discussion is illustrated in figure 3.4.

It may be noted that the best achievable lateral resolution of each projection corresponds to the lateral resolution of the array in focus, which is inversely proportional to the product of the center frequency and the width of the active aperture [106], if the defocussing effects due to refraction are ignored. The resulting isotropic resolution of the acoustic speed tomogram will thus be defined by these two parameters of the linear array. Another implication of this limitation on the lateral resolution of each projection is that the highest spatial frequency occurring in each projection is thus proportional to the center frequency of the array and the width of the active aperture. Assuming the width of the active aperture to be not controllable in case of reception with a commercial ultrasound system, the highest spatial frequency will ultimately be proportional to

\[
\frac{1}{2\Delta_{lat}} \left( \frac{\pi}{n_{proj}} \right)
\]

\[
\frac{1}{\Delta_{lat}} \frac{1}{n_{beams}}
\]
the center frequency of the transducer. The resulting resolution of the speed of sound
tomogram will be directly proportional to the center frequency of the transducer.

### 3.2.4 Validity and Limitations of the Model

There are several limiting factors in the applicability of the model introduced in the pre-
ceding sections. The Assumptions 3.1 listed above are only partially met in the imaging
of realistic objects like human tissue. Following is a brief account of the limitations in
the applicability of the above model in realistic situations.

1. The attenuation modeled as the ratio of the amplitudes, or that of energies, of
the received signals after propagation through the object and through water has
several shortcomings. This ratio of the maximum amplitudes will approximate the
attenuation quite well if the bandwidth of the signal is substantially narrower than
that of the transfer function of the tissue possessing a uniform acoustic impedance.
For a normal case of the tissue transfer function possessing a bandwidth compa-
rable to that of the signal, the error of estimation can be as big as 10% for 1 cm
thick fairly uniform liver tissue layer [65].

- Attenuation caused by the **reflection on the front and back end** of the
  object is not accounted for.
- The object to be imaged does not possess a uniform acoustic impedance. The
  non-uniform impedance will give rise to reflection, so that only a fraction of
  the incident energy is transmitted across the boundary of the interface given
  by the **unknown transmission factor** of equation 2.8.
- The varying speed of sound inside the object and the finite size of the receive
  aperture cause **phase cancelation** after reception, thus distorting magnitude
  of the the maximum amplitude.

2. The attenuation model does not take care of attenuation caused by out of plane
reflection, refraction and scattering.

3. The straight line propagation assumption neglects the bending effect of refraction.
   This topic is discussed in detail in chapter 5.
4. Multiple reflections and reverberations are the most difficult to model and are ignored in almost every propagation model. Nonetheless, they may cause substantial distortions.

### 3.2.5 Refinements and Extensions in the Model

#### Generalized Model

A more comprehensive model of the wave propagation may be arrived at by incorporating as many of the factors addressed above into it as possible within the framework of the straight line propagation. Let there be \( n \) regions of differing acoustic impedances along the propagation path inside the object shown in figure 3.3. Let the frequency dependent attenuation coefficient of the \( i \)th region be \( \alpha_i(\omega) \) and that of water be \( \alpha_{water} \) and the

![Figure 3.5: A generalized model for ultrasound transmission CT](image-url)
transmission factor between the interface of \((i-1)\)th and \(i\)th region be \(t_i\). The frequency dependence of the attenuation coefficient of water is given by a square law [97], but the relative magnitude of the attenuation coefficient is a few orders of magnitude lesser than that of the tissue [33], which legitimizes the dropping of frequency dependence. The spatial dependence of the attenuation coefficient on arguments \(x\) and \(y\) will not be given here for brevity of notation. The effect of the phase cancelation due to finite receive aperture size and due to axial shift resulting from refraction is modeled as a an unknown scalar \(k_{ph}\). If the effect of diffraction loss along the propagation path is modeled as a linear transfer function \(H(\omega)\), the Fourier transform \(Y_{obj}(\omega)\) of the electrical signal received after propagation through the object

\[
Y_{obj}(\omega) = X(\omega) H_1(\omega) e^{-(\alpha_{water}+j\beta_{water})L_1} \left( \prod_{i=1}^{n} t_i e^{-\alpha_i(\omega)+j\beta_i(\omega)} \Delta l_i \right) t_{n+1} e^{-(\alpha_{water}+j\beta_{water})L_2} H_2(\omega) k_{ph}(\omega) H_{scatter}(\omega) \tag{3.17}
\]

and that after propagation through water alone \(Y_{water}(\omega)\) in the absence of the object

\[
Y_{water}(\omega) = X(\omega) H_1(\omega) e^{-(\alpha_{water}+j\beta_{water})(L_1+L_2)} H_2(\omega) H(\omega) \tag{3.18}
\]

where \(\beta_i(\omega)\) is the phase coefficient of the \(i\)th region and \(\beta_{water}\) is that of water, \(H_1(\omega)\) and \(H_2(\omega)\) are the transfer functions of the transmitting and receiving transducer respectively and \(H_{scatter}(\omega)\) denotes the attenuation characteristics due to scattering. The necessity to take into account the acoustic speed dispersion is shown in [50, 102] for the reason of the causality. It was, however, shown in [65] that the non-causal energy is less than 0.001% of the total energy for the range of attenuation occurring in the tissue [33]. The effect of refractional bending can not be included in such a model based on straight line propagation assumption. By dividing equation 3.17 by 3.18, the effects of the propagation trajectory outside the object and those of transduction cancel out. The absolute value of the ratio of the Fourier transforms of the two signal is thus given by

\[
\left| \frac{Y_{obj}(\omega)}{Y_{water}(\omega)} \right| = e^{-\int_L (\alpha(\omega)-\alpha_{water})dl} \left| \left( \prod_{i=1}^{n+1} t_i \right) k_{ph} H_{scatter}(\omega) \right| \tag{3.19}
\]

where the summation is again generalized to integration, to allow variable attenuation coefficient within a region of a given acoustic impedance, \(t_{n+1}\) is the transmission coefficient between that boundary of the object with water as the ultrasound beam leaves the object. The line integral of the difference of the attenuation coefficient of the object
and that of water may thus be given by

\[
\int_{L} (\alpha (\omega) - \alpha_{water}) \, dl = \ln \left( \prod_{i=1}^{n+1} t_i \right) + \ln |k_{ph}| + \ln |H_{scatter} (\omega)| - \ln \left| \frac{Y_{obj} (\omega)}{Y_{water} (\omega)} \right| \]

(3.20)

As the transmission factors \( t_i \), the scalar term \( k_{ph} \) denoting the effect of phase cancelation and the scattering attenuation are unknown, it is impossible to find out the exact value of the right hand side. There are several practical approaches to avoid this limitation, albeit partially. One of them is to neglect the effect of the unknown factors and to approximate the line integral only through the logarithm of the ratio \( \frac{Y_{obj} (\omega)}{Y_{water} (\omega)} \) [65]. The other more accurate approach is to separate the frequency dependent terms from the frequency independent ones. It is evident from equations 2.5 and 2.8 that the scalar \( k_{ph} \) modeling the phase cancelation phenomenon and the transmission factors are independent of frequency. Rewriting equation 3.20 and dropping the transfer function \( H_{scatter} (\omega) \) yields:

\[
H_t (\omega) = \ln \left| \frac{Y_{obj} (\omega)}{Y_{water} (\omega)} \right| = \ln \left( \prod_{i=1}^{n+1} t_i \right) + \ln |k_{ph}| - \int_{L} (\alpha (\omega) - \alpha_{water}) \, dl \]

(3.21)

The first two terms on the right hand side are frequency independent.

### An Indirect Measure for Attenuation

Now instead of calculating the line integral of \( \alpha (\omega) \), we introduce the line integral of another quantity derived from \( \alpha (\omega) \). The method was introduced in [71] for continuous wave measuring system and is extended here to broadband signals by subtracting the averaged spectrum of the transfer function \( H_t (\omega) \) around two different frequencies \( \omega_1 \) and \( \omega_2 \) over a window of \( 2\Delta \omega \), as a result of which the frequency independent terms will cancel out. Changing the order of integration results in

\[
\int_{L} \left( \int_{\omega_1 - \Delta \omega}^{\omega_1 + \Delta \omega} \alpha (\omega) \, d\omega - \int_{\omega_2 - \Delta \omega}^{\omega_2 + \Delta \omega} \alpha (\omega) \, d\omega \right) \, dl = \int_{\omega_2 - \Delta \omega}^{\omega_2 + \Delta \omega} H_t (\omega) \, d\omega - \int_{\omega_1 - \Delta \omega}^{\omega_1 + \Delta \omega} H_t (\omega) \, d\omega. \]

(3.22)

The tomographic reconstruction of the quantity \( \int_{\omega_1 - \Delta \omega}^{\omega_1 + \Delta \omega} \alpha (\omega) \, d\omega - \int_{\omega_2 - \Delta \omega}^{\omega_2 + \Delta \omega} \alpha (\omega) \, d\omega \) is thus possible under less restrictive assumptions.
A fairly valid simplification

The utility of the formula derived above is limited by the fact that it is only a qualitative estimate of the attenuation in a given frequency band. Another practical approach is to make two fairly valid assumptions, and to try to reconstruct the attenuation itself, rather than its frequency derivative.

**ASSUMPTIONS 3.2 (Spectral Behavior of Signals).**

1. The attenuation coefficient may be approximated by a linear frequency dependence.
2. The power spectral density of the received signal through water may be approximated to be of Gaussian shape.

Equation 3.19 is rewritten in the following form to this end.

\[
|Y_{\text{obj}}(\omega)|^2 = T e^{-\int L \alpha(\omega) dl} |Y_{\text{water}}(\omega)|^2
\]  

(3.23)

where \( T = \left| \prod_{i=1}^{n+1} t_i \right|^2 \) denotes the frequency independent part of the transfer characteristic, the scattering term being dropped again. Moreover, the attenuation coefficient of water is ignored here due to the reason described above. It can be shown, as in Appendix A.2, that the required line integral is given by

\[
\int_L \alpha(f) df = f \frac{f_{c,\text{water}} - f_{c,\text{obj}}}{2\sigma^2}
\]

(3.24)

where \( f = \omega/2\pi \). \( f_{c,\text{water}} \) and \( f_{c,\text{obj}} \) are the center frequencies of the power spectral densities of the two signals and \( \sigma^2 \) the variance of \( |Y_{\text{obj}}(\omega)|^2 \). The standard deviation \( \sigma \) may also be interpreted as 4.34 dB bandwidth of the power spectral density in this case. In terms of the energy ratio \( \frac{E_{\text{obj}}}{E_{\text{water}}} \) of the two signals, the integral can also be shown to be

\[
\int_L \alpha(f) df = f \sqrt{f_{c,\text{water}}^2 - 2\sigma^2 \ln \frac{E_{\text{obj}}}{E_{\text{water}}}}
\]

(3.25)

The above discussion makes it clear that it is very difficult to separate the effect of attenuation coefficient \( \alpha \) from the other attenuating effects. As a result it becomes quite difficult to interpret the reconstruction of the attenuation from a transmission measurement.
3.3 Alternative Tomographic Models and their Limitations

3.3.1 Iterative Inverse Scattering Approach

While the term inversion is applicable to all reconstructive imaging techniques including the tomographic techniques discussed above, inverse scattering is an alternative approach for reconstructing the acoustic parameters. While the approaches discussed above use some simplifying assumptions to circumvent the explicit use of the wave equation as a propagation model the inverse scattering approach applies the wave equation itself as a propagation model. The fundamentals of the methods will be outlined below in order to appreciate the prohibiting factors in taking up this technique for the current work. The approximate framework of inverse problem solving may be sketched as an iterative method consisting of the following steps.

- **The forward problem** consists of solving the wave equation based on a known or guessed acoustic parameter distribution to find out the field quantities.

- **The inverse problem** consists of comparing the computed field quantities with the measured ones and to minimize the difference between the two in such as way as to find out a correction for the previous parameter distribution.

More formally, it is usual in the literature to start with Helmholtz equation of the form \[ \nabla^2 P(r) + \gamma^2 P(r) = 0 \] (3.26) for harmonic pressure fields \( P(r) \), where \( \gamma = \frac{\omega}{c} + j\alpha \) is the complex propagation coefficient modelling the lossy medium. The total field at every spatial point is assumed to be composed of an incident component \( P_i(r) \) and a scattered component \( P_s(r) \), i.e.

\[ P(r) = P_i(r) + P_s(r) \] (3.27)

In addition, the Sommerfeld radiation condition is included in order to insure that the scattered field is outgoing [28]. The problem is usually restated as a Lippmann-Schwinger integral equation and solved as a nonlinear or linearized optimization problem [64][103][46] with the finite element method. This technique suffers from the severe
limitation of unmanageably high computational times which become even unrealistic for the usual range of ultrasonic frequencies used in medical imaging due to numerical stability constraint on the element size $h$ of the finite element mesh given by

$$ h < \frac{\lambda}{2\pi} $$

where $\lambda$ is the wavelength of the time harmonic field. The other, probably less restrictive problem is that the methods are developed for continuous wave (CW) fields and may not be applicable to a broadband field chosen for the current work. This latter prohibiting factor also applies to the finite differences variant of the inversion algorithm [87].

### 3.3.2 Diffraction CT

![Diagram of Diffraction CT](image)

Figure 3.6: Schematic illustration of Diffraction CT

The basic idea behind the Diffraction CT is to invoke either of the weak scattering approximations, i.e. Born or Rytov approximation to reduce the Helmholtz equation 3.26 to an explicit solution known as Fourier diffraction theorem [66] illustrated in figure 3.6. The object is now illuminated with a time harmonic plane wave, the fourier transform of the forward scattered field measured by the receiver gives a qualitative estimate of the object properties along semicircles of radius $2\pi/\lambda$ passing through the origin in the spatial frequency domain.

This technique has some major drawbacks in addition to its validity only within the framework of the weak scattering approximation which make it quite unattractive for the current work. Firstly it delivers only qualitative estimates of the acoustic parameters [32]. Secondly, again, application of broadband signals make the reconstruction procedure quite involved [81, 82], which is not a fruitful proposition if there is little gain in it.
Apart from these considerations, the potentially achievable resolution of the tomography is limited by the lateral resolution of the linear array employed here as a receiver. For example for a signal of 5 MHz center frequency, the lateral resolution will be a few millimeters. Spatial Fourier transform of the projection received with linear array will thus possess a Nyquist frequency of the order of several thousand radians per meter which compared to the radius of the arc $2\pi/\lambda \approx 20000$ radians per meter will at the most be $1/5$th to $1/10$th. The effective portion of the arc contribution to the final image formed by the two dimensional inverse Fourier transform will thus consist only of the small central segments of the arcs near the origin which may be considered as a straight line for practical purpose. The Fourier Diffraction theorem thus effectively reduces to Fourier slice theorem in this case. A detailed discussion on this topic is out of the scope of this work and may be found in [66].

3.4 Tomography with Echo Data

The echo data acquired from around the object lends itself to be processed further for the extraction of useful information otherwise not visible in individual B-scans.

3.4.1 Spatial Compounding

Following shortcomings of the B mode imaging prompt a further processing of the echo data.

- The speckle, as described in section 2.3.3, is an inherent artifact present in B mode images. It blurs the underlying details.
- The lateral resolution of the B-mode images is much worse than its axial one for the reasons discussed in section 2.2.
- The attenuation makes it difficult to image the axially deeper details.

The most intuitive approach is to add the individual B mode images, taken from around the object, together. Originally inspired due to the partially uncorrelated nature of the speckle which will get reduced as a result of summation, the other two limitations outlined above are also partially taken care of due to the full angular coverage of the object.
If the underlying morphological form of the object is described by a two dimensional object function \(a(x, y)\), and the point spread function of the B mode imaging system is given by \(h_{\text{bmode}}(x, y)\), the 360° compound image \(b_{\text{comp}}(x, y)\) will be given by

\[
b_{\text{comp}}(x, y) = \int_{0}^{2\pi} a(x, y) * h_{\text{bmode}}(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta) d\theta \tag{3.29}
\]

where ** denotes the two dimensional convolution. As the object function is invariant with respect to the angle of incidence \(\theta\), equation 3.29 may be rewritten as follows.

\[
b_{\text{comp}}(x, y) = a(x, y) * \int_{0}^{2\pi} h_{\text{bmode}}(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta) d\theta \tag{3.30}
\]

It is thus evident that the point spread function of the compound image \(h_{\text{comp}}(x, y)\) is given by

\[
h_{\text{comp}}(x, y) = \int_{0}^{2\pi} h_{\text{bmode}}(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta) d\theta \tag{3.31}
\]

Changing the coordinate system from rectangular to polar ones with the transformation \(x = r \cos \phi\) and \(y = r \sin \phi\) equation 3.31 becomes

\[
h_{\text{comp}}(r, \phi) = \int_{0}^{2\pi} h_{\text{bmode}}(r \cos (\phi - \theta), r \sin (\phi - \theta)) d\theta \tag{3.32}
\]

Now the variable of integration may be changed from \(\theta\) to \(\dot{\theta} = \theta - \phi\) so that the above equation becomes

\[
h_{\text{comp}}(r, \phi) = \int_{-\phi}^{2\pi - \phi} h_{\text{bmode}}(r \cos \dot{\theta}, -r \sin \dot{\theta}) d\dot{\theta} \tag{3.33}
\]

As both \(\cos \dot{\theta}\) and \(\sin \dot{\theta}\) are 2\(\pi\) periodic the limits of integration may be changed to 0 and 2\(\pi\) respectively. The result is shown below

\[
h_{\text{comp}}(r) = \int_{0}^{2\pi} h_{\text{bmode}}(r \cos \dot{\theta}, -r \sin \dot{\theta}) d\dot{\theta} \tag{3.34}
\]

The point spread function of compound imaging being a function of radial coordinate \(r\) only thus possess a rotational symmetry. This shows that the spatial resolution of compound imaging will be isotropic in contrast to the B mode imaging. The problem of attenuation in the deeper regions will also be alleviated with the rotational symmetry of the point spread function of equation 3.34.
3.4.2 Axial Correction in Compounding based on Echo Data

In addition to enhancing the morphological information of a single B-mode image, the echo data acquired over a full circular aperture offers another possibility to correct the refraction artifacts without reconstructing the speed of sound. The axial correction will be detailed later in chapter 5.

As the full angle echo data of an object is available to carry out the spatial compounding described above, it is worthwhile to try to extract some more information from the data. As the data acquisition system assumes the speed of sound in tissue to be constant, the echo data acquired from each pair of opposite directions will not register accurately in the compound image due to variations in the speed of sound. If there are enough tissue interfaces causing echo amplitudes greater than the noise floor in both the B-mode images taken from opposite directions, the time of flight may be estimated as illustrated in figure 3.7. The example of figure 3.7 shows a medium consisting entirely of water except the shaded region possessing a speed of sound higher than that of water. In the absence of the shaded region the time of flight measured between points \(z_0\) and \(z_3\) will be given by equation 3.35.

\[
t_{\text{water}} = 2 \int_{z_0}^{z_3} \frac{1}{c_0} dz = 2 \frac{z_3 - z_0}{c_0} \tag{3.35}
\]

The times of flight \(t_1\) and \(t_2\) shown in the figure are given by equations 3.36 and 3.37 respectively.

\[
t_1 = 2 \int_{z_0}^{z_1} \frac{1}{c} dz \tag{3.36}
\]

\[
t_2 = t_{\text{water}} - 2 \int_{z_1}^{z_3} \frac{1}{c} dz \tag{3.37}
\]

It is clear that the equations will hold for any medium. Subtracting equation 3.37 from 3.36 and rearranging yields the following equation.

\[
\int_{z_0}^{z_3} \left( \frac{1}{c} - \frac{1}{c_0} \right) dz = \frac{1}{2} (t_2 - t_1) \tag{3.38}
\]

This relationship is equivalent to equation 3.15 for the case of transmission data acquisition system. For the estimation of the time delay \((t_2 - t_1)\) cross correlation method outlined in section 4.3.2 is used. The estimated time delay is used to align A-line pairs taken from opposite direction. This method is subject to the assumptions discussed in the transmission case, in addition to the following additional ones.
Figure 3.7: Schematic Illustration of the time of flight estimation from echo data. The top and the bottom panels show a transducer at two positions 180° apart. The middle two panels show the corresponding A-lines plotted as amplitude variation against time.

3.4 Tomography with Echo Data
Assumptions 3.3 (Axial Time Delay Estimation from Echo Data).

1. The estimation of time delay is complicated due to the fact that not all interfaces are perpendicular to the direction of propagation and therefore the echo signal may not reach the transducer from both or either direction. This limitation is tackled by first choosing only those echoes which possess an amplitude greater than a certain noise threshold. The further selection is carried out by choosing only those echoes from both the directions which possess correlation coefficient greater than a threshold value.

2. As the ultrasound system assumes the speed of sound to be constant, it is bound to acquire A-lines whose spatial length is not equal to each other in each case. The integral on the right hand side will thus have slightly different integration limits for each A-line. This is an inherent problem with echo data acquisition.

3.5 Chapter summary

The computed tomography is the mathematical proposition of reconstructing a spatial object function from its projections, the latter being defined as line integrals of the object functions along a known straight line through the object function. This is one of the models of the tomographic reconstructive systems. The reconstruction itself is based on Fourier slice theorem which states that the one dimensional Fourier transform of a projection data is a slice through the two dimensional Fourier transform of the object function. It is, however, much more efficient to implement the filtered back projection algorithm also based on Fourier slice theorem. In the context of ultrasound, line integrals of the speed of sound and that of attenuation may be evaluated from the measured transmission data under several simplifying assumptions. The sampling requirements of the data is dictated by the sampling theorem. Some of the difficulties may be overcome in case of attenuation by reconstructing its frequency dependence rather than the attenuation itself. The other approaches like inverse scattering and diffraction CT were discussed only insofar as to demonstrate their inapplicability in the currently considered case. The chapter concludes with a short description of the 360° compounding appended with some concepts to correct the refraction artifacts based on information extracted from echo signals.
4 Time of Flight Estimation

An accurate time of flight estimation plays a central role in a speed of sound tomography system. The accuracy and precision of the time of flight estimation depends primarily on the data acquisition scheme itself. The data acquisition schemes implemented in this work employ a standard ultrasound system. The details of the realized systems will be discussed in chapter 6. Presuming a sufficiently high precision on part of the mechanical drive system, the accuracy of the reconstructed speed of sound tomogram is essentially dependent on that of the time of flight (TOF). The TOF data estimation is carried out on the basis of the first arrival signal reaching the receiver. Physical effects like multiple reflections, attenuation, refraction and speckle contribute negatively to the constitution of the received signal, thus complicating the task of TOF data estimation. Conventional techniques like threshold detection and cross correlation often fail to provide adequate accuracy of the estimation. Another important factor which has to be taken into account in the current measuring schemes is the use of a conventional ultrasound system driven in its normal pulse echo B-mode. Effect of beamforming, which being primarily optimized for the B-mode, can not be compensated. This introduces further difficulties in the TOF estimation [71]. In the following sections several methods for estimation of TOF including threshold method and cross correlation method will be discussed. Some of the problems posed by the effects outlined above may be overcome by incorporating a priori information about the signal into the estimation algorithm itself. The ability of the active contour models to allow prior information to be formulated in the form of a cost functional to be minimized was exploited to get around some of the the above mentioned problems.
4 Time of Flight Estimation

4.1 Required Accuracy of the TOF Estimation

4.1.1 Straight Line Propagation Model

When reconstructing the acoustic speed tomographically, the factors that contribute towards the final spatial and contrast resolution are the precision of the mechanical positioning system and the accuracy of time of flight data. If the positioning system is assumed to be sufficiently precise, as will be discussed in chapter 6, the only factor which may be influenced positively to enhance the reconstruction accuracy is the TOF data itself. To get a first estimate as to how the time of flight data influences the reconstruction accuracy, the effects of attenuation, refraction and multiple reflections are ignored and a rectangular inclusion of side length $d$ parallel to the direction of transmission with acoustic speed $c_1$ is considered, where the acoustic speed through water is $c_0$. The difference $\delta_{\text{TOF}_{\text{ideal}}}$ in the time of flight caused by the inclusion is given by:

$$\delta_{\text{TOF}_{\text{ideal}}} = d \left( \frac{1}{c_1} - \frac{1}{c_0} \right)$$ (4.1)

Considering ideal conditions, accuracy of the time of flight $\delta_{\text{TOF}_{\text{ideal}}}$ is determined by the sampling frequency $f_s$ of the digitized signal, the former being a function of the speed of sound $c_1$ of the inclusion, the background acoustic speed $c_0$ as well as the size

![Figure 4.1: Time of flight through a rectangular inclusion](image)
4.1 Required Accuracy of the TOF Estimation

Figure 4.2: Difference between time of flight through a rectangular inclusion and through water $\delta_{\text{TOF}_{\text{ideal}}}$, assuming straight path model, as a function of inclusion size $d$ and acoustic speed difference $c_1 - c_0$. The contours show the simulated time of flight difference in terms of sampling interval at integral values. The labels on the contours denote the time difference in number of samples.

of the inclusion $d$. The dependence of $\delta_{\text{TOF}_{\text{ideal}}}$ on the two parameters, $d$ and $c_1 - c_0$, is shown in figure 4.2 in the form of a contour plot for a background acoustic speed of $c_0 = 1490$ m/s and a sampling frequency of $f_S = 1/T_S = 36$ MHz. It is evident from figure 4.2 that a sub sample resolution is required to achieve useful contrast and spatial resolution. If the temporal resolution is limited to a full sample length, one can see from figure 4.2 that it is impossible to discern between two objects with a speed of sound difference of 10 m/s or lesser if they possess a size smaller than 1 cm.
4.1.2 Effects of Refraction and Receive Beamforming

The discussion of the preceding section was based on equation 4.1 which does not take into account the effects of refraction and assumes that the receive transducer samples the transmitted pulse without performing any beamforming. Some of the problems introduced by refraction and receive beamforming will be pointed out here, though a detailed discussion of the refraction effects will be postponed until chapter 5. As is evident from the law of refraction, equation 2.5, the extent of bending of ultrasound beam depends on the ratio of the speeds of sound of the two mediums as well as the angle of incidence \( \theta_i \). In order to demonstrate the effect of refraction in such a way that a considerable range of the angles of incidence is covered, the best choice for lesion shape is spherical. The data acquisition setup used here is, however a planar one, for which a lesion of cylindrical shape is sufficient to demonstrate the effect of refraction. The out of plane reflection and refraction can not be achieved with such a choice as far as the base of the right cylindrical lesion is parallel to the imaging plane. Let the diameter of the cylinder be \( d \) and the speed of sound through it be \( c_1 > c_0 \). Due to discrete values of speeds of sound and a well defined boundary between the two mediums, the path of a ray through the cylinder may be traced by employing the bending method based on Snell’s...
4.1 Required Accuracy of the TOF Estimation

law of equation 2.5. Considering the geometry of the Figure 4.3, the essential equations for ray path may be derived in the following manners. Origin of the coordinate system in the imaging plane is chosen to be at the center of the circular cross section of the cylinder without any loss of generality. Let the angle of first incidence on the circle be $\theta_i$ and the corresponding angle of refraction be $\theta_r$. Since $OP_2P_3$ is an isosceles triangle, the angle of second incidence will be $\theta_r$ and, due to reciprocity, the corresponding angle of refraction will be $\theta_i$. Application of the Snell’s law is however limited to the region for which the first angle of incidence is less than the critical angle.

$$\theta_{critical} = \arcsin\left(\frac{c_0}{c_1}\right)$$

(4.2)

Angles of deviation from the original direction of propagation along $P_1P_2$ in Figure 4.3 may easily be shown to be $(\theta_r - \theta_i)$ and $2(\theta_r - \theta_i)$ after the first and the second refraction at $P_2$ and $P_3$ respectively. Both the angles $\theta_i$ and $\theta_r$ and coordinates of points $P_2$, $P_3$ and $P_4$ in case of refraction for beams located in the right half, i.e. $x > 0$, are given by the following equations. Results for $x < 0$ may be obtained from symmetry of the problem about the y-axis.

$$\theta_i = \arcsin\left(\frac{2x_1}{d}\right)$$

(4.3)

$$\theta_r = \arcsin\left(\frac{2x_1 c_1}{d c_0}\right)$$

$$x_2 = x_1$$

(4.4)

$$y_2 = \frac{1}{2}\sqrt{d^2 - 4x_1^2}$$

$$x_3 = \frac{y_3 - c}{m}$$

(4.5)

$$y_3 = \frac{m\sqrt{d^2 (1 + m^2) - 4c^2} + c}{1 + m^2}$$

$$x_4 = (y_3 + d_2)\tan(2(\theta_r - \theta_i)) + x_3$$

$$y_4 = -d_2$$

(4.6)

where $m = -1/\tan(\theta_r - \theta_i)$ and $c$ are slope and intercept of the straight line through $P_2P_3$. The zone inside the cylinder through which no rays pass due to total reflection is limited for the practical case where difference of the the speed of sound inside the lesion and the background acoustic speed does not exceed 7% [44]. Figure 4.4 shows a typical case of ray paths through a lesion having 15 mm diameter and an acoustic speed of 1594 m/s. It is clear from figure 4.4 that the rays refracted through the lesion may get spread as much as two times the diameter of the lesion depending on the acoustic speed and the location of the lesion. Postponing the effect of lesion location to chapter 5, deviations from the straight ray model regarding the spreading of the rays and the variation of the time of flight difference from the one calculated on the basis of equation 4.1 will be discussed. A formula for the current case, similar to equation 4.1 for the ideal
Figure 4.4: Ray paths through a cylindrical lesion of 15 mm diameter having an acoustic speed of 1594 m/s (7% higher than that of the background). The refracted rays are spread to about double the diameter of the lesion.

\[ \delta_{\text{TOF}_{\text{refr}}} = \frac{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}}{c_1} + \frac{\sqrt{(x_3 - x_4)^2 + (y_3 - y_4)^2 + |y_2 - y_1|}}{c_0} \] (4.7)

where the x’s, y’s and other symbols are in accordance with figure 4.3. Figure 4.5 shows the deviation of the time of flight due to refraction from the ideal behavior as a function of the lateral coordinate. It is evident that the actually measured data is much different from the ideal data which is assumed to have been measured. Strategies to compensate the anomalies due to that will be a topic of discussion for chapter 5.

The degree of departure in the time of flight data due to refraction may be depicted as a contour plot shown in figure 4.6. This figure shows the contour plot of the standard deviation of the difference \( \delta_{\text{TOF}_{\text{ideal}}} - \delta_{\text{TOF}_{\text{refr}}} \) as a function of lesion size and lesion acoustic speed difference to the background acoustic speed. It is quite clear that the departure from the ideally assumed time of flight due to refraction remains insignifi-
4.1 Required Accuracy of the TOF Estimation

Figure 4.5: Deviation of the actual time of flight from straight ray assumption for a cylindrical lesion of 1 cm radius, $c_1 = 1594 m/s$. The upper half shows the $\delta_{\text{TOF,ideal}}$ and $\delta_{\text{TOF,refr}}$ according to equations 4.1 and 4.7 in dotted line and solid line respectively. The lower half shows the difference between the two over the lateral coordinate.

Significantly small only for smaller lesion sizes and lower acoustic speed contrasts. This may be identified by the relatively large area under the contour $T_S = 1$ in figure 4.6, where $T_S$ is the sampling interval. For the rest of cases involving larger lesion sizes or higher acoustic speed contrasts or both, the deviation from the ideal behavior is large enough to have detrimental effect on the subsequent reconstruction, which assumes a straight line propagation.

The other factor that plays an important role in defining the achievable accuracy of the time of flight data is the receive beam forming. This was also studied by means of a simulation. The data acquisition will be carried out with a commercial ultrasound system driven in its usual pulse echo mode, which performs dynamic receive beamforming. Though the receive beamforming plays a vital role towards increasing the lateral resolution of the B mode images, it has an undesirable effect on the accuracy of the
TOF data [34]. The anomaly in time of flight data introduced by this factor might be overcome if the receive beamforming could be turned off and single channel data be acquired without beamforming. As this is not possible in the considered case, only its effect will be outlined briefly. The time delays to be applied to individual channel data before their summation, in case of the receive beamforming, are calculated under the assumption of a uniform speed of sound of 1540 m/s. This adds to a source of inaccuracy in the measurement of the time of flight data. The deviation introduced by this process depends on the acoustic speed of the lesion as well as the size of the active sub aperture used for reception [51, 52]. The distance between the transmit and the receive transducer also plays a role, which will not be discussed here, as it can not be varied in the realized system. As with all the simulations performed in this section, attenuation, diffraction and scattering were not taken into account to facilitate the study of the effects of refraction and receive beam-forming only. Figure 4.7 shows a comparison of the
4.2 Characteristics of the Received Signal

Figure 4.7: Comparison of simulated times of flight with beamforming and refraction to the ideal one for a sub aperture size of 15\(\lambda\). Rest of the parameters are the same as in figure 4.5.

three cases for a lesion size of 15 mm and a lesion acoustic speed of 1594 m/s. Figure 4.8 shows the corresponding contour plot of the additional deviation in the time of flight introduced by the beamforming as a function of the aperture size and the acoustic speed difference of the lesion with the background.

4.2 Characteristics of the Received Signal

In order to devise an effective algorithm to estimate the time of flight accurately, the acoustic properties of the received signal are first considered.

4.2.1 Prior Information

Two properties of the signal derived from a priori knowledge about it may be formulated as follows.

The maximum variation in the acoustic speed of different tissue types is limited. For example, the maximum difference between the acoustic speed of carcinomas and that of healthy breast tissue is reported to be about 70 m/s [44, 74, 96]. Furthermore, the size and number of the lesions with an acoustic speed different from that of the background may also reasonably be assumed to be limited, it may be postulated that the approximate arrival time of the received signal is easily predictable. Moreover, it
4 Time of Flight Estimation

Figure 4.8: Contour plot of the simulated standard deviation of the difference between the time of flight with receive beamforming and that without it. The contours show quotient of the standard deviation of the difference in TOFs and the sampling interval as a function of the sub-aperture size in wavelengths and speed of sound difference of the lesion with the background. The labels on the contours denote the time difference in number of samples.

It sounds realistic to assume that the breast tissue does not possess any abrupt changes in the acoustic speed, because a breast carcinoma usually exhibits serrated boundaries [80]. It may, therefore, be assumed that the lateral course of the reflector echo in the B-mode image is sufficiently smooth. This may be summarized as follows.

Property 1 Approximate arrival time of the first arrival pulse is predictable.

Property 2 The lateral course of the first arrival pulse is sufficiently smooth.

4.2.2 Acoustic Properties

In addition to the two properties formulated above, some more characteristics of the first arrival pulse arising from a specific measuring scheme may be described below. In case of the reflex-transmission setup, the signal is characterized by the acoustic properties of the metallic reflector as well as those of the medium between the transducer and the metallic reflector. The huge difference between the acoustic impedances of the metallic reflector and that of water gives rise to a large reflection factor. This can be noticed from
4.3 Conventional TOF Estimation

4.3.1 Threshold Detection

Threshold method is the simplest of the TOF estimation methods. Based on the supposition that the signal level will always be markedly above the noise floor, the accuracy of the method depends on the threshold set to distinguish the signal from the noise as well as on the properties of the signal. Keeping property 4 in mind, it is sometimes more effective to detect the threshold of the gradient of the signal. The signal is rectified before subjecting it to the threshold detection. Figure 4.9 shows typical rectified signals and the detected time of flight profile overlayed on the data imaged on gray scale. The shown data was acquired for a cylindrical phantom of 22 mm diameter having an acoustic speed of 1545 m/s, the background speed of sound being 1495 m/s. It is evident from the figure that the threshold has to fulfil paradoxical requirements to be able to detect the first arrival pulse in every lateral position satisfactorily. This may, to some extent, be avoided by normalising each line and then choosing a single threshold for every line. Partly due to attenuating effect of refraction on the signal passing the lesion extremities and partly due to varying attenuation of the signal along its lateral course, the threshold detection method lacks smoothness and possesses an inadequate
### 4 Time of Flight Estimation

<table>
<thead>
<tr>
<th>Lateral Distance [cm]</th>
<th>Time [µs]</th>
</tr>
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<tbody>
<tr>
<td>-1.5</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>-0.5</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
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<tr>
<td>0.5</td>
<td>0.5</td>
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<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1.5</td>
<td>1.5</td>
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</tbody>
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<table>
<thead>
<tr>
<th>Normalized Amplitude</th>
</tr>
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<tbody>
<tr>
<td>Threshold</td>
</tr>
<tr>
<td>Signal 1</td>
</tr>
<tr>
<td>Signal 2</td>
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</tbody>
</table>

Figure 4.9: Threshold Detection of Time of Flight Data. The upper half shows the two signals taken from the lateral positions shown in the lower half.

...accuracy, especially with respect to its qualitative appearance. In the worst case it may end up in a strongly spiked shape thus rendering the method unusable.

#### 4.3.2 Cross Correlation Detection

The cross correlation approach is based on the assumption that the shape of the signal to be detected is known. In case of an entirely linear wave transmission, the signal sent from the transducer will retain its shape and thus it will be possible to detect the signal transmitted through a medium by cross correlating it with the sent signal. The only two changes that will take place in an idealized linear case will be the amplitude
4.4 TOF Estimation with Active Contour Models

attenuation and a phase shift. The phase shift can be estimated with the help of the cross correlation. If the signal sent by the transducer is $x_1(t)$, the received signal may be denoted by $x_2(t) = ax_1(t - \tau)$, where $a$ and $\tau$ signify attenuation and time shift respectively. It may easily be seen that the cross correlation function $\langle x_1, x_2 \rangle$ of $x_1(t)$ and $x_2(t)$ is readily reduced to auto correlation function $\langle x_1, x_1 \rangle$, as shown below.

$$\langle x_1, x_2 \rangle(t) = \int_{-\infty}^{+\infty} x_1(t') x_2(t' + t) dt'$$ (4.8)

$$= a \int_{-\infty}^{+\infty} x_1(t') x_1(t' - \tau + t) dt'$$ (4.9)

$$= a \cdot \langle x_1, x_1 \rangle(t - \tau)$$ (4.10)

It is straightforward to show, see for example [26], that the autocorrelation function $\langle x_1, x_1 \rangle(t)$ is an even function possessing its maximum at $t = 0$. The maximum of the cross correlation function $\langle x_1, x_2 \rangle$ will thus lie at $t = \tau$. This last statement forms the core of the computational algorithm. This method of signal detection is sometimes also called matched filtering. It is necessary to cross correlate the two signals and find the phase of the maximum amplitude. This procedure may also be replaced by a much faster one based on ‘phase root seeking’ [93]. In this work the reference signal $x_1(t)$ was obtained via a reference measurement through water at known temperature and was cross correlated with the actual measurement made on the object. Figure 4.10 shows the result of estimation on the data as in the above example. This approach is generally more robust than the preceding one. However, like threshold method, this one also exhibits spikes, especially in the proximity of a lesion interface, which can be traced back to the phase distortions caused by refraction and total reflection deforming the signal shape acutely and thereby rendering the approach unattractive.

4.4 TOF Estimation with Active Contour Models

Since their introduction by Kaas and Witkin [67] in 1987, the active contour models, also known as snakes, have been extensively utilized for various segmentation tasks in image processing. The conventional methods pointed out above are short of the capability to take into account the a priori information 1 and 2 [22, 23]. The active contour models, on the other hand, are able to incorporate this information into the estimation process, thus overcoming partly the limitations posed by the various effects such as multiple
reflections, refraction and attenuation. An active contour model consists of a curve with its parametric equation

\[ v(s) = (x(s), z(s)) \]  

(4.11)

where \( x \) and \( z \) are the coordinates in lateral and axial directions respectively, and

\[ 0 \leq s \leq 1. \]  

(4.12)

Starting from an initial guess for the contour based on \textit{a priori} information of property 1, it is successively deformed into the final curve under certain restraints on the process of the deformation. The deformation amounts to an iterative minimization of a cumulative cost functional embodying the prior information, as well as the measured data itself. While the part of the cost functional incorporating the prior information is customarily known as internal energy denoted here by \( E_{\text{int}} \), the other part of the cost functional dependent on the measured data itself is known as external energy, \( E_{\text{ext}} \). In current specific case, the minimization process is applied only in the axial direction, since the TOF profile is not a closed curve.

### 4.4.1 External Energy

In current context, the external energy is exclusively derived from the measured data. It is evident in view of the properties 3 and 4 that the absolute value of the amplitude
as well as that of the gradient of the sought-after pulse will be markedly higher than elsewhere in its close neighborhood. In view of that, the external energy is defined to be the weighted sum of the square of both these terms,

\[ E_{\text{ext}} = -\int_0^1 (k_{\text{amp}} |e(s)|^2 + k_{\text{grad}} |\nabla z e(s)|^2) \, ds = -\int_0^1 e_{\text{ext}} \, ds \quad (4.13) \]

where

\[ e(s) = e(x(s), z(s)) \quad (4.14) \]

denotes the measured signal as a function of lateral and axial coordinates respectively, and \( k_{\text{amp}} + k_{\text{grad}} = 1 \) with \( k_{\text{amp}} \ll k_{\text{grad}} \). As the external energy \( E_{\text{ext}} \) will be highest at the position of the desired signal in view of the properties 3 and 4, the negative sign of the integral, equation 4.13, will lead to a minimization instead of a maximization. This formulation of the external energy will be further refined in section 4.5.1.

### 4.4.2 Internal Energy

A purposeful formulation of the internal energy should incorporate a priori information about smoothness into the cost functional. The smoothness of a curve can be described in terms of, and controlled on the basis of its slope and curvature, the two terms being the first and the second derivatives of the parametric representation of the contour 4.11. One can combine the two terms in such a way that the required smoothness of the sought contour in the B-mode image may be controlled. The two energy terms are formally defined in equations 4.15 and 4.16. It may be noted that the first and the second derivative of the curve are squared in order to force them to possess positive values at all coordinates.

\[ E_{\text{slope}} = \frac{1}{2} \int_0^1 \alpha(s) \left| \frac{\partial v(s)}{\partial s} \right|^2 ds \quad (4.15) \]

\[ E_{\text{curvature}} = \frac{1}{2} \int_0^1 \beta(s) \left| \frac{\partial^2 v(s)}{\partial s^2} \right|^2 ds \quad (4.16) \]

The weights \( \alpha(s) \) and \( \beta(s) \) are to be selected according to the significance of the respective terms. The cumulative effect of the two terms constitutes the entire internal energy \( E_{\text{int}} \) given by the following equation.

\[ E_{\text{int}} = E_{\text{slope}} + E_{\text{curvature}} \quad (4.17) \]
A careful selection of the two parameters has an adequate smoothening effect, thus rendering a subsequent application of any further filtering operation, such as median filtering to eliminate any residual spikes from the TOF data, entirely superfluous. The smooth lateral course springs largely from the slope term, equation 4.15, of the internal energy, which is readily explicable on account of the one-dimensional nature of the minimization as well as the bounded nature of the expected variations in the acoustic speed.

### 4.4.3 Minimization of the cost functional

The cost functional to be minimized may be written down as follows,

$$ E_{\text{total}} = w_{\text{ext}} E_{\text{ext}} + E_{\text{int}} $$

where $w_{\text{ext}}$ is the weighting factor used to control the influence of the external energy on the minimization process. Now equations 4.13, 4.15 and 4.16 may be substituted into equation 4.18 to arrive at the cost functional to be minimized.

$$ E_{\text{total}} = \int_0^1 \left( w_{\text{ext}} e_{\text{ext}} + \frac{1}{2} \left( \alpha(s) |v_s(s)|^2 + \beta(s) |v_{ss}(s)|^2 \right) \right) ds $$

where $v_s$ and $v_{ss}$ are abbreviated notations for the first and second derivatives of $v$ with respect to $s$ and $w_{\text{ext}}$ is the weight of the total external energy. This cost function may be rewritten in the following form.

$$ E_{\text{total}} = \int_0^1 F(s, v_s, v_{ss}) \, ds $$

where,

$$ F(s, v_s, v_{ss}) = w_{\text{ext}} e_{\text{ext}} + \frac{1}{2} \left( \alpha(s) |v_s(s)|^2 + \beta(s) |v_{ss}(s)|^2 \right) $$

As is evident from the form of the equations 4.20 and 4.21, the estimation of the time of flight has been reduced to a classical problem of the calculus of variations. In addition to classical solution, there exists another approach to minimize this cost functional. These approaches will be outlined below.
4.4 TOF Estimation with Active Contour Models

Classical Solution

It is well established in the calculus of variations that every function \( F \) that minimizes the

\[
\int_0^1 F(x, y, y', y'', \ldots, y^{(n)}) \, dx \tag{4.22}
\]
must satisfy the differential equation [36]

\[
\sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k} F_{y^{(k)}} = 0 \tag{4.23}
\]

where \( y', y'', \ldots, y^{(n)} \) are, respectively, the first, second, \ldots, \( n \)th derivatives of \( y \) with
respect to \( x \) and \( F_{y^{(k)}} \) are partial derivatives of the \( F \) with respect to \( y, y', \ldots, y^{(n)} \) for
\( k = 0, 1, \ldots, n \). The differential equation 4.23 of \( 2n \)th order is known as Euler-Poisson
equation whose integration yields the desired minimization of the variation functional
of equation 4.22. This solution may also be applied to the minimization problem of
equations 4.20 and 4.21 to yield the following Euler-Poisson equation of fourth order
associated with it.

\[
F_v - \frac{\partial}{\partial s} F_{v_s} + \frac{\partial^2}{\partial s^2} F_{v_{ss}} = 0 \tag{4.24}
\]

Generally \( v \) is a vector \((x, z)\), see equation 4.11 also. In the current case the minimization
process has to take place only along the axial direction \( z \). The partial differential
equation 4.24 may be reduced to the following equation of fourth order, after substitut-
ing the appropriate derivatives of \( F \) from equation 4.21 and assuming \( \alpha \) and \( \beta \) to be
constant with respect to the parameter \( s \).

\[
w_{\text{ext}} \frac{\partial e_{\text{ext}}}{\partial z} - \alpha \frac{\partial^2}{\partial s^2} + \beta \frac{\partial^4}{\partial s^4} = 0 \tag{4.25}
\]

This differential equation may be solved numerically after being converted to the follow-
ing difference equation [67].

\[
w_{\text{ext}} g_z (x, z_i) + \alpha_i (z_i - z_{i-1}) - \alpha_{i+1} (z_{i+1} - z_i) +
\beta_{i+1} (z_{i+2} - 2z_{i+1} + z_i) - 2\beta_i (z_{i-1} - 2z_i + z_{i+1}) + \beta_{i-1} (z_i - 2z_{i+1} + z_{i+2}) = 0 \tag{4.26}
\]

where

\[
g_z (x, z_i) = e_{\text{ext}} (x, z_i) - e_{\text{ext}} (x, z_{i-1}) \tag{4.27}
\]
is a representation of the first difference of $e_{\text{ext}}(x, z)$ with respect to $z$ and corresponds to the $z-$gradient of the external energy $\frac{\partial e_{\text{ext}}}{\partial z}$ in equation 4.25. Collecting coefficients, equation 4.26 becomes:

$$a_i z_{i-2} + b_i z_{i-1} + c_i z_i + d_i z_{i+1} + e_i z_{i+2} + w_{\text{ext}} g_z(x, z_i) = 0$$

(4.28)

where

$$a_i = \beta_{i-1}$$

(4.29)

$$b_i = -2\beta_i - 2\beta_{i-1} - \alpha_i$$

(4.30)

$$c_i = \alpha_i + \alpha_{i+1} + \beta_{i-1} + 4\beta_i + \beta_{i+1}$$

(4.31)

$$d_i = -\alpha_{i+1} - 2\beta_i - 2\beta_{i+1}$$

(4.32)

$$e_i = \beta_{i+2}$$

(4.33)

After laying down the number of points the lateral course of the time of flight projection should possess to be equal to $N$, an equation for each chosen lateral coordinate $x$ can be written down in the manner of equation 4.28. This system of $N$ simultaneous linear equations may be written down in the following matrix form:

$$Az + w_{\text{ext}} g_z(x, z) = 0$$

(4.34)

where $z$ is a $N \times 1$ column vector comprising the axial coordinates of the time of flight profile at each chosen lateral coordinate, $g_z(x, z)$ is a matrix containing the discrete values of the $z-$gradient of the external energy and $A$ is a pentadiagonal $N \times N$ matrix comprising the coefficients $a_i, \ldots, e_i$. The vectors $[a_2, a_3, \ldots, a_N]$, $[b_1, b_2, \ldots, b_N]$, $[c_0, c_1, \ldots, c_N]$, $[d_1, d_2, \ldots, d_N]$, and $[e_2, e_3, \ldots, e_N]$ form the diagonals of the matrix $A$ starting from two diagonals below the main diagonal. The solution of equation 4.34 for $z$ will constitute the required time of flight profile which will not only be based on the acoustic properties 3 and 4 but will also incorporate the smoothness prior described as property 2 above. The equation is, however, not directly solvable if taking into account the practical fact, to be discussed below in more detail, that the gradient of the external energy has to be weighted with a very small factor $w_{\text{ext}}$ thus rendering the solution of the equation rather instable. To avoid this, another way is to find out a regularized solution to this equation which consists of introducing a regularization parameter $\gamma > 0$ into equation 4.34. The product of $\gamma$ and the finite difference $z_t - z_{t-1}$ of the two consecutive iterations is added to the left hand side of equation 4.34 giving the following equation.

$$Az_t + w_{\text{ext}} g_z(x, z_{t-1}) + \gamma (z_t - z_{t-1}) = 0$$

(4.35)
4.4 TOF Estimation with Active Contour Models

The subscripts \( t \) and \( t - 1 \) signify the iteration steps. As the step size \( \gamma \) of iterative integration will be chosen to be very small, as mentioned above, the gradient \( g_z \) is chosen from the preceding iteration step to facilitate a fast iteration. This will give rise to the following update equation for each iteration.

\[
z_t = (A + \gamma I)^{-1} (z_{t-1} - w_{ext} g_z (x, z_{t-1})) \quad (4.36)
\]

The penta-diagonal banded matrix \( A + \gamma I \) has to be inverted only once thus saving substantial computational cost. Despite its numerical efficacy, the method suffers from drawbacks regarding its convergence and the congruence of the solution arrived at with the one sought for. It is easy to comprehend that the selection of the weighting parameter \( w_{ext} \) plays a vital role in view of the points raised above. The influence of the measured data itself on the solution has to be controlled very precisely to arrive at a desired solution. This demands a new choice of the weighting factor for every data set. A too low value of the weighting factor leads to a very weak influence of the data and the smoothness constraint will tend to make the solution a straight line. On the other hand a slightly too high value of \( w_{ext} \) leads to undesirably high influence of the noisy data undermining the smoothness constraints and even forcing the solution to oscillate about the desired coordinates. A very high value of the weighting factor may even lead to a severe dysfunction of the algorithm.

**Dynamic Programming**

Another approach was suggested in [1] which applies dynamic programming to the solution of the variational problem of equation 4.19. The dynamic programming implementation of the minimization necessitates the discrete form of the cost functional 4.19 including the integration shown in equation 4.37.

\[
E_{total} (v_1, v_2, \ldots, v_n) = \sum_{i=2}^{n-1} e_{ext} (v_i) + \frac{1}{2} (\alpha_i |v_i - v_{i-1}|^2 + \beta_i |v_{i+1} - 2v_i + v_{i-1}|^2) \quad (4.37)
\]

The above equation may also be reformulated as.

\[
E_{total} (v_1, v_2, \ldots, v_n) = \sum_{i=2}^{n-1} E_i (v_{i-1}, v_i, v_{i+1}) \quad (4.38)
\]

where

\[
E_i (v_{i-1}, v_i, v_{i+1}) = e_{ext} (v_i) + \frac{1}{2} (\alpha_i |v_i - v_{i-1}|^2 + \beta_i |v_{i+1} - 2v_i + v_{i-1}|^2) \quad (4.39)
\]
The minimization of this equation consists of finding the lateral path of minimum energy in such a way that the smoothness condition imposed by slope and curvature terms are fulfilled. Starting from an initial arbitrary guess for the time of flight profile consisting of a set of points across the whole lateral range, an optimum path of total minimum energy has to be decided upon. Instead of carrying out the minimization through exhaustive enumeration taking into account every possible combination, which suffers from severe computational inefficiency, the technique of dynamic programming is used. Drawing on the conclusions from the preceding section and recalling that the slope term in the cost functional accounts for the most of energy so that the curvature term may be dropped. This reduces the dependence of the cost functional of equation 4.37 from three axial coordinates to only two adjacent ones. This will reduce equations 4.38 and 4.39 to equations 4.40 and 4.41 respectively.

\[
E_{total}(v_1, v_2, \ldots, v_n) = \sum_{i=1}^{n-1} E_i(v_{i-1}, v_i) \quad (4.40)
\]

\[
E_i(v_{i-1}, v_i) = e_{ext}(v_i) + \alpha_i |v_i - v_{i-1}|^2 \quad (4.41)
\]

Dynamic programming procedure is based on recursive calculation. In the discrete decision making process each lateral coordinate signifies a decision making stage. As is evident from equation 4.41 the energy at every coordinate is a weighted sum of the external energy associated to that coordinate and the square of the slope of the line joining that point to a point on the adjacent lateral position. Starting with the first axial coordinate of the first lateral position \((x_1, z_1)\), the slope term \(\alpha_i (z_1 - z_i)^2\) is calculated for every possible axial coordinate \(z_i\) of the next lateral position lying within the axial target region laid down on the basis of the \(a\ priori\) information. The minimum energy comprising the weighted sum is assigned to the the point \((x_1, z_1)\). This procedure is repeated for all axial coordinates \((x_1, z_2), (x_1, z_3), \ldots, (x_1, z_n)\). In the second and every succeeding stage, this procedure is repeated with one addition. To the weighted sum of equation 4.41 the corresponding minimum energy of the adjacent point from the previous stage is added. Exactly this is the recursion mentioned above. The recursion formula may be written down as.

\[
f_k(v_{k+1}) = \min_{v_k} e_{ext}(v_k) + \alpha_i |v_{k+1} - v_k|^2 + f_{k-1}(v_k) \quad (4.42)
\]

where \(f_k(v_{k+1})\) is the minimum energy of an axial coordinate of the \(k\)th stage to be used later recursively in the \((k+1)\)th stage. Now it is evident that the minimum energy of
every step as well as the axial coordinate minimizing the energy has to be saved for every
axial coordinate at each stage. In the above discussion of the minimization process, only
the slope term was considered and the curvature term was discarded. If the curvature
term were also included the state variable will no longer be a single point but a pair of
points. The computational complexity will thus increase to $o(N \times n^3)$ instead of the
present $o(N \times n^2)$ and the storage requirements will also increase to $o(N \times n^2)$ instead
of the present $o(N \times n)$, where $N$ is the number of stages and $n$ is the discrete length
of the axial target region.

The advantage of this approach over the first one is that it always converges and needs no
start contour to be specified. Only the region of interest has to be specified to minimize
storage requirements and computational cost. Another advantage is that no matrix has
to be inverted. The advantages of the classical minimization scheme are, however, its
ability to reach the desired contour ‘from above’ so as to force it to lock always on to the
first arrival pulse on one hand and to achieve a sub-sample resolution on the other. The
dynamic programming procedure lacks these two characteristics due to its global search
within the region of interest and its discrete nature respectively. In the context of the
present work, this approach is occasionally applied to arrive at the first approximation
of the desired time of flight. This first approximation is then taken as the first iteration
of the time of flight for the other minimization technique. In the following discussion,
influence of limiting factors on the classical solution will be discussed.

4.5 Discussion

The correct detection of the first arrival pulse depends on several factors. In addition
to the algorithm parameters $\alpha$, $\beta$ and $w_{ext}$, the choice of the image energy also plays
a deciding role in detection as well as convergence of the algorithm. The acquired
radio frequency transmission data in its raw shape does not constitute a suitable image
energy function for several reasons. Largely due to refraction and attenuation of the
signal, the definition of the first arrival pulse becomes difficult. Along the lateral course
of the signal, it is attenuated differently. These effects are evident in figure 4.11 showing
measured data in gray scale.
Figure 4.11: A typical transmission data set. The radio frequency data acquired with a transducer of center frequency 7.5 MHz is shown in gray scale. The middle portion of the pulse was transmitted through a naked mouse. The effects of refraction and multiple reflection are evident at the arrow locations. The middle portion of the signal, which encodes most of the information of the acoustic characteristics of the imaged object, is attenuated differently at different locations.

4.5.1 Choice of Image Energy

As discussed in section 4.4.3, the external energy term, also known as image energy term, is defined by the actual data on which detection has to take place. For the reasons described above, the acquired radio frequency data alone in its raw form is not a suitable image energy function, as it is difficult to define the first arrival pulse from this data. To refine this data, it is normalized on a line wise basis for each lateral coordinate. The resulting normalized data for the example of figure 4.11 is shown in figure 4.12. Although the first arrival pulse may be more clearly noticeable from normalized data, the effects of refraction have become more prominent than in the original data. To find a compromise between the two, a weighted average of the two is taken as a more suitable measure for the image energy. The weights $w_{\text{norm}}$ and $w_{\text{raw}}$ of the normalized energy term $E_{\text{norm}}$ and the original energy term $E_{\text{raw}}$, respectively, have to be established empirically for a given measurement once. The external energy is thus given by:

$$E_{\text{ext}} = \frac{1}{w_{\text{norm}} + w_{\text{raw}}} (w_{\text{norm}} E_{\text{norm}} + w_{\text{raw}} E_{\text{raw}}) \quad (4.43)$$
In the context of the transmission data measured in a bi-static setup, the gradient of the signals does not possess additional information as is the case with a mono-static setup with a passive metallic reflector.

### 4.5.2 Choice of Algorithm Parameters

The internal energy chosen for the current case according to equation 4.17 comprises a slope and a curvature term. These terms are weighted with parameters $\alpha$ and $\beta$ respectively. The choice of these two terms plays a vital role in the definition of the detected first arrival pulse. Keeping in mind that the slope term forces the curve to be a straight line and the curvature term allows it to bend across corners, the two energy terms may be weighted in accordance with the expected shape of the curve. For the example discussed above, only the middle portion possesses curvature, while the right and the left parts are expected to be just straight lines. The parameters $\alpha$ and $\beta$ may, therefore, be allowed to vary across the lateral course of the pulse. The three portions can be identified on the basis of average signal level at a given lateral coordinate. Due to the gradual transition from the linear to the curvilinear portion evident from the example shown in figure 4.11 the segmentation need not be highly accurate. The required segmentation may be implemented by deciding whether the level of the signal raises above a certain threshold $e_{th}$ or not.

\[
Level = \text{sgn} \left( \text{med}_x \left( E_z |e_{\text{raw}}(x,z)| \right) - e_{th} \right)
\]  

(4.44)
$E_z$, $e_{raw}(x, z)$, $med_x$ and $sgn$ denote the expected value operator along the axial direction $z$, the acquired raw transmission data, the median filtering along the lateral coordinate $x$, and the sign function respectively. Having decided the three portions across lateral course of the pulse, it is clear that the middle portion being curved should be governed by a relatively larger value of $\beta$ than the linear portions on both sides. In fact the linear portions can be detected quite easily by assigning them $\alpha = 1$ and $\beta = 0$. To facilitate a smooth curvilinear course, $\alpha$ for the middle portion has to be chosen considerably smaller than its $\beta$ counterpart.

In addition to defining the characteristics of the curve on the basis of parameters $\alpha$ and $\beta$, another important way to make use of a priori information is to weight the image energy in equation 4.34 with a weighting vector $w_{ext}$ instead of a weighting scalar. The segmentation carried out on the basis of equation 4.44 can again be used to allocate different values of weighting factor to different portions of the curve to be detected. In this case the middle portion may be assigned higher values of the factor compared to the linear portions.

### 4.5.3 A Note on Convergence

The discussion of section 4.1.2 points out to the fact that, even in the ideal case, it is difficult to estimate the correct time of flight data on the basis of the setup used here, largely due to refraction and to a lesser extent due to receive beam-forming. This will be further investigated in chapter 5. For the current discussion, it suffices to point out that refraction causes phase distortions in the transmitted pulse, that makes definition of the first arrival pulse difficult. This can be seen in figure 4.12 clearly. Now, in the absence of an unambiguous definition of the first arrival, it is hard to prescribe a unique criterion to which the detected lateral course of time of flight should conform. The only meaningful alternative is to choose some data sets at random and establish a set of algorithm parameters and other weighting factors discussed above in such a way that the shape of the detected time of flight projection is reasonably acceptable. Using this heuristic measure, the rest of the data is analyzed for the time of flight. Due to the iterative nature of the algorithm, it is necessary to define a stop criterion to avoid unnecessary computational cost. This stop criterion should make sure that the shape of the time of flight projection has ‘converged’ to the final shape in accordance with the algorithm parameters and has ceased to change substantially from one iteration to the
next. The stop criterion chosen for the current work is outlined below.

1. First, the algorithm is allowed to iterate for a minimum number of times $n_{\text{min}}$ to avoid the additional computational cost of applying the stop criterion in the beginning.

2. The shape of the time of flight after each iteration is saved, so that the Euclidean distance $\delta t_{x,i}$ between two consecutive TOF estimates $t_{x,i}$ and $t_{x,i-1}$ may be calculated as

$$\delta t_{x,i} = |t_{x,i} - t_{x,i-1}|$$

(4.45)

where $i$ and $i-1$ denote iteration numbers and $x$ is the lateral coordinate. Sum of these absolute difference $\text{SAD}_x (i)$ over the lateral coordinate $x$ for each projection forms one of the two measures of convergence.

$$\text{SAD}_x (i) = \sum_x \delta t_{x,i}$$

(4.46)

3. The iteration is stopped and the algorithm is accepted to have converged when either of the following criteria is fulfilled,

$$\text{SAD}_x (i) < \begin{cases} \epsilon_1 \\ \epsilon_2 \text{Median} (\text{SAD}_x (i - n_{\text{min}} + 1), \ldots , \text{SAD}_x (i)) \end{cases}$$

(4.47)

where $\epsilon_1$ and $\epsilon_2$ are small numbers empirically established with the help of the heuristic method described above.

An example of the time of flight estimation with this method can be seen in figure 4.13. The time of flight data estimated with conventional methods was median filtered with a window size of five. The spikes are still present everywhere. The first arrival pulse could be successfully estimated only with active contours.

### 4.6 Chapter Summary

Estimation of time of flight is of primary importance in an ultrasound transmission tomography system. Conventionally the time of flight is estimated with the help of a
threshold detection method which differentiates an incoming signal from noise whenever the signal level raises over a specified threshold. This method suffers from lack of smoothness and inadequate accuracy. The more robust of the conventional approaches is cross correlation. Though this method yields more accurate estimates of the time of flight, it also exhibits spikes in the lateral course. Whereas these problems are usually minimized by low pass filtering of the extracted time of flight profiles, the achieved accuracy has quite drastic effects on subsequent reconstruction [60]. A new approach was developed and tested to cope with these problems. This approach is a one dimensional variant of the deformable active contours that allows incorporation of the *a priori* information such as smoothness into the detection algorithm itself. This is reduced to an optimization problem in the sense of the calculus of variations. The minimization can be carried out with Euler-Poisson equation associated with the problem as well as with the help of dynamic programming. The former approach yields smooth TOF projections and is capable of detecting the true first arrival pulse, though it suffers from its strong dependence on the choice of weighting factors which influence the convergence of the iterative solution. The dynamic programming solution converges always, though it cannot distinguish between first arrival pulse and a multiple reflection. The *a priori* information is incorporated into the detection algorithm by selecting the algorithm parameters as well as the external energy function carefully.
5 Refraction Artifacts

An obvious choice for a reconstruction technique is based on the assumption that sound waves travel along straight lines. Some marginally more accurate models for reconstruction also exist, which are based either on Born or Rytov Approximation [30, 32, 81, 99] or on a direct inversion of the wave equation [28]. The limitations of such approaches were pointed out in section 3.3.1. The three mechanisms responsible for deviation of sound waves from straight line propagation behavior are refraction, diffraction and scattering. The latter two mechanisms contribute only marginally to the deviation from the straight path, the former one can be shown to be chiefly responsible for it [83, 90]. It was shown in section 4.1.2 for a special case of a cylindrical lesion possessing an acoustic speed higher than that of the background, that refraction manifests itself in the form of a substantial spreading of the time of flight profile compared to the one supposed to have been measured under the assumption of straight line propagation on one hand and in a considerable deviation of the time of flight from the ideal behavior on the other. The difficulty in accounting for these adverse effects of refraction lies in the fact that they depend on the underlying speed of sound distribution itself which is to be reconstructed. One of the approaches taken to tackle this situation is to estimate the speed of sound based on the straight line propagation assumption and to compensate for the effect of refraction iteratively [14, 29]. In the following sections an approach will be developed to carry out the iterative correction for the general case of a fairly arbitrary speed of sound distribution. As pointed out earlier, use of a commercial ultrasound scanner was opted also due to the feasibility of the acquisition of echo data simultaneous to the transmission data, the former to be used to perform reflection mode tomography. The refraction will be shown to play an unfavorable effect in that case, too [21] and two schemes for its correction will be investigated.
5 Refraction Artifacts

5.1 Effects of Refraction on Reconstruction

The theoretical study of the refraction artifacts will be carried out within the framework of geometrical acoustics assuming a piecewise linear propagation that obeys Snell’s Law at medium boundaries. Such ray-like behavior may be justified for frequencies of 7 MHz and above when the smallest lesion size is not smaller than 2 mm. Even in this limiting case the wavelength will be about 1/10th of the lesion size. The results of the simulations thus carried out were found to be comparable to the ones obtained with the field theoretical model detailed in appendix A.1.

A ray tracing algorithm will be derived below that is capable of tracing a ray through multiple cylindrical objects lying at arbitrary positions and possessing arbitrary constant speeds of sound. This algorithm will be used to calculate the time of flight taking into consideration only the effect of refraction. The reconstruction will, however, be carried out under the assumption of the straight line transmission thus simulating the effect of refraction on the straight ray reconstruction approach. Figure 5.1 illustrates the possible scenarios of an acoustic ray getting either refracted or totally reflected while impinging on the surface of the cylinder from inside or from outside. To develop an algorithm for ray tracing, the objects used here are first defined formally. As only a 2D case will be considered, the circles characterizing the right circular cylinders need be defined.

Circle $C_k^{(a_k,b_k,r_k)}(x,y)$: The $k$th of the $n$ different circles having a radius $r_k$ and centered at $(a_k,b_k)$ will be given by.

$$C_k^{(a_k,b_k,r_k)}(x,y) = (x - a_k)^2 + (y - b_k)^2 - r_k^2 = 0 \quad (5.1)$$

The Speed of sound $c_k$ is constant for all points $(x, y)$ such that $C_k^{(a_k,b_k,r_k)}(x,y) \leq 0$.

Ray $R_{j}^{(x_j,y_j),(\alpha_j,\beta_j)}(x,y)$: The parametric equation of a ray starting at a point $P(x_j,y_j)$ and propagating in a direction given by the direction cosines $(\alpha_j, \beta_j)$ can be written as follows,

$$R_{j}^{(x_j,y_j),(\alpha_j,\beta_j)}(x,y) = \{(x, y) : x = x_j + t\alpha_j, \ y = y_j + t\beta_j, \ t \in \mathbb{R}, \ t \geq 0\} \quad (5.2)$$

where $\alpha_j$ and $\beta_j$ are cosines of the angles the ray $R_{j}(x,y)$ makes with positive $x$- and $y$- axis, respectively.

Points of intersection of $C_k$ and $R_j$: The points of intersection of a circle $C_k^{(a_k,b_k,r_k)}(x,y)$ and a ray $R_{j}^{(x_j,y_j),(\alpha_j,\beta_j)}(x,y)$ are given by.

$$t = \frac{(x_j - a_k) \alpha_j + (y_j - b_k) \beta_j \pm \sqrt{r_k^2 (\alpha_j^2 + \beta_j^2) - ((y_j - b_k) \alpha_j + (x_j - a_k) \beta_j)^2}}{\alpha_j^2 + \beta_j^2} \quad (5.3)$$
5.1 Effects of Refraction on Reconstruction

A ray $R_j$ will be entering or leaving a circle $C_k$ when the point $(x_j, y_j)$ lies within or outside it:

$$C_k^{(a_k, b_k, r_k)}(x_j, y_j) \begin{cases} > 0, & \text{if } R_j \text{ is entering } C_k; \\ < 0, & \text{if } R_j \text{ is leaving } C_k. \end{cases} \quad (5.4)$$

And finally the direction cosines $(\alpha_{n_j}, \beta_{n_j})$ of the unit normal vector $\hat{n}_j$ at the point of incidence $P_j$ lying at $C_k$ will be given by:

$$\alpha_{n_j} = \frac{x_j - a_k}{r_k} \cdot \text{sgn} \left( C_k^{(a_k, b_k, r_k)}(x, y) \right) \quad (5.5)$$

$$\beta_{n_j} = \frac{y_j - b_k}{r_k} \cdot \text{sgn} \left( C_k^{(a_k, b_k, r_k)}(x, y) \right) \quad (5.6)$$

Now the algorithm may be described as follows.

1. Begin with $j = 0$ from the transmitter. The point $P_0(x_0, y_0)$ is where the beam
5 Refraction Artifacts

originates and its direction cosines \((\alpha_0, \beta_0)\) are known, so that the ray \(R_0^{(x_0, y_0), (\alpha_0, \beta_0)}(x, y)\) is fully known. Increment \(j\) by 1.

2. Find the points of intersection \(P_j(x_j, y_j)\) of ray \(R_{j-1}\) with all possible circles \(C_k(x, y)\) using equation 5.3 and with the straight line representing the receiver. Minimize \(t\) to find the first object the ray comes across. If this object is the receiver, the complete path has been traced, otherwise proceed with the following steps.

3. Calculate the angle of incidence \(\theta_{ij}\):

\[
\cos \theta_{ij} = -\alpha_n \alpha_{j-1} - \beta_n \beta_{j-1} \quad (5.7)
\]

4. Denoting the medium containing \(P_{j-1}\) as medium 1 and the medium that the ray would enter if refracted as medium 2 and the corresponding acoustic speeds as \(c_{\text{medium}_1}\) and \(c_{\text{medium}_2}\) respectively, the ray will get totally reflected at \(P_j\) for \(c_{\text{medium}_2} > c_{\text{medium}_1}\) if the angle of incidence of equation 5.7 exceeds the critical angle. Otherwise the ray will be refracted.

5. If the ray is totally reflected, the direction cosines \((\alpha_j, \beta_j)\) of the reflected ray \(R_j\) will be determined by the solution of the linear equations 5.8 and 5.9. The ray \(R_j\) together with point \(P_j(x_j, y_j)\) found in step 2 is thus completely known. Increment \(j\) by 1 and go to step 2.

\[
\cos 2\theta_{ij} = -\alpha_{j-1} \alpha_j - \beta_{j-1} \beta_j \quad (5.8)
\]
\[
\cos \theta_{ij} = \alpha_n \alpha_j + \beta_n \beta_j \quad (5.9)
\]

6. If the ray is refracted, the direction cosines \((\alpha_j, \beta_j)\) of the refracted ray \(R_j(x, y)\) will be determined by the solution of the linear equations 5.10 and 5.11. The ray \(R_j(x, y)\) together with point \(P_j(x_j, y_j)\) found in step 2 is thus completely known. Increment \(j\) by 1 and go to step 2.

\[
\cos (\theta_{ij} - \theta_r) = \alpha_{j-1} \alpha_j + \beta_{j-1} \beta_j \quad (5.10)
\]
\[
\cos \theta_r = -\alpha_n \alpha_j - \beta_n \beta_j \quad (5.11)
\]
5.1 Effects of Refraction on Reconstruction

The above algorithm delivers the coordinates of all the \( m \) points \( P_j \) where the ray changes its direction due to refraction or total reflection. The algorithm also stores the acoustic speed of the medium between every pair of points \( P_{j-1} \) and \( P_j \) as \( c_j \). The time of flight is thus given by:

\[
TOF = \sum_{j=1}^{m} \frac{1}{c_j} \sqrt{(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2}
\]

(5.12)

5.1.1 Refraction Artifacts in Transmission Tomography

The reconstruction assumes a straight line propagation. The sampling requirements are met to facilitate interpretation. In all cases the distance between the transmitter and the receiver is 8 cm, both the transmitter and the receiver arrays are also 8 cm wide, diameter of the cylindrical lesion is 1 cm. The speed of sound inside the lesion is 1550 m/s and outside it is 1490 m/s. A schematic illustration of the simulation setup is shown in figure 5.2.

Figure 5.2: Illustration of the simulation setup used for the results shown in this section.

Lesion of the first example is given by \( C_1^{(0,0.5 \text{ mm})1550\text{m/s}} \). Figure 5.3 shows reconstruction.
Figure 5.3: Reconstructed speed of sound for a cylindrical lesion of 1 cm diameter lying at the center of rotation. The plots on the left and bottom show slices through reconstruction.

The reconstructed lesion size is much larger than the actual one. The extent of widening of the lesion is dependent on several factors like its size, shape, the ratio of its acoustic speed to that of the background and its orientation. The widening will increase for higher acoustic speed contrasts and for larger reconstruction circles. It may be seen from the vertical as well as the horizontal slice that the reconstructed speed falls almost linearly for the region beyond the actual lesion size. Two more cases are shown in figures 5.4 and 5.5 where the cylindrical lesion is located at a distance of 15 mm and 30 mm, respectively, above the center of rotation. There are two crescent shaped artifacts beside the image of the lesion to the right and left of it. The thickness of the crescent artifacts varies directly with the distance of the lesion from the center of rotation. Time of flight projections at different angles of view are considered to explain these artifacts. The widening of the lesion size in a given projection depends on the distance covered by the rays after getting refracted from the lesion. As the distance covered by the refracted rays for the two extreme positions of the receiver ($\theta_{rotation} = 0^\circ$ and $\theta_{rotation} = 180^\circ$) is
5.1 Effects of Refraction on Reconstruction

![Graph showing reconstructed speed of sound for a cylindrical lesion of 1 cm diameter lying at a distance of 15 mm from the center of rotation.]

Figure 5.4: Reconstructed speed of sound for a cylindrical lesion of 1 cm diameter lying at a distance of 15 mm from the center of rotation.

different, the widening of the lesion size in the projection data will also be considerably different from each other. In the ideal case assumed by the reconstruction process the two time of flight projections taken from the opposite directions shown in figure 5.6 for lesion of figure 5.5 should be mirror image of each other. The upper half shows the time of flight projections from \( \theta_{\text{rotation}} = 0^\circ \) and \( \theta_{\text{rotation}} = 180^\circ \) while the lower half shows the corresponding projections from \( \theta_{\text{rotation}} = 90^\circ \) and \( \theta_{\text{rotation}} = 270^\circ \). While the projection pair taken from \( \theta_{\text{rotation}} = 90^\circ \) and \( \theta_{\text{rotation}} = 270^\circ \) shows a mirror symmetry, the projections from \( \theta_{\text{rotation}} = 0^\circ \) and \( \theta_{\text{rotation}} = 180^\circ \) possess unequal lesion width. The width varies for varying angles and may be shown to have a sinusoidal character.

5.1.2 Resolution and Definition

Reconstruction with straight line propagation assumption enlarges areas with higher speeds of sound and shrinks the ones with lower speeds of sound. The actual lesion size may only be guessed from the knowledge of the reconstructed speed of sound in the light
5 Refraction Artifacts

Figure 5.5: Reconstructed speed of sound for a cylindrical lesion of 1 cm diameter lying at a distance of 30 mm from the center of rotation.

of these conclusions as long as adjacent areas do not merge in the reconstruction [16].

To study the merging effect, two cylindrical lesions possessing the same speeds of sound are considered. Acoustic speed of the background is again 1490 m/s and reconstruction circle has a diameter of 8 cm. Figure 5.7 shows the zoomed in cutouts of reconstructions for lesion with different speeds of sound and radii. The axis labels show the position of the lesion. For all cases in figure 5.7, the separation between the boundaries of the inclusion is one millimeter. It is evident that the definition of the reconstruction deteriorates with increasing acoustic speeds and decreasing radii of the lesions. For lesions of smaller sizes lying near each other, the refraction artifacts almost burry the whole area around the two lesions thus imparting the reconstruction an appearance of a single lesion. The definition for larger speeds gets improved with increasing lesion sizes.

The asymmetrical location of the lesions with respect to the center of rotation results in reconstructions shown in figure 5.8. Though the results may look quite similar to the previous case, specially the general appearance of the reconstruction for lower speeds or for larger lesion sizes, yet a close inspection will reveal several differences. The crescent
5.1 Effects of Refraction on Reconstruction

Figure 5.6: Effect of transducer distance from the lesion on time of flight data. The upper half shows the two time of flight projections corresponding to the two transducer positions $\theta_{\text{rotation}} = 0^\circ, 180^\circ$ while the lower half shows projections for the transducer positions $\theta_{\text{rotation}} = 90^\circ, 270^\circ$.

Artifacts noted in the previous section are no more visible, instead of that the lesions appear to be slightly rotated unsymmetrical ovals. The degree of rotation depends on the distance from the center of rotation as well as the lesion size. It may however be noted that the reconstruction of the lesions with higher acoustic speeds is much superior to its symmetric case counterpart above. It may readily be explained by observing that the receiver will have a minimum distance from the lesion for quite a range of time of flight projections so that the effect of refraction on these measurements will be considerably lower than for the one of the previous case.

It may be concluded from the above discussion that the refraction affects the contrast resolution more adversely for smaller lesions of higher contrasts, especially when they are located in the vicinity of the center of rotation of the transducers. Though the broadening of the features possessing higher speeds of sound is exhibited for all locations, it has the most adverse effect on the contrast resolution when the neighboring features possessing higher speeds of sound lie near the center of rotation.
Figure 5.7: Straight ray reconstruction of two lesions of different sizes having 1 mm distance between their boundaries and located symmetrical with respect to center of rotation. Both the lesions in each case possess the same speed of sound indicated on the vertical axis. The shown maps are cutouts from the reconstruction around the lesion area.
5.1 Effects of Refraction on Reconstruction

Figure 5.8: Straight ray reconstruction of two lesions of different sizes having 1 mm distance between their boundaries and located 30 mm apart from the vertical line through the center of rotation. Both the lesions possess the same speed of sound in each case indicated on the vertical axis. The shown maps are cutouts from the reconstruction around the lesion area.
5.2 Refraction Correction in Transmission CT

The refraction artifacts may partially be corrected through the iterative process of reconstructing a first estimate of the speed of sound under the assumption of straight line propagation, tracing rays through it and reconstructing the speed of sound along these rays. The ray tracing can either be carried out using Snell’s law or Fermat’s principle. The former approach was taken in section 5.1 to derive the ray tracing algorithm for the special case of cylindrical lesions. This method is however not suitable for the distributions of gradually changing acoustic speeds containing lesions of irregular shapes and unknown boundaries. In the following section an algorithm will be derived from the Fermat’s principle.

5.2.1 Ray Tracing Algorithm

Due to the two dimensional nature of the data acquisition setup the algorithm will be limited to the two dimensional case of a known acoustic speed distribution $c(x, y)$. Let

![Figure 5.9: Schematic diagram of a ray between points A and B.](image)

the acoustic speed between the two points $A$ and $B$ in figure 5.9 be known everywhere. Let the path taken by the ray between two points be as shown in the figure. First an expression will be derived to calculate this path as a function of the acoustic speed. The acoustic waves will take the path of minimum time between the two points $A$ and $B$ according to Fermat’s principle. In other words the line integral of $1/c(x, y)$ over the ray should be minimum. If the infinitesimal length of the ray at any arbitrary point $(x, y)$ be $dl$, the line integral will be given by.

$$t_{A\rightarrow B} = \int_{A}^{B} \frac{1}{c(x, y)} dl$$

As the ray starting from point $A$ and traveling towards point $B$ will be traveling forwards, a parameter $u$ may be introduced which is equal to 0 at point $A$ and 1 at point $B$ and increases monotonously as the wave propagates along the path shown in figure 5.9 from.
5.2 Refraction Correction in Transmission CT

A to B. The ray coordinates may then be written down as $(x(u), y(u))$ so that $dl$ will be given by

$$dl = \sqrt{\dot{x}^2 + \dot{y}^2} du$$

(5.14)

where $\dot{x} = dx/du$ and $\dot{y} = dy/du$. Substituting from equation 5.14 to 5.13, the following expression is obtained for the time of flight between points $A$ and $B$ to be minimized.

$$t_{A \rightarrow B} = \int_{0}^{1} \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{c(x(u), y(u))} du$$

(5.15)

The integral to be minimized has the form $\int_{0}^{1} F(x, y, \dot{x}, \dot{y}) du$, where $F = \sqrt{\dot{x}^2 + \dot{y}^2}/c(x, y)$, so that the minimization problem of equation 5.15 can be solved with the method of the calculus of variations [36]. The time $t_{A \rightarrow B}$ will be minimum if the following Euler equations hold (cf. equations 4.22 and 4.23).

$$\frac{\partial F}{\partial x} - \frac{d}{du} \left( \frac{\partial F}{\partial \dot{x}} \right) = 0$$

(5.16)

$$\frac{\partial F}{\partial y} - \frac{d}{du} \left( \frac{\partial F}{\partial \dot{y}} \right) = 0$$

(5.17)

Substituting for $F$ and simplifying yield

$$-\frac{1}{c^2} \frac{\partial c}{\partial x} = \frac{d}{\sqrt{\dot{x}^2 + \dot{y}^2}} du \left( \frac{1}{c} \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right)$$

(5.18)

$$-\frac{1}{c^2} \frac{\partial c}{\partial y} = \frac{d}{\sqrt{\dot{x}^2 + \dot{y}^2}} du \left( \frac{1}{c} \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right)$$

(5.19)

Substituting from 5.14 and recognizing $\dot{x}/\sqrt{\dot{x}^2 + \dot{y}^2}$ and $\dot{y}/\sqrt{\dot{x}^2 + \dot{y}^2}$ to be $\cos \theta$ and $\sin \theta$ from figure 5.9, the above two equations reduce to

$$-\frac{1}{c^2} \frac{\partial c}{\partial x} = \frac{dl}{dl} \left( \frac{1}{c} \cos \theta \right)$$

(5.20)

$$-\frac{1}{c^2} \frac{\partial c}{\partial y} = \frac{dl}{dl} \left( \frac{1}{c} \sin \theta \right)$$

(5.21)
Performing the differentiation on the right hand sides of both the equations and multiplying equations 5.20 and 5.21 by $-\sin \theta$ and $\cos \theta$ respectively and adding the results yield

$$\frac{d\theta}{dl} = \frac{1}{c} \left( \frac{\partial c}{\partial x} \sin \theta - \frac{\partial c}{\partial y} \cos \theta \right). \quad (5.22)$$

Equation 5.22 has to be integrated numerically in order to find the ray paths. A pixel-driven algorithm is not feasible as the ray will always encounter either a horizontal or a vertical boundary between two pixels which will give rise to misleading results [101]. In addition, the acoustic speed $c(x, y)$ should be at least once differentiable along both the spatial coordinates as the gradients $\partial c/\partial x$ and $\partial c/\partial y$ will be needed at each step. As will be seen in section 5.2.3, the speed of sound will have to be interpolated at arbitrary points. There are different ways to do that, the most appropriate one is, however, the use of some basis functions that meet the requirement of differentiability. The interpolation techniques such as nearest neighbor or bilinear interpolation result in problems similar to those found in pixel driven algorithm [29]. Due to their attractive behavior regarding differentiability and smoothness on edges, bivariate bicubic B-splines will be used for this purpose. The speed of sound will therefore be approximated by the following tensor product

$$c(x, y) = \sum_{m=0}^{3} \sum_{n=0}^{3} b_{mn} B_m(x) B_n(y). \quad (5.23)$$

The basis functions $B_m(x)$ and $B_n(y)$ are the univariate splines, and $b_{mn}$ are the coefficients to be calculated in accordance with the underlying speed of sound data. The spline basis functions approximate a mean free Gaussian with a standard deviation of $1/3$ clipped to zero beyond the extremities. At the most four splines are needed, each being a polynomial of a maximum of third degree in present case. In the case of uniformly spaced data, the computation with B-splines becomes more attractive. The algorithm to compute the spline values and their derivatives is detailed in [25, 27, 40].

A uniform step size Runge-Kutta algorithm of second order was used to integrate equation 5.22.

$$k_1 = \frac{\Delta l}{c(x_i, y_i)} [g_x(x_i, y_i) \sin \theta_i - g_y(x_i, y_i) \cos \theta_i] \quad (5.24)$$

$$k_2 = \frac{\Delta l}{c(x_i, y_i)} [g_x(x_i, y_i) \sin (\theta_i + k_1) - g_y(x_i, y_i) \cos (\theta_i + k_1)] \quad (5.25)$$

$$\theta_{i+1} = \theta_i + \frac{1}{2} (k_1 + k_2) \quad (5.26)$$
where $g_x$ and $g_y$ are the numerically computed gradients of the acoustic speed in $x$ and $y$ direction respectively and $\Delta l$ is the step size. The ray is approximated as a spline of first order between two steps so that the ray coordinates may be calculated at each integration step from the following linear relationships:

$$x_{i+1} = \begin{cases} x_i + \Delta l \cos \theta_{i+1} & \text{if } 3\pi/2 < (\theta_i \mod 2\pi) < \pi/2, \\ x_i - \Delta l \cos \theta_{i+1} & \text{else.} \end{cases}$$  \hspace{1cm} (5.27)

$$y_{i+1} = y_i + (x_{i+1} - x_i) \tan \theta_{i+1}$$  \hspace{1cm} (5.28)

The algorithm derived above is a bending algorithm as it is derived directly from the Fermat’s principle in contrast to the one described in section 5.1 based on the law of refraction. Although the latter is much more faster than the former, its applicability is, nonetheless, limited to the case of regular geometrical objects possessing a constant speed of sound. The bending algorithm just derived is applicable to any two dimensional case.

### 5.2.2 Validation of the Algorithm

The efficacy of the implemented algorithm was tested on a numerical medium known as Maxwell’s fish eye whose refractive index distribution is defined by [2]

$$n = \frac{n_0}{1 + \frac{r^2}{a^2}}$$  \hspace{1cm} (5.29)

where $n_0$ the peak value of the refractive index, $r$ is the distance of a given point from the origin and $a$ is an arbitrary constant. This medium has the property that every ray entering the medium will end up in a closed curve terminating in itself. Figure 5.10 shows the result of some of the tests with the Maxwell’s fish eye phantom of equation 5.29. It is clear that each ray closes in itself irrespective of where it starts. In all other tests as well the ray always closed in itself independent of the radius $r$ and the constant $a$ for step sizes as big as unity.

### 5.2.3 Iterative Correction of the Reconstruction

The artifacts resulting due to refraction may partially be corrected in an iterative manner summarized below.

1. Straight ray reconstruction.
5 Refraction Artifacts

Figure 5.10: Validation of the ray tracing algorithm on Maxwell fisheye numerical phantom. Each ray terminates in itself for step size as big as one. The underlay shows the distribution of the refractive index according to the gray scale shown on the right.

2. Ray tracing.

3. Reconstruction along ray paths of step 2.

4. Iteration of Steps 2 and 3 with some termination criterion.

Virtually all the iterative correction algorithms proposed so far [3, 29] are based on the Kaczmarz projection method for the approximate solution of a system of linear equations elaborated in [110]. A very brief description follows. Each line integral
\[ p_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{c(x,y)} \delta(r_i(x,y)) \, dx \, dy \]
representing a projection along a bent ray \( r_i(x,y) = 0 \) may be discretized to an algebraic linear equation
\[ p_i = \sum_k \frac{\Delta l_{ik}}{c_k} \]  
(5.30)
where the summation is carried out along the ray \( r_i(x,y) = 0 \) consisting of points \((x_k, y_k)\) found out from equations 5.27 and 5.28 and \( \Delta l_{ik} \) is the step size used therein.
5.2 Refraction Correction in Transmission CT

The acoustic speed \( c_k = c(x_k, y_k) \) is calculated from the spline approximation of equation 5.23. If the acoustic speed distribution \( c_1, c_2, \ldots, c_{MN} \) over the entire region of reconstruction \( m = 1, \ldots, M \) and \( n = 1, \ldots, N \) is taken as a single point in an \( MN \) dimensional space, each of the equations 5.30 will represent a hyperplane in that space. A Kaczmarz iteration consists of projecting the initial solution guess \( c_j^{(i)} \) to one of these hyperplanes and taking the projected point as the next iteration value \( c_j^{(i+1)} \) to be projected to the next hyperplane. The concept is illustrated in the following figure for a trivial case consisting only of two acoustic speed values \( c_1 \) and \( c_2 \) determined from two linear equations shown in figure 5.11. The iterative correction may be proved [66] to be given by

\[
\frac{1}{c_j^{(i+1)}} = \frac{1}{c_j^{(i)}} + \frac{p_i - \sum_k \frac{\Delta l_{ik}}{c_k^{(i)}} \Delta l_{ij}}{\sum_k \Delta l_{ik}^2 \Delta l_{ij}}. \tag{5.31}
\]

It is evident from figure 5.11 that the correction will always converge to a solution if it exists. Two interesting cases can be considered. Firstly, if the two lines are parallel, the solution does not exist and thus the iteration can not converge. Secondly, if the two lines are orthogonal to each other, the solution will always converge in two iterations irrespective of the initial guess. This fact was utilized in the practical implementation of the algorithm by taking two consecutive projections which are approximately orthogonal to each other.

![Figure 5.11: Illustration of the Kaczmarz projection method for two unknowns.](image)
5.3 Refraction Correction for Compounding

The nature of refraction artifacts in the echosonography and in compound images calculated from the former is different from the ones discussed above.

5.3.1 Refraction Artifacts in B-mode and Compound Images

The departure of the acoustic speed of a medium from the uniformity assumption has the following major implications for the B-mode imaging.

1. The calculation of the depth from the arrival times of the echo signals under the assumption of a uniform tissue acoustic speed results in registration errors.

2. The transmit and receive beamforming is based on the time delays calculated on the basis of the uniformity of acoustic speed assumption. Refraction will have an effect on beamforming, which will impair the lateral resolution in the worst case.

3. Refraction of the acoustic beam out of the imaging plane will result in a shadow behind the refracting interface.

Figure 5.12: Schematic Sketches of the objects whose images appear in figure 5.13. The dotted rectangular area in each sketch corresponds to the image.

Two typical examples of the refraction artifacts are shown in figure 5.13. The imaged objects are illustrated in figure 5.12. The registration error stated above may have little significance regarding the visual appraisal of an individual B-scan. Its effect on
5.3 Refraction Correction for Compounding

Figure 5.13: Refraction artifacts in echosonography. The left two panels show the registration error due to refraction. The point in the left most B-scan is a wire orthogonal to the imaging plane, which is displaced laterally as well as axially in the second panel from left, when an object possessing a higher speed of sound is placed before it. The phantom image of the third panel is shadowed in the forth one when an object before it causes an out of plane refraction.

reflection mode tomography or on compounding of individual scans may nevertheless be considerably detrimental to the spatial and contrast resolution. To demonstrate and study the effect, a cylindrical vessel was filled with ethylene glycol which possesses a substantially higher speed of sound (1664 m/s at 22°C) than water (1490 m/s at 22°C). Four polypropylene fiber pieces were tightly fixed inside the vessel at a distance of 1 cm from each other, parallel to each other and orthogonal to the imaging plane. The compound images calculated from 72 individual B-scans under the assumption of a uniform speed of sound equal to that of the surrounding medium water exhibit substantial registration errors visible in Figure 5.14.

5.3.2 Correction Schemes

The defocussing problem stated above can only be tackled if the complete data sets consisting of all possible combinations of the transmitter and receiver needed to correct the phase aberration are available [58, 89]. Also, there is no way to compensate for the out of plane refraction in a two dimensional measurement scheme. The only corrective measure that can be taken on the basis of the available two dimensional beamformed...
5 Refraction Artifacts

Figure 5.14: An example of refraction artifacts in compound images. The right panel shows the compounded image of the phantom illustrated in the schematic sketch in the left panel.

echo data is the correction of the false registration. Even this will only be possible if the speed of sound distribution in the medium is known. In the following discussion it will be assumed that the speed of sound distribution of the medium is available. The correction itself can be undertaken either only along the axial direction or along the refracted ray directions.

Axial Time Delay Correction

This approach was proposed in [59] and is based on approximating the refraction with an axial shift expressed in equation 4.1 thus ignoring the bending of rays due to refraction. Instead of assuming a uniform speed of sound and calculating the depth from the arrival time on the basis of $z = c_0 t/2$ the depth will be calculated from $z_k = \frac{t_k}{2} \sum_{i=1}^{k} c_i$ if the signal is received at $k$th sample and the speed of sound associated with the depth $z_i$ is $c_i$. It was postulated that bending does not contribute significantly to refraction errors and simply the axial shifting of the respective A-lines according to the underlying speed of sound distribution is sufficient to correct the refraction errors. It will be shown below that the applicability of this method is limited by the location of the features and the speed of sound distribution of the medium [19, 20]. To illustrate this point, let speed of sound inside the cylindrical object of radius $r$ be $c_1$ and that of the background be $c_0$, see figure 5.15. Let there be a single scatterer at location $P_1 (x_1, z_1)$ inside the object. The part of the backscattered signal reaching the active aperture of the transducer will be refracted at $(x, z)$ after covering a distance $d_0$ through the object. The commercial
5.3 Refraction Correction for Compounding

Figure 5.15: Path of the beam scattered from a scatterer at $P_1$ to the transducer. Points $P_2$ and $P_3$ correspond to interpreted scatterer location after the axial correction and without correction, respectively, in the individual B-mode image.

The ultrasound system will interpret the signal coming from point $P_1$ to be emanating from point $P_3$ shown in figure 5.15. The axial correction will correct along the axial direction only and will move the scatterer position from $P_3$ to $P_2$. To see what happens to the compound image computed with such a correction, a reverse ray tracing will have to be carried out. Neither the shooting method of section 5.1 nor the bending method of section 5.2 is applicable to this situation as the initial direction of the scattered field reaching the transducer is unknown. To cope up with this situation a reverse ray tracing algorithm is derived in appendix A.3. Figure 5.16 shows the results of an exact simulation based on the results of Appendix A.3. The speed of sound inside the object of 6 cm diameter is 1550 m/s and that of the background is 1490 m/s. A single scatterer located at (0,-2 cm) is simulated here. The actual paths of rays are shown in (a). The paths calculated by considering a uniform speed of 1490 m/s everywhere are shown in (b) while (c) shows the axial correction along these paths. The lower half of the figure shows the enlarged views of the single scatterer compound image. In the uncorrected version, the single point will appear as an ellipse with axes equal to 2.32 mm and 1.76 mm, the axial correction is limaçon like curve extending 0.88 mm and 1.18 mm in each direction respectively. It is evident that the axial correction does not eliminate the effect of refraction in this case. In a compound image calculated from measured data, the shape shown in panel (f) will not be visible due to the resolution limitations.
Figure 5.16: The axial time delay correction in compounding. Panel (a) shows the refracted beams inside the cylindrical object from a few angles of incidence, (b) and (c) is the depiction of the respective beams without correction and with axial correction, respectively. Panel (d) shows the ideal case and (e) and (f) show the enlarged views of the scatterer compound image without correction and with axial correction respectively.

of the individual images. The point spread function of the axial correction is also a function of the scatterer coordinates. In case of speeds of sound lower than that of the background the spreading is predominantly in the increasing radial direction while for higher acoustic speeds the converse is true. The point spread function of the uncorrected case is generally an ellipse symmetrical about the radial line passing through its center. This elliptical form approaches circular one in the vicinity of the center of rotation. This is where the axial correction is most accurately. The effect of acoustic speed and radial distance of the scatterer from the center of rotation is shown in figure 5.18. It may be noted that the radial spreading in the uncorrected case is independent of the radial coordinate and is determined alone on the basis of the refractive index. The axial correction bears fruits only in the vicinity of the center of rotation. For scatterers lying
5.3 Refraction Correction for Compounding

Figure 5.17: Point spread function of compounding in dependence of the axial distance of the scatterer from the center of rotation without (dotted line) and with axial time delay correction (continuous line). The upper half is for $c_1 < c_0$ while the lower half for $c_1 > c_0$.

near the periphery of the cylindrical object of figure 5.15 the axial time delay correction proves almost useless, at least regarding correction of the azimuthal spreading. It may be pointed out that this discussion is based on the object geometry of figure 5.15.

**Refraction Correction**

Considering the results of the preceding discussion it becomes clear that the axial correction works for lower speed contrast and at the best in the vicinity of the center of rotation. It may therefore be appreciated that a full compensation of the refraction based on tracing of beam paths through the medium and aligning the individual A-lines along them becomes necessary in many cases. The differences between the two approaches will become evident in the next section.
Figure 5.18: Maximum radial and azimuthal extents of spreading of a point and effect of axial correction. The dashed lines show speeds of sound lower than the that of background while continuous ones show the converse case.

5.3.3 Experimental Validation

Figure 5.19 shows the results of compounding with axial and with refraction correction carried out on data measured on the phantom described above in section 5.3.1. The lower panels of the figure show the magnified images of the compound image with axial and with refraction correction respectively. The results of the simulation model are overlayed to the measurement results. It may be noted that the results of simulation are in conformity with those of the measurement regarding the width of the point spread function. The small visual discrepancy, which is irrelevant for the current discussion, is due to the fact that the point spread function of the B-mode imaging system was supposed to be ideal. The experimental validation shows the model assumptions to be
5.3 Refraction Correction for Compounding

Figure 5.19: Panels (a), (b) and (c) show the uncorrected, axially corrected and refraction corrected compound of the data measured on phantom described in the previous section. The lower part of the figure shows enlarged views of the corresponding upper part, overlayed with the corresponding simulation results.

well placed. Regarding the validity of the quantification shown in the curves of figure 5.18, it may be noted that they are applicable to the simulated case, in strict manner. It may however be pointed out that the qualitative results summarized below, remain valid for the most other cases possessing some other speed of sound distribution.

- Refraction tends to produce circular, elliptical and arc shaped artifacts with decreasing azimuthal extents for higher radial coordinates.

- The axial time delay compensation works at the best at locations where the PSF of the uncorrected compound image has a symmetrical circular shape. In the worst case of higher radial coordinates, or generally, at locations where the PSF of uncorrected spatial compounding deviates substantially from the circular shape, the axial correction may even increase the PSF width.
5.4 Chapter Summary

Ultrasound CT differs substantially from x-ray CT due to strong refracting and diffracting characteristics of acoustic waves. As a result, the reconstruction based on the assumption of straight line propagation does not provide enough accuracy. Though the exact wave equation inversion technique would be the method of choice in such a situation, it can not be investigated within the framework of the current work due to unavailability of the single channel data required by it on one hand and due to the fact that the currently known methods of full inversion are computationally unviable for practical purposes. Effects of refraction on the straight line reconstruction were quantified in this chapter for transmission CT and compounding. For that, several ray tracing techniques were developed. The effects of speed of sound variations and lesion size on the reconstruction were also investigated. It was shown that regions possessing higher speeds of sound appear to be larger than their actual size, and, regions with lower speeds of sound appear to be smaller than they really are. An iterative correction technique based on a bending ray tracing algorithm was discussed. Towards the end of the chapter, effects of refraction on compounding were discussed and the efficacy of two correction techniques was compared on the basis of simulations. The chapter concluded with the experimental validation of the algorithms.
6 Experimental Investigation

A modular measurement setup was designed and constructed in order to investigate the methods and algorithms elucidated in the preceding chapters experimentally. One of the possible approaches may have been a fully custom designed setup comprising customized electronics to switch and trigger the ultrasound transducers in a desired way. Despite being an attractive approach it lies beyond the scope of this work due to its multifaceted and expansive nature. The more viable alternative is to utilize a commercial ultrasound scanner as data acquisition system. This will be possible as far as the ultrasound system (US system) allows access to its raw data. Though this approach has several drawbacks due to limited access to the US systems, it offers an attractive alternative due to its comparatively speedy implementation. The other advantage of using a commercial US system is the availability of the technically perfected pulse echo data which might scarcely be equalled by a custom designed setup. Finally the use of a commercial scanner instead of a custom hardware promises an easier clinical applicability. Considering these factors the approach seems to be the only plausible solution commensurate with the current work. The only minor modification in case of through transmission mode, which does not affect the operation of the ultrasound system in any way, is to tap off the trigger signal as well as the unmodulated radiofrequency signal and to feed the former into a custom designed trigger interface circuit for further processing. Two different modes can be implemented using the setup.

1. **Transmission mode** is implemented around an analog US system allowing access to its trigger signal in order to acquire transmission and echo data simultaneously.

2. **Reflection mode** is implemented around a high end digital US system, with no access to its trigger signal, to acquire only the echo data.
6.1 Measurement Setup

Apart from the commercial scanner, the setup consists of a mechanical subsystem, a custom software and external hardware to be described below. Depending on the type of application one of the two modes can be activated in conjunction with the respective US system.

6.1.1 Mechanical Add-on Applicator

As the mechanical system is common to both the modes, it will be described first. This subsystem is illustrated in figure 6.1. It comprises primarily a horizontally rotatable platform (RP) that can also be translated vertically with high precision servo motors (SM), the two movements being independent of each other. The platform is fitted with two brackets (BR) to hold the two ultrasound transducers (UST). The transducers can be fixed into the brackets in such a way that they are facing each other. The brackets are hinged at three points to allow for a manual adjustment in such a manner that the two ultrasound arrays are placed in the same plane and are facing each other. Each motor possesses a base angular step size of 10.8'. Both the axes are driven via timing belts (TB). Besides the timing belt transmission ratios an additional gear transmission (GT) on the rotary axis and the pitch of the translational spindle (TS) result in rotational and translational step sizes of 7.2" and 0.25 $\mu$m respectively. It may, however, be noted that the actual resolution is also a function of the pulse encoder of the servo motors and the accuracy of the PID controllers. The whole setup including the rotating platform and the ultrasound transducers is built inside a water tank (WT) which in its turn is fitted beneath a modified patient couch (MPC) in such a way that the organ or the object to be imaged may hang from above inside the water bath between the two transducers looking into each other. In reflection mode there will be a single transducer which can also easily be reconfigured to reflex-transmission mode [10, 11, 12] by replacing the transmitting transducer of the transmission mode with a metallic plate. The servo motors are controlled via an external PC with the help of digital PID controllers. Some important specifications of the mechanical setup are summarized in table 6.1. Depending on the mode of operation the external hardware will be different.
Figure 6.1: Schematic Sketch of the realized mechanical applicator.
6 Experimental Investigation

Table 6.1: Salient features of the mechanical setup

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transmission ratio for vertical motion</td>
<td>1:5</td>
</tr>
<tr>
<td>Transmission ratio for horizontal motion</td>
<td>2:15</td>
</tr>
<tr>
<td>Spindle pitch</td>
<td>2.5 mm</td>
</tr>
<tr>
<td>Maximum allowable radius of the object to be imaged</td>
<td>12 cm</td>
</tr>
<tr>
<td>Maximum allowable height of the object to be imaged</td>
<td>16 cm</td>
</tr>
</tbody>
</table>

Figure 6.2: Photos of the realized mechanical add-on applicator. (a) Overall-view of the system fitted into a modified patient couch. (b) Top close-up of the water tank. (c) Side close-up of the water tank.

RP  Rotatable Platform
SM  Servo Motors
BR  Bracket
UST Ultrasound Transducer
GT  Gear Transmission
SP  Spindle
WT  Water Tank
TB  Timing Belt
MPC modified patient couch
6.1 Measurement Setup

6.1.2 Transmission Mode

In case of transmission data acquisition through an analog US system the external hardware will consist of a pulser/receiver, custom made trigger interface [88] and an analog-to-digital converter (A/D Converter) [13, 15]. The trigger interface extracts the shot trigger information used to trigger the data acquisition via a PCI A/D Converter. A linear array ultrasound transducer identical to the one connected to the US system is driven by the pulser triggered with the delayed trigger signal mentioned above and is designed to send short Gaussian pulses. All the channels of the linear array are connected parallel to each other so that the pulses sent by the array will give rise to plane waves in the imaging plane. This forms the core of the concept behind the transmission measurement with a commercial US system. Now, the functionality of the transmission mode may be summarized as under. One of the transducers connected to the ultrasound system is driven in its normal B-mode whilst short pulses of the same center frequency are sent from the other with an external wave generator triggered with delayed version of the system trigger signal. The acquired radiofrequency (RF) echo data thus contains the transmitted data as well. The delay between the shot trigger of the two transducers is selected in such a way that the transmitted pulse, appearing as a narrow white line in the B-mode image, may be positioned in an echo poor region. A typical B-mode frame acquired with this method is shown in figure 6.7. The cylindrical polyvinyl alcohol phantom possessing a higher acoustic speed than that of water has a hole inside it. The transmitted wave front shown on the right depicts clearly the distortion caused by the variation in the acoustic speed.

6.1.3 Reflection Mode

The external hardware in case of the acquisition of echo data with a high-end US System is limited to a PC and the ECG Physio module of the Siemens Sonoline Antares® equipped with an ultrasound research interface (URI). The URI can save the 16 bit beamformed RF data of each frame sampled at 40 MHz sampling rate. In addition to the settings possible from the front control panel of the US System, it is possible to change some of the beamforming parameters with a special interface. The acquisition is triggered with the help of the ECG trigger feature of the US system. The data sets are stored on the US system dard disk and can be downloaded afterward to the PC via local area network (LAN) for further processing. Figure 6.4 gives a schematic overview
6 Experimental Investigation

Figure 6.3: Schematic block diagram of the transmission mode.

6.1.4 Custom Software

A special software was written in C++ to synchronize the operations of individual frame data acquisition in a given position, servo motion of the gantry and most importantly, the relative clocking of the two ultrasound transducers in case of transmission mode. In both modes the acquired data is automatically saved to the hard disc of either the US System or the external PC. The software was named SynchroSuite.
6.2 Calibration of the Setup

It is clear from the description of the previous section that the ultrasound transducer has to be mounted anew before every measurement. This necessitates determination of the center of rotation every time the transducers are mounted afresh. Another reason to calibrate the system is the use of an external AD-Converter for data acquisition in case of an analog access to the ultrasound system. Unlike digital US systems which save header information for each acquired frame allowing the calculation of the geometrical coordinates of each sample, the aspect ratio of each individual pixel will be unknown for the analog case. This is the other parameter which has to be determined besides...
the center of rotation. The simple method described in [60] is thus not sufficient for our purposes.

A polypropylene yarn phantom consisting of at least one yarn thread stretched orthogonal to the imaging plane and fixed at a position lying inside the reconstruction circle of the CT system is used for calibration. The image of the thread will appear as a point whose shape will roughly correspond to the point spread function of the B-mode image. Consider a single frame of echo data shown in figure 6.5. The data are saved using natural matrix coordinates having origin at the top left corner. We choose the origin of the cartesian coordinate system to coincide with the unknown center of rotation.

6.2.1 Deterministic Approach

Let the point corresponding to the thread image be \( P \) with matrix coordinates \((x_{\text{mat},c}, y_{\text{mat},c})\) and the pixel aspect ratio for the matrix coordinates be \( m \), i.e. \( m = \frac{\delta x_{\text{mat}}}{\delta y_{\text{mat}}} \). The relationship between the matrix coordinates and cartesian coordinates of point \( P \) is given
6.2 Calibration of the Setup

by the following transformation $T_{\text{mat2cart}}$.

$$
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
m & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
x_{\text{mat}} - x_{\text{mat,c}} \\
y_{\text{mat}} - y_{\text{mat,c}}
\end{bmatrix}
$$ (6.1)

Now consider a frame number $n$ at an arbitrary angle $\theta_n$ to the starting frame number 1 as shown in figure 6.6. The coordinate systems of frame $n$ and 1 may be transformed to each other by the transformation matrix of equation 3.5, so that the cartesian coordinates of the point $P$ appearing in frame $n$ with respect to frame 1 are given by the following relationship

$$
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
\cos \theta_n & \sin \theta_n \\
-\sin \theta_n & \cos \theta_n
\end{bmatrix}
\begin{bmatrix}
m & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
x_{\text{mat}} - x_{\text{mat,c}} \\
y_{\text{mat}} - y_{\text{mat,c}}
\end{bmatrix}
$$ (6.2)

Figure 6.6: Determination of the center of rotation.

The matrix equation 6.2 may be simplified to the following two scaler equations.

$$
x = m (x_{\text{mat}} - x_{\text{mat,c}}) \cos \theta_n - (y_{\text{mat}} - y_{\text{mat,c}}) \sin \theta_n
$$ (6.3)

$$
y = -m (x_{\text{mat}} - x_{\text{mat,c}}) \sin \theta_n - (y_{\text{mat}} - y_{\text{mat,c}}) \cos \theta_n
$$ (6.4)

As the two equations contain three unknown parameters $x_{\text{mat,c}}$, $y_{\text{mat,c}}$ and $m$ to be calibrated and two unknown variables $x$ and $y$, any three arbitrary frames can be used.
to arrive at the required parameters. Let the three frames be taken from angles $\theta_m$, $\theta_n$ and $\theta_p$. Equating the cartesian coordinates of the point with respect to initial frame 1 will yield six equations of the form $m (k_1 + k_2 x_{mat,c}) + k_3 y_{mat,c} + k_4 = 0$. Any three of them can be solved simultaneously to arrive at the required parameters. A specially attractive case arises for $\theta_m = 0^\circ$, $\theta_n = 90^\circ$ and $\theta_p = 180^\circ$, which result in the following three simple calibration equations.

\[
x_{mat,c} = (x_{mat,m} + x_{mat,p}) / 2 \tag{6.5}
\]
\[
y_{mat,c} = (y_{mat,m} + y_{mat,p}) / 2 \tag{6.6}
\]
\[
m = (y_{mat,n} - y_{mat,c}) / (x_{mat,m} - x_{mat,c}) \tag{6.7}
\]

The actual calibration is based on sixty frames each measured at an angular displacement of 6°. As the point spread function of B-mode imaging is anisotropic, a centroid approach was taken to calculate the individual matrix coordinates of the points depicting the polypropylene yarn threads. If the image of the thread is bounded by a rectangle whose upper left and lower right corner have matrix coordinates $(x_{mat,n_1}, y_{mat,n_2})$ and $(x_{mat,n_3}, y_{mat,n_4})$, the matrix coordinates of the thread will be calculated as under.

\[
x_{mat,thread} = \frac{\sum_{i=n_1}^{n_3} \sum_{j=n_2}^{n_4} x_i b(x_i, y_j)}{\sum_{i=n_1}^{n_3} \sum_{j=n_2}^{n_4} b(x_i, y_j)} \tag{6.8}
\]
\[
y_{mat,thread} = \frac{\sum_{j=n_2}^{n_4} \sum_{i=n_1}^{n_3} y_j b(x_i, y_j)}{\sum_{i=n_1}^{n_3} \sum_{j=n_2}^{n_4} b(x_i, y_j)} \tag{6.9}
\]

where $b(x_i, y_j)$ is the amplitude of the demodulated signal at matrix coordinates $(x_i, y_j)$.

### 6.2.2 Least Square Estimation

The least square estimation is based on taking into account the complete data set consisting of the matrix coordinates of known points in all the frames in different angular positions and minimizing the least square error of estimation. The transformation of equation 6.2 may be rewritten in the usual form using $3 \times 3$ transformation matrices $T_{mat2cart}$, $T_{move2center}$ and $T_{\theta_n}$ used to denote the three operations of scaling, translation and rotations respectively.

\[
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix} = T_{\theta_n} T_{mat2cart} T_{move2center} \begin{bmatrix}
x_n \\
y_n \\
1
\end{bmatrix} \tag{6.10}
\]
where \( x_n \) and \( y_n \) are the matrix coordinates of point \( P \) in the \( n \)th frame.

\[
T_{\text{mat}2\text{cart}} = \begin{bmatrix}
m & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] (6.11)

\[
T_{\text{move}2\text{center}} = \begin{bmatrix}1 & 0 & -x_c \\
0 & 1 & -y_c \\
0 & 0 & 1\end{bmatrix}
\] (6.12)

\[
T_{\theta_n} = \begin{bmatrix}
\cos \theta_n & \sin \theta_n & 0 \\
-\sin \theta_n & \cos \theta_n & 0 \\
0 & 0 & 1
\end{bmatrix}
\] (6.13)

Similarly the matrix coordinates \( x_1 \) and \( y_1 \) may be transformed to cartesian coordinates through \( T_{\text{mat}2\text{cart}}T_{\text{move}2\text{center}}[x_1, y_1, 1]^T \). The estimation consists of minimizing the sum of squares of Euclidean distance between points \( P_1 \) and \( P_i \) where \( i = 2, \ldots, n \). This will result in a nonlinear least square estimation problem, which may be reduced to a linear one with the introduction of a new parameter vector to be estimated. So, instead of estimating \([m, x_c, y_c]^T\) directly, a new vector \([p_1, p_2, p_3]^T = [mx_c, m, y_c]^T\) will be estimated. The detailed derivation of appendix A.4 shows that the following linear matrix equation can be used to estimate the required parameters.

\[
\begin{bmatrix}
2 \sum (1 - \cos \theta_i) \\
\sum (x_1 + x_i)(\cos \theta_i - 1) \\
0
\end{bmatrix} \begin{bmatrix}
\sum (x_1 + x_i)(\cos \theta_i - 1) \\
\sum (x_1^2 + x_i^2 - 2x_1x_i \cos \theta_i) \\
\sum (x_1 - x_i) \sin \theta_i
\end{bmatrix} \begin{bmatrix}
0 \\
2 \sum (1 - \cos \theta_i) \\
\sum (x_1 - x_i) \sin \theta_i
\end{bmatrix} = \begin{bmatrix}
p_1 \\
p_2 \\
p_3
\end{bmatrix}
\] (6.14)

\[
\begin{bmatrix}
\sum (y_1 - y_i) \sin \theta_i \\
\sum (x_1y_i - x_iy_1) \sin \theta_i \\
\sum (y_1 + y_i) (1 - \cos \theta_i)
\end{bmatrix}
\]

### 6.3 Phantom Experiments

Several phantom results of the current work were already reported by the author in several publications [10]-[23]. One of those results was also discussed in chapter 5. This
6 Experimental Investigation

Table 6.2: Comparison of the two calibration methods

<table>
<thead>
<tr>
<th>Method</th>
<th>$x_c$</th>
<th>$y_c$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deterministic method</td>
<td>69.244</td>
<td>124.234</td>
<td>23.237</td>
</tr>
<tr>
<td>Estimation method</td>
<td>69.109</td>
<td>124.011</td>
<td>23.261</td>
</tr>
</tbody>
</table>

Figure 6.7: Phase jitter in acquired data. The left figure shows the B-scan and the right one the zoomed up portion of transmitted pulse, both with jitter. More prominent jitter can be seen at the locations pointed out by arrows.

Section will discuss one more phantom result with the primary purpose of illustrating the efficacy of the algorithms developed in the previous chapters as well as highlighting some of the practical problems which remained undiscussed as yet. The phantom chosen for this discussion is sufficiently well defined in its acoustic properties and geometry so that the discussion remains manageable within the framework set out in chapter 1. The phantom was made of polyvinyl alcohol with a known speed of sound of 1520 m/s. It has a simple right cylindrical geometry with a hole in the middle. The diameter of the phantom and that of the hole are 19 mm and 4 mm respectively. The data were acquired with the transmission setup described in section 6.1.2. A short description of some practical problems and the proposed solutions are as follows.

### 6.3.1 Phase Jitter

The acquired data is afflicted with phase jitter of the data acquisition card. This can be seen from a typical data frame acquired with this setup shown in figure 6.7.

Though the phase jitter can be seen overall in the B-scan itself, there are some points
6.3 Phantom Experiments

Figure 6.8: Acquired data after jitter correction. The left figure shows the B-scan and the right one the zoomed up portion of transmitted pulse, both after jitter correction.

Figure 6.9: Illustration of jitter correction. Panel (a) shows the trigger signal added to the RF signal in gray scale. Panel (b) shows a zoomed up area of panel (a). Panel (c) plots the same data as shown in gray scale in panel (b).

at which the jitter deteriorates also the transmission data so intensely that it is even visible to the naked eye. This visible portion can be corrected partially by introducing a simple trick. A small portion of the trigger signal is added to the acquired RF data so that it will appear right below the ring down area of the B-scan. Now the trigger can be corrected by means of cross correlation. To make the correction more precise, the signal caused by trigger signal is interpolated to a finer sampling grid. The correction concept is demonstrated in figure 6.9. The correction is only partially efficient as the phase jitter does not have to synchronize with the trigger signal. This will also be seen shortly in the following discussion. The corrected data set is shown in figure 6.8. It may be seen from the zoomed up transmission data that, at least, the prominent jitter present in figure 6.7 have been reduced.
6.3.2 Trend Removal From Estimated TOF

As described in section 6.1, the alignment of the two transducers is carried out manually at the beginning of each data acquisition. Despite careful alignment, a mutual tilt of the transducers is not guaranteed. This has a far reaching adverse effect on the reconstruction of speed of sound. In order to demonstrate this point the two transducers were not precisely aligned which resulted in the time of flight shown in figure 6.10 as a dotted line. The detrending is carried out on the basis of least square estimation of the slope and intercept parameters of the inclination and then subtracting the trend from the time of flight.

6.3.3 Sinogram Correction

After the sinogram has been calculated, there are certain factors that may be corrected more easily in sinogram space rather than in the individual projection data itself. The first factor may be understood in conjunction with the fact that every curve in the sinogram should be sinusoidal. This property is limited by the distortions introduced by refraction as already discussed in detail in chapter 5. Nevertheless a visual inspection of the sinogram may always be helpful to assure that there are no randomly introduced errors arising from errors in time of flight estimation. Another source of random errors is the random motion of the object during data acquisition. While the latter is very difficult to be removed, the former may often be corrected directly in the sinogram space, so that the curves are sinusoids. The procedure of correction consists of locating a major landmark clearly visible in each projection and aligning the neighboring projections in
6.3 Phantom Experiments

Figure 6.11: Sinograms as surface plots, left: before correction and right: after correction.

Figure 6.12: Schematic illustration of the phantom.

such a way that this landmark is as close to sinusoidal curve as possible. In the worst case the concerned projection is spotted and its time of flight is estimated again. If that also does not alleviate the problem, the projection is estimated linearly on the basis of adjoining projections. This process can easily be carried out automatically.

Another important factor that may also be corrected in sinogram space is the phase jitter that has not yet been removed completely due to the reason mentioned above. The reminiscent jitter exhibits itself in a systematic way thus causing creases in it. This results in a cogwheel-like artifact. A local moving average filter is employed to smooth out the local peaks resulting in fine creases and wrinkles in the sinogram.
The difference between the reconstructed speed of sound from uncorrected and from corrected sinogram is visible in figure 6.13.

The reconstructed speed of sound map was also corrected for refraction according to the method described in chapter 5. Due to the symmetry of the problem about the center of the phantom, the result is presented more compactly as a one dimensional slice through the reconstruction to facilitate a visual inspection in figure 6.14. The tapering effect at edges due to refraction has been corrected to a large extent after three iterations.

### 6.4 Small Animal Imaging

One of the major focuses of small animal imaging is preclinical cancer research. Importance of animal models lies in the fact that a better understanding of cancer can contribute to its optimal treatment and prevention. That the animals have to be sacrificed at some stage of the preclinical research to obtain histological evidence is one of the prohibiting factors in achieving the potential advantages of animal models. There is a great need to develop functional, molecular, morphologic and histologic imaging techniques. While the former two techniques should help understand biochemical and genetic processes in vivo they are out of scope of the current work. The importance of the latter ones is their capability to locate and differentiate between tissues of different types and between different tissue changes, if the imaged information has a quantitative content of some histological relevance [17, 18]. Consequently, the development of such
imaging tools can help use the same animals over much longer periods of time. It is highly desirable to develop ultrasound based imaging tools for in vivo imaging of small animals due to non ionizing nature of the acoustic field, besides other usual advantages.

In addition to the goals described above, the reason to choose a naked mouse for the current work, in the first instance, is the fact that acoustic properties of mouse tissue are quite similar to those of human tissue. The only appropriate ultrasound transducer available for the case of bi-static transmission setup has a center frequency of 7.5 MHz. The penetration depth of the acoustic field is limited to a few centimeters which is not adequate for the case of a female breast. The naked mouse being easier to image with ultrasound due to absence of fur was chosen to demonstrate the applicability of the concepts developed in this work. A linear transducer with a center frequency of about 1-2 MHz will arguably yield at least comparable results for the case of female breast. Due to this reason this section will present some results of quantitative and qualitative small animal imaging. The imaging results presented here were carried out on naked mouse cadaver provided by Institute of Molecular Gastroenterological Oncology of the Ruhr-University Bochum.

Figures 6.16, 6.17 and 6.18 show the results of small animal imaging carried out in the lower thorax region of a fresh naked mouse cadaver. All the data acquisition for the
results presented in this section was carried out with the setup shown in figure 6.3. The angular resolution of the acquisition was 2 degrees so that a total of 180 frames were taken from around the animal. The water temperature was 20° and the cadaver was that of a two week old naked mouse. Figure 6.16 shows the reconstructed speed of sound. The left panel shows the reconstruction based on straight ray inversion model, while the right panel shows the refraction corrected speed of sound map after three iterations. Both the kidneys and the spine are all merged together in a big clump showing an average acoustic speed of 1572.3 m/s. In the refraction corrected version the kidneys and the spine are separately visible each showing approximately the same average speed as the bigger cluster in the uncorrected version. The acoustic speed values for the case of mouse tissue mentioned in literature are 1586 ± 10.7 m/s for kidneys, 1536-1543 m/s for skin [33], and 1580-1690 m/s for cartilage [56]. The average reconstructed acoustic speed of skin was found to be 1535.3 m/s and the rest of the thorax/abdomen has an average speed of sound of 1551.9 m/s. It may be noted that the reconstructed speeds of sound are in very good conformity with the values known from standard literature. As the highest achievable spatial resolution of the system is given by the lateral resolution of the ultrasound system in the focal position, it will be realistic to assume an average
Figure 6.16: Reconstructed speed of sound in naked mouse. The left panel shows the reconstruction with straight line propagation model while the right one shows the reconstruction after three iterations of refraction correction. Both the reconstructions are color coded in m/s according to the color bar shown below the figures.

Figure 6.16: Reconstructed speed of sound in naked mouse. The left panel shows the reconstruction with straight line propagation model while the right one shows the reconstruction after three iterations of refraction correction. Both the reconstructions are color coded in m/s according to the color bar shown below the figures.

Lateral resolution of about 0.3 mm in the B scan itself. Figure 6.15 shows the estimated time of flight from one of the angular position to help visualize the data from which the reconstruction was carried out. Also taking into account the effect of refraction, the actually achievable resolution will be even lower. The two kidneys, the spine and the skin are clearly demarcated in the acoustic speed tomogram.

In addition to the good agreement between the reconstructed values with those mentioned in published literature, another indirect measure to judge the quality of the reconstruction is to correct the refraction artifacts in the individual B scans, also obtained simultaneously with the transmission data. The degree to which the artifacts have been corrected in the compound images reconstructed from the corrected B scans will correspond to the reconstruction accuracy for acoustic speed in terms of quantitative information and the spatial resolution. Both the correction schemes, the axial correction and the complete refraction correction with the help of ray tracing, were applied. It may be noted from figure 6.18 even visually, that the axial correction, despite having alleviated some of the arc and circle artifacts, could not perform as good as the refraction correction. The compound image corrected with the latter method shows clearly the organs whose positions are labeled in the B scan. The boundaries of the reconstructed
organs, specially the kidneys and the spine, have become very clear after the correction. It may however be noted that not all the refraction artifacts have been removed after correction. This is partially also due to the fact that the measurement setup was not primarily designed for small animal imaging. The minimum distance between the two transducers can not be reduced below 7 cm, which, for an average mouse diameter of 2-3 cm, is too large. The refraction artifacts can be minimized even in the acquisition stage by keeping the distance between the two transducers to the minimum dictated by the diameter of the given animal. Another way to reduce the refraction artifacts is to design a much smaller water tank and to use an appropriate fluid which has an acoustic speed matched to the average acoustic speed of the mouse.

The reconstruction of attenuation is shown in figure 6.17. The left panel shows the reconstructed attenuation coefficient based on a frequency shift method assuming a linear frequency dependance of attenuation coefficient on frequency according to the model of equation 3.24. The right panel shows the reconstruction with energy ratios. The reconstructed attenuation of skin in both the reconstructions, 11.2 dB/cm and 13.7 dB/cm respectively at 7.5 MHz, correspond good to that found in literature [33]. The energy ratio method reconstruction does not seem to work elsewhere. The strongly reflecting portions of the mouse, according to this reconstruction, show a strong attenuation. The reconstructed attenuation is therefore due to the reflection of the signal. On the other hand the frequency shift method also works quite good in the spine region with an average reconstructed attenuation coefficient of 12.9 dB/cm which is slightly lower than
the values (14-24 dB/cm) mentioned in literature [56]. As expected, the liver has a substantially lower attenuation than the rest of tissue. The strongly reflecting areas do not show a strong attenuation in this reconstruction. While no data was available for the attenuation coefficient in kidneys in the literature, it will not be discussed here. In addition the kidneys are not separately visible in the reconstruction.

As already discussed in section 3.2.4, the modeling of the attenuation is far more difficult than that of the speed of sound. Not only the simplest models like maximum signal ratio model, or more generally the energy ratio model, but also more sophisticated models using the frequency shifting property due to attenuation has severe limitations also listed in section 3.2.4. The experimental results presented here show that the attenuation tomography within the framework of Born approximation works only moderately good. This is in line with the findings of other researchers [31, 59, 65, 66].
6.5 Breast Imaging

As mentioned before, the ultrasound transducers available for the through transmission system have too high frequency range to achieve the penetration depth needed to scan a female breast. The breast imaging was therefore limited to the echo measurement using the setup of figure 6.4. In a clinical study carried out by Huber et al. [57], it was found that the spatial compounding may enhance the potential of B-mode imaging for the differential diagnostics by increasing the conspicuity of low contrast lesions and improving the delineation of capsular margins beside more vivid depiction of systemic contents due to speckle reduction. A full angle compounding technique implemented in this work has two more advantages.

- The standardized acquisition technique gives a full view of the section of the breast being imaged. The technique may even be extended to acquire 3D data.
- The alternative technique developed in section 3.4.2 to correct for the refraction artifacts on the basis of the acquired echo data itself may enhance the resolution of the compound image.

Result of an imaging experiment carried out on the right breast of a 27 year old healthy volunteer is shown in figure 6.19. The center frequency of the transducer was set to 5.5 MHz. The compound image shown in the figure is of a section about 25 mm from the chest wall in face down position. The diameter of the breast in this section is about 85 mm. The compound was computed from a set of 72 B scans acquired over full angle with an angular resolution of five degrees.

The left panel of the figure shows the uncorrected compound image, while the right panel shows the compound image after carrying out the axial correction with the technique of section 3.4.2. The most visible part in both the images is the fat tissue just beneath the skin. At the first glance there seems to be little difference between the two images before and after the correction. It may however be noted after a close inspection of the regions bounded by rectangles that the corrected images does show more detail than the uncorrected one. It is evident from the zoomed in view shown in figure 6.20 that the arc like refraction artifacts seen in the uncorrected image are formed into well defined points in the corrected one. This shows the potential of the new correction scheme to enhance the image definition.
6.6 Chapter Summary

Figure 6.19: Spatial compound of the right breast of a healthy volunteer. The left panel shows the uncorrected spatial compound, while the right panel shows the axially corrected spatial compound according to the technique of section 3.4.2. The rectangular windows are shown on a larger scale in figure 6.20.

The image quality of spatial compound may also be assessed statistically by using the coefficient of variation \( c_v = \sigma / \mu \), where \( \sigma \) and \( \mu \) are the standard deviation and the arithmetic mean of the image data [56]. The coefficient of variation of the zoomed areas shown in figure 6.20 improved from 20.1% for the uncorrected case to 17.2% for the corrected one.

6.6 Chapter Summary

This chapter described the realized measurement system with two variants. The first one is a through transmission setup to acquire both the transmission and the echo data simultaneously and the second is a setup to acquire echo data. The two presented calibration methods were shown to possess comparable accuracy. The acquired data was analyzed in detail regarding the practical problems it posed to its further processing for tomographic reconstruction. Results of a phantom experiment were discussed in detail to illustrate the problems mentioned above. Correction of phase jitter due to data acquisition board, removal of trend from the data due to transducer tilting and sinogram
correction were discussed. The actual reconstruction results of the acoustic speed according to the methods described in previous chapters were also presented. Due to the unavailability of ultrasound transducers for the through transmission setup in the frequency range appropriate for breast imaging, a naked mouse cadaver was used to test the efficacy of the developed methods for real tissue having characteristics similar to those of the female breast. The reconstructed speed of sound show a good spatial resolution and an excellent agreement of the quantitative results with the literature data. Speed of sound reconstructions were also used to correct the compound images using both the methods investigated in chapter 5. Again, the corrected compound images also show a good quality when compared to the uncorrected ones thus indirectly verifying the quality of the speed of sound reconstruction. The reconstruction of the attenuation coefficient for the mouse tissue was moderately good for the reasons already discussed in chapter 3. The major reasons is the use of Born approximation neglecting multiple reflections, refraction and multi-path effects on one hand and the strong frequency dependance of the attenuation on the other. At the end of the chapter a compound image of a female breast in vivo was corrected with the alternative approach discussed in chapter 3.
7 Summary and Outlook

In the preceding chapters, different signal processing concepts and measurement strategies for the tomographic reconstruction of speed of sound and acoustic attenuation were presented.

Some of the important aspects of the imaging with ultrasound were presented in the beginning to provide a basis for the material dealt with in subsequent chapters. Physical concepts underlying transduction, focussing and field-medium interaction phenomena were discussed. A special emphasis was laid on highlighting the assumptions that have to be made for a viable mathematical modeling suitable to develop the algorithms. Different sources of signal attenuation and the extent to which they can be separated from each other were also discussed in order to underline the difficulties associated with the reconstruction of the attenuation within the framework of Born Approximation. Also within the same framework, the reconstructive ultrasonic tomography was dealt with in detail. Solution to some of the problems specific to ultrasound computed tomography were also presented. An accurate estimation of the time of flight is crucial to subsequent reconstruction of the acoustic speed. The data acquired with the measurement setup described towards the end of this work proved to be error prone when subjected to conventional estimation tools like threshold detection or matched filtering. A new time of flight detection approach based on deformable splines was developed and tested. The deformable splines, also known as active contour models, allow a direct incorporation of the a priori knowledge about the signals into the algorithm. The incorporated a priori information consists of smoothness constrains, lower and upper bounds of the expected time of flight and some other signal properties arising from measurement electronics. One of the two implemented approaches in solving the formulated variational problem was via Euler-Poisson equation associated with it. It yielded a temporal resolution below ultrasound pulse duration and proved to be capable of detecting the true first arrival pulse. The influence of parameter values and image energy was discussed in detail and appropriate stop criterion was presented.
The nature and shape of refraction artifacts were studied in detail with the help of a new shooting algorithm developed for ray tracing. It was noted that refraction artifact are spatially variant. The lesions that lie too closer together seem to be merged in a larger clump in the reconstruction so that the individual lesions are hard to differentiate. Several simulations showed that the overlap of the two lesions starts to recede as the distance between the two becomes larger than the mean of the diameter of the two lesions. The iterative refraction correction technique developed for transmission tomography is applicable to all cases. It may, however, be noted that the lesions clumped together due to refraction effects may not always be recovered as separate lesions with the iterative correction depending on their shapes and acoustic speeds. The two correction algorithms developed for spatial compounding, one of them using a newly developed bending algorithm for ray tracing, seem to be quite efficient in the light of the results presented in the last chapter.

In the last chapter the measurement setup and its calibration was discussed in detail. The reconstruction and the practical problems posed by the data measured with the setup were demonstrated with the help of a phantom experiment. Owing to the unavailability of ultrasound transducers for through transmission setup in the frequency range appropriate for breast imaging, a naked mouse cadaver was used to test the efficacy of the developed methods for real tissue having characteristics similar to those of female breast. The reconstructed speed of sound show a good spatial resolution and an excellent agreement of the quantitative results with the literature data, in addition to its efficacy to correct the refraction artifacts in compound images. The reconstruction of the attenuation coefficient for mouse tissue was moderately good for the reasons discussed in chapter 3. The use of Born approximation which ignores multiple reflections and the complex frequency dependance of the attenuation makes the applicability of a linearized approach only moderately successful. A compound image of a female breast in vivo and its corrected version on the basis of the echo data as already discussed in chapter 3 concluded the chapter.

The applicability of the achieved small animal results in a more realistic setting is quite promising, specially for the imaging of tumors that lie quite next to the skin. The motion artifacts expected in the in vivo setting will have to be taken care of using an ECG triggered acquisition. The acoustic properties like speed of sound and tissue attenuation of mouse tumors need to be studied to show their histological relevance for differentiating between different tissue changes.
The acquisition time for a single set consisting of 180 angular positions is kept at 2.6 minutes though a much shorter time of about $2 \times 12 \text{ cm}/1500 \text{ ms}^{-1} \times 160 \text{ lines} \times 180 \text{ positions} = 4.6 \text{ seconds}$ is physically needed to acquire the data of 180 frames. The reason to choose the much longer time was to avoid the motion artifacts caused by the movements of the transducer assembly in water. For the case of the breast the acquisition time for 72 frames was about 10 minutes. One of the reasons for this was a very slow frame rate set on the ultrasound system to avoid multiple reflections.

In the conclusion, it may be said that this work is the first one to implement a versatile tomography system based on a commercial ultrasound system. Due to the accurate time of flight estimation algorithm and the refraction correction the achieved results show a very good quality of speed of sound reconstruction. The diagnostic value of the speed of sound in combination with attenuation coefficient is well established [44, 96] for breast tissue differentiation. An appropriate ultrasound transducer with a center frequency of about 1-2 MHz will yield similar results for in vivo breast imaging.

A combination of the developed setup with a photo acoustic tomography setup may also provide additional diagnostic value. The current design of the add-on scanner is capable of integrating such a system into it. The compound image correction technique based on echo data may prove useful for the reasons given in section 6.5. It may be noted that a clinical study has to be carried out to find out the clinical relevance of the corrected compound imaging alone.

Another approach which could not be tested experimentally as yet, is to extend the limited angle multi-static synthetic aperture approach presented by Krüger [74] to a full angle tomographic technique. It was shown in simulations not included in this work that a data acquired from only 18 angular positions is capable of reconstructing the speed of sound whose accuracy will be comparable to the one carried out from normal data acquisition over 90 angular positions. That can substantially contribute to reduce motion artifacts. Such an approach will have a much improved uniform spatial resolution and due to the availability of single channel data, it will be possible to correct the refraction artifacts more effectively. Though such a system was simulated during the current research, that could not be tested experimentally due to the absence of suitable ultrasound system which allows access to its single channel data.
A Appendices

A.1 A Field-theoretical Simulation Algorithm for Ultrasound Propagation

A simulated signal is often very useful particularly for studying the effects of individual parameters which may otherwise not be realizable in a real measurement. The other reason to model is to compare the results of other approximations used in this work with the field theoretical solution. For this purpose an algorithm for the simulation of the ultrasonic propagation will be presented in this section. To simulate propagation of a broadband signal on a domain of the order of several hundred wavelengths accurately becomes inefficient in terms of manageable computational costs if finite element or finite difference methods are employed. Approximating the derivative with local low order spatial and temporal derivatives distorts signals severely over large distances of propagation, particularly in terms or their phase [78].

A feasible solution to the problem lies in the combination of the pseudo spectral method with perfectly matched layer (PML) boundaries first proposed in [85, 107, 108]. A detailed treatment of the pseudo spectral method and the PML can be found in [43, 109]. It is useful to write down the linearized field equations instead of the wave equation, for example from [37].

\[
\begin{align*}
\rho \frac{\partial \mathbf{v}}{\partial t} &= -\nabla p \\
\kappa \frac{\partial p}{\partial t} &= -\nabla \cdot \mathbf{v}
\end{align*}
\]

where \(\rho\) and \(\kappa\) denote the medium parameters density and compressibility respectively, and \(p\) and \(\mathbf{v}\) denote the field quantities pressure and particle velocity respectively. In
case of a two dimensional problem the equations may be rewritten as follows:

\[
\frac{\partial v_x}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \\
\frac{\partial v_z}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial z} \\
\frac{\partial p}{\partial t} = -\frac{1}{\kappa} \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} \right)
\]

(A.3) \hspace{1cm} (A.4) \hspace{1cm} (A.5)

where \(v_x\) and \(v_z\) are the \(x\) and \(z\) components of the particle velocity \(v\). The pseudo spectral method consists of taking the fourier transform in the space domain so that the above partial differential equations (PDEs) are reduced to the following ordinary differential equations (ODEs) in time:

\[
\frac{\partial V_x}{\partial t} = -\frac{1}{\rho} j\omega_x P \\
\frac{\partial V_z}{\partial t} = -\frac{1}{\rho} j\omega_z P \\
\frac{\partial P}{\partial t} = -\frac{1}{\kappa} \left( j\omega_x V_x + j\omega_z V_z \right)
\]

(A.6) \hspace{1cm} (A.7) \hspace{1cm} (A.8)

Equations A.6, A.7 and A.8 may be integrated with a standard ODE solver after being discretized to some suitable cartesian grid. The incident waves are taken as the initial conditions. The problem that still remains to solve is the periodic nature of the pseudo spectral method [43] resulting in wraparounds at boundaries. To overcome this problem the perfectly matched layer (PML) boundaries will be used. Adapted for the case of acoustic propagation in [109] for finite differences simulation, the PML boundary conditions are essentially absorbing boundary conditions. The wave incident on the PML consisting of several nodes near the boundary with gradually increasing attenuation coefficient will be attenuated twice, on leaving the boundaries as well as on re-entering them. The governing equations at the PML boundaries may be generalized from equations A.6-A.8 by introducing an attenuating term. The PML equations for two dimensional acoustic case are obtained by splitting pressure \(P\) into \(P_x\) and \(P_z\) [109].
and adding a damping term to the resulting equations.

\[
\begin{align*}
\frac{\partial V_x}{\partial t} &= -\frac{1}{\rho} j \omega_x P - \frac{1}{\kappa} \alpha V_x \\
\frac{\partial V_z}{\partial t} &= -\frac{1}{\rho} j \omega_z P - \frac{1}{\kappa} \alpha V_z \\
\frac{\partial P_x}{\partial t} &= -\frac{1}{\kappa} j \omega_x V_x - \frac{1}{\kappa} \alpha P_x \\
\frac{\partial P_z}{\partial t} &= -\frac{1}{\kappa} j \omega_z V_z - \frac{1}{\kappa} \alpha P_z
\end{align*}
\] (A.9, A.10, A.11, A.12)

The system of these four ODEs may also be rewritten as a matrix ODE A.13 if a state variable \(X = [V_x, V_z, P_x, P_z]^T\) is introduced.

\[
\frac{\partial X}{\partial t} = AX
\] (A.13)

where \(A\) is given by

\[
A = -\frac{1}{\rho c^2} \begin{bmatrix}
\alpha & 0 & j \omega_x c^2 & j \omega_x c^2 \\
0 & \alpha & j \omega_z c^2 & j \omega_z c^2 \\
-j \omega_x & 0 & \alpha & 0 \\
0 & -j \omega_z & 0 & \alpha
\end{bmatrix}
\] (A.14)

where \(\kappa = \rho c^2\). The implementation of the algorithm is straightforward. After initializing the state variable \(W\) for any given excitation on a suitable cartesian grid, spatial derivatives are calculated using fast Fourier transform (FFT). The ODE A.13 is solved with a standard solver one time-step yielding the new state variable. The last two steps are repeated as long as dictated by the problem. The PML matrix equation A.13 is used also for the interior of the domain by setting \(\alpha\) equal to zero. The relaxation parameter responsible for absorption [86] in case of a fairly uniform medium may also be incorporated into the algorithm to simulate an attenuating medium.
A.2 Estimation of Attenuation Line Integral

Starting with equation 3.23 modeling the transmission of ultrasound taking into account the most important factors affecting attenuation, also reproduced below as equation A.15 for simplicity, a relationship will be derived for the estimation of the line integral $\int_L \alpha(f) \, dl$. This derivation is akin to the treatment presented in [31][95].

\[ |Y_{\text{obj}}(f)|^2 = T e^{-2\int_L \alpha(f) \, dl} |Y_{\text{water}}(f)|^2 \]  \hspace{1cm} (A.15)

Approximating the power spectral density of the signal transmitted through water $|Y_{\text{water}}(f)|^2$ as a Gaussian function having a mean $f_{c,\text{water}}$ and a variance $\sigma^2$ results in

\[ |Y_{\text{water}}(f)|^2 = P_{Y_{\text{water, max}}} e^{-\frac{(f-f_{c,\text{water}})^2}{2\sigma^2}} \]  \hspace{1cm} (A.16)

Invoking a simple linear frequency dependence for the tissue attenuation in the form of $\alpha = \alpha_0 f$, equation A.15 reduces to

\[ |Y_{\text{obj}}(f)|^2 = T P_{Y_{\text{water, max}}} e^{-2fI - \frac{(f-f_{c,\text{water}})^2}{2\sigma^2}} \]  \hspace{1cm} (A.17)

where $I = \int_L \alpha_0 \, dl$ denotes the line integral of the normalized attenuation coefficient. Now it is elementary algebra to rearrange the the exponential term $-2fI - \frac{(f-f_{c,\text{water}})^2}{2\sigma^2}$ as $-(f-(f_{c,\text{water}}-2\sigma^2))^2 - f_{c,\text{water}}^2 + f_{c,\text{water}}(2\sigma^2)^2$. Here two different approaches can be taken to estimate the line integral $I$.

A.2.1 Direct Estimation From Frequency Shift

If the second term in the above simplified expression is treated as a constant and merged together with $P_{Y_{\text{water, max}}}$ into the scaler $T$ so that equation A.17 takes on the following form.

\[ |Y_{\text{obj}}(f)|^2 = T e^{-\frac{(f-(f_{c,\text{water}}-2\sigma^2))^2}{2\sigma^2}} \]  \hspace{1cm} (A.18)

It is evident that the center frequency of the power spectral density $|Y_{\text{obj}}(f)|^2$ is lowered by $2I\sigma^2$. This shifted central frequency $f_{c,\text{obj}}$ may be estimated from the power spectral density numerically and will be related to $f_{c,\text{water}}$ through

\[ f_{c,\text{obj}} = f_{c,\text{water}} - 2I\sigma^2 \]  \hspace{1cm} (A.19)
A.2 Estimation of Attenuation Line Integral

The line integral $I = \int_L d\alpha_0$ is thus given by

$$I = \int_L d\alpha_0 = \frac{f_{c,\text{water}} - f_{c,\text{obj}}}{2\sigma^2}$$  \hspace{1cm} (A.20)

The required integral $\int_L \alpha(f) dl$ under the above linearity assumption thus becomes

$$\int_L \alpha(f) dl = \frac{f_{c,\text{water}} - f_{c,\text{obj}}}{2\sigma^2}.$$  \hspace{1cm} (A.21)

A.2.2 Indirect Estimation from Total Signal Energies

The other way to estimate integral $I$ is to calculate the total signal energies from equations A.16 and A.17 given by.

$$E_{\text{water}} = \int_{-\infty}^{\infty} |Y_{\text{water}}(f)|^2 df = \sqrt{2\pi \sigma} P Y_{\text{water,max}}$$  \hspace{1cm} (A.22)

$$E_{\text{obj}} = \int_{-\infty}^{\infty} |Y_{\text{obj}}(f)|^2 df = \sqrt{2\pi \sigma} P Y_{\text{water,max}} T e^{-\frac{f_{c,\text{water}} - 2f_{c,\text{water}} - 2\sigma^2}{2\sigma^2}}$$  \hspace{1cm} (A.23)

where $\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sqrt{2\pi \sigma}$.[26] Dividing equation A.23 by A.22, simplifying the exponential term, taking natural logarithm of the resulting equation and rearranging results in the following quadratic equation, which can be solved to find out the value of the integral $I$.

$$I^2 - \left(\frac{f_{c,\text{water}}}{\sigma^2}\right) I + \left(\frac{1}{2\sigma^2} \ln T - \frac{1}{2\sigma^2} \ln \left(\frac{E_{\text{obj}}}{E_{\text{water}}}\right)\right) = 0$$  \hspace{1cm} (A.24)

In this case the transmittance $T$ will be set equal to one so that the above equation has the following solution.

$$I = \frac{f_{c,\text{water}} - \sqrt{f_{c,\text{water}}^2 - 2\sigma^2 \ln \frac{E_{\text{obj}}}{E_{\text{water}}}}}{2\sigma^2}$$  \hspace{1cm} (A.25)
A.3 Reverse Tracing of the Beam Path in a B-mode Image

Consider the following figure where the field emanating from the scatterer located at \((x_1, z_1)\) reaches the active aperture of the transducer at \((x, z_0)\) after being refracted at \((x, z)\). The time \(t\) taken by the backscattered waves to reach the transducer is given by.

\[
t = \frac{z - z_0}{c_0} + \frac{\sqrt{(x - x_1)^2 + (z - z_1)^2}}{c_1} = \frac{z - z_0}{c_0} + \frac{d_0}{c_1}\tag{A.26}
\]

The point \((x, z)\) that minimizes this time of arrival will satisfy the Fermat’s law. That the point \((x, z)\) lies on the circle \(x^2 + z^2 - r^2 = 0\) may be taken care of by formulating it as a constraint in the minimization problem with an introduction of a Lagrange multiplier \(\sigma\). The functional \(F(x, z, \sigma)\) to be minimized is thus given by

\[
F(x, z, \sigma) = t + \sigma \left( x^2 + z^2 - r^2 \right)\tag{A.27}
\]

While the condition \(\partial F/\partial \sigma = 0\) will result in the equation of the circle, the other two conditions \(\partial F/\partial x = 0\) and \(\partial F/\partial z = 0\) will yield the following system of equations.

\[
\frac{x - x_1}{c_1d_0} + 2\sigma x = 0 \text{ and } \frac{1}{c_0} + \frac{z - z_1}{c_1d_0} + 2\sigma z = 0\tag{A.28}
\]
Eliminating $\sigma$ from these two equations yields equation A.29 which can be solved together with the equation of the circle to determine the point $(x, z)$.

$$c_0 (x_1 z - x z_1) + c_1 x d_0 = 0 \quad (A.29)$$

This is a fourth order equation which will yield four possible solutions for the point $(x, z)$. As $(x, z)$ is physically the point of intersection of a straight line with a circle, either the solutions will be pairwise identical or two of the solutions will be complex. Sorting out the complex solutions and minimizing the distance between points $P_1$ and $P_0$ will yield the required solution as the other solution will be the intersection of the straight line with the upper half of the circle in figure A.1. The coordinates of the points $P_2$ and $P_3$ will be given by A.30 and A.31.

$$P_2 : (x, z + d_0) \quad (A.30)$$

$$P_3 : \left( x, z + d_0 \frac{c_0}{c_1} \right) \quad (A.31)$$
A.4 A Least Square Estimator for Calibration

Consider the following figure modified from figure 6.6. The three transformations involved here are translation, scaling and rotation denoted by the matrices $T_{\text{move2center}}$, $T_{\text{mat2cart}}$ and $T_{\theta_i}$ respectively, given by the following equations.

$$
T_{\text{mat2cart}} = \begin{bmatrix}
m & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$  \hspace{1cm} (A.32)

$$
T_{\text{move2center}} = \begin{bmatrix}
1 & 0 & -x_c \\
0 & 1 & -y_c \\
0 & 0 & 1
\end{bmatrix}
$$  \hspace{1cm} (A.33)

$$
T_{\theta_i} = \begin{bmatrix}
\cos \theta_i & \sin \theta_i & 0 \\
-\sin \theta_i & \cos \theta_i & 0 \\
0 & 0 & 1
\end{bmatrix}
$$  \hspace{1cm} (A.34)

Now, let the matrix coordinates of the point $P$ in frame 1 be denoted by $P1 = [x_1, y_1]^T$.
and those of frame \( n \) be \( P_1 = [x_i, y_i]^T \). The cartesian coordinates corresponding to the above matrix coordinates with respect to frame 1 will be given by \( T_{\text{mat}_2\text{cart}}T_{\text{move}_2\text{center}}[P_1^T, 1]^T \) and \( T_{\theta_1}T_{\text{mat}_2\text{cart}}T_{\text{move}_2\text{center}}[P_1^T, 1]^T \) respectively. The respective coordinates may now be subtracted to get the error between the two frames.

\[
T_{\text{mat}_2\text{cart}}T_{\text{move}_2\text{center}}[P_1^T, 1]^T - T_{\theta_1}T_{\text{mat}_2\text{cart}}T_{\text{move}_2\text{center}}[P_1^T, 1]^T
\]

This may easily be reduced to the following form after multiplying the individual matrices denoting affine transformations.

\[
T_1[P_1^T, 1]^T - T_i[P_i^T, 1]^T
\]

where \( T_1 \) and \( T_i \) are given by

\[
T_1 = \begin{bmatrix}
m & 0 & -m x_c \\
0 & -1 & y_c \\
0 & 0 & 1
\end{bmatrix}
\]

(A.37)

and

\[
T_i = \begin{bmatrix}
m \cos \theta_i & -\sin \theta_i & -m \cos \theta_i + y_c \sin \theta_i + m x_c \\
-m \sin \theta_i & -\cos \theta_i & m x_c \sin \theta_i + y_c \cos \theta_i \\
0 & 0 & 1
\end{bmatrix}
\]

(A.38)

It is evident from the last two equations that the estimation of \( m, m x_c \) and \( y_c \) in the sense of least square minimization will reduce this nonlinear estimation problem to a linear one. We, therefore, define three parameters to be estimated as \( p_1 = m x_c, p_2 = m \) and \( p_3 = y_c \). Substituting these values and equations A.37 and A.38 in A.36, and summing the Euclidean distances between respective pairs of frames will yield the following quantity to be minimized.

\[
f(p) = \sum_{i=2}^{n} (A_i p - b_i)^T (A_i p - b_i)
\]

(A.39)

where

\[
A_i = \begin{bmatrix}
(1 - \cos \theta_i) & (x_1 \cos \theta_i - x_i) & \sin \theta_i \\
\sin \theta_i & -x_1 \sin \theta_i & -(1 - \cos \theta_i)
\end{bmatrix}
\]

(A.40)

\[
b_i = \begin{bmatrix}
-y_1 \sin \theta_i \\
-y_1 \cos \theta_i + y_i
\end{bmatrix}
\]

(A.41)

\[
p = \begin{bmatrix}
p_1 \\
p_2 \\
p_3
\end{bmatrix}
\]

(A.42)
The functional \( f(p) \) will be minimum, for \( \partial f / \partial p_i = 0 \), for \( i = 1, 2, 3 \). The three resulting linear equations are given by the following matrix equation which may be solved to arrive at the desired parameter estimates.

\[
\begin{bmatrix}
2 \sum (1 - \cos \theta_i) & \sum (x_1 + x_i)(\cos \theta - 1) & 0 \\
\sum (x_1 + x_i)(\cos \theta - 1) & \sum (x_1^2 + x_i^2 - 2x_1x_i \cos \theta_i) & \sum (x_1 - x_i) \sin \theta_i \\
0 & \sum (x_1 - x_i) \sin \theta_i & 2 \sum (1 - \cos \theta_i)
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
p_3
\end{bmatrix}
= \begin{bmatrix}
\sum (y_1 - y_i) \sin \theta_i \\
\sum (x_1 y_i - x_i y_1) \sin \theta_i \\
\sum (y_1 + y_i)(1 - \cos \theta_i)
\end{bmatrix}
\tag{A.43}
\]

All the summations in the above equation are carried out from \( i = 2, \ldots, n \).
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